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Rota-Baxter Family Systems and Gröbner-Shirshov Bases

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Abstract Motivated by the concept of Rota-Baxter family algebras arising from associative Yang-Baxter family equations and Volterra integral equations, we introduce the notion of a Rota-Baxter family system which generalizes the Rota-Baxter system proposed by Brzeziński. We show that this notion is also related to an associative Yang-Baxter family pair and the pre-Lie family algebras. Furthermore, as an analogue of Rota-Baxter family system, we introduce a notion of averaging family system and prove that an averaging family system induces a dialgebra family structure. We also study Rota-Baxter family systems on a dendriform algebra and show how they induce quadri family algebra structures. Finally, we give a linear basis of the Rota-Baxter family system by the methods of Gröbner-Shirshov bases.

Keywords Rota-Baxter family systems; Rota-Baxter family algebras; Gröbner-Shirshov bases

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1. Introduction

The concept of Rota-Baxter family algebra was proposed by Guo, which arises naturally in renormalization of quantum field theory [1]. Based on his excellent work, free Rota-Baxter family algebras and (tri)dendriform family algebras [2], free (tri)dendriform family algebras [3,4] appeared successively. Recently, Das showed that Rota-Baxter family algebras have some new motivations from associative Yang-Baxter family equations [5]. More precisely, Das [5] called the corresponding equation

$$r^{13}_{\alpha\beta}r^{12}_{\alpha} - r^{12}_{\alpha}r^{23}_{\beta} + r^{23}_{\beta}r^{13}_{\alpha\beta} = 0$$

an associative Yang-Baxter family equation (AYBFE). Paralleled to the fact that solutions of the associative Yang-Baxter family equation naturally give Rota-Baxter family operators, the Rota-Baxter family operators are determined by solutions of an AYBFE [5]. Considering the Rota-Baxter system is another generalization of the Rota-Baxter algebra, it is especially interesting to study Rota-Baxter system in the context of family version. This is one of our purposes of this paper.

The concept of Rota-Baxter systems was introduced by Brzeziński [6] in the study of the Jackson q-integral as a Rota-Baxter operator and the extended connections between Rota-Baxter

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algebras [7, 8], dendriform algebras and infinitesimal bialgebras. Motivated by the terminology of Rota-Baxter family algebras [1] and Rota-Baxter systems, different from [9], we introduce the notion of Rota-Baxter family systems and give some characterizations and new constructions of it, we modify the definition of Rota-Baxter family systems by adding a curvature term and then derive the conditions that the curvature has to satisfy in order to make a connection with the pre-Lie family algebra.

Our second source of motivation comes from the Gröbner-Shirshov bases, which is an effective tool to construct a free object for some algebraic structures. In 2009, Bokut, Chen and Qiu [10] established the Composition-Diamond lemma for associative algebras with multiple linear operators. As a consequence, they obtained Gröbner-Shirshov bases of free Rota-Baxter algebras. This method has been successfully used to construct some other free objects, such as free Rota-Baxter algebras [11], free Ω-Lie algebras [12], free intergro-differential algebras [13], free differential algebras [14], free pre-associative algebras [15], replicated algebras [16], free Lie differential Rota-Baxter algebras [17], free Rota-Baxter systems [18], operated algebraic structures [19] and so on. In this paper, we construct free objects in the category of Rota-Baxter family system, via a method of Gröbner-Shirshov bases.

The paper is organized as follows. In Section 2, we first recall some basic concepts that will be used in this paper. In Section 3, we introduce the notion of Rota-Baxter family systems and give some characterizations and new constructions. In Section 4, as a particular case of Rota-Baxter family system, we introduce a notion of averaging family system and prove that an averaging family system induces a dialgebra family structure. In Section 5, we consider Rota-Baxter family systems on a dendriform algebra and show how they induce quadri family algebra structures. In Section 6, we construct free objects in the category of Rota-Baxter family systems, via a method of Gröbner-Shirshov bases.

Throughout this paper, let $\mathbf{k}$ be a unitary commutative ring unless the contrary is specified. It will be the base ring of all modules, algebras, tensor products, as well as linear maps.

2. Preliminaries

In this section, we recall some useful definitions which will be used in this paper from [2, 4].

Definition 2.1 *Let Ω be a semigroup and $\lambda \in \mathbf{k}$. A Rota-Baxter family algebra of weight λ is a pair $(A, \{R_\omega | \omega \in \Omega\})$ consisting of an associative algebra A and a collection of linear operators $\{R_\omega | \omega \in \Omega\}$ such that*

$$R_\alpha(x)R_\beta(y) = R_{\alpha\beta}(xR_\beta(y) + R_\alpha(x)y + \lambda xy), \quad \forall\, x, y \in A, \alpha, \beta \in \Omega. \tag{2.1}$$

Definition 2.2 *Let Ω be a semigroup. A dendriform family algebra is a $\mathbf{k}$-module D together with a collection of binary operations $\{\prec_\omega, \succ_\omega | \omega \in \Omega\}$, such that, for $x, y, z \in D$ and $\alpha, \beta \in \Omega$, there is*

$$(x \prec_\alpha y) \prec_\beta z = x \prec_{\alpha\beta} (y \prec_\beta z + y \succ_\alpha z), \tag{2.2}$$

$$(x \succ_\alpha y) \prec_\beta z = x \succ_\alpha (y \prec_\beta z), \tag{2.3}$$

$$(x \prec_\beta y + x \succ_\alpha y) \succ_{\alpha\beta} z = x \succ_\alpha (y \succ_\beta z). \tag{2.4}$$

We now recall the concept of a pre-Lie family algebra and its construction by typed rooted trees which was used by [4] to obtain that the operad of pre-Lie family algebras is isomorphic to the operad of typed labeled rooted trees.

Definition 2.3 *Let Ω be a commutative semigroup. A (left) pre-Lie family algebra is a pair $(A, \{*_\omega|\omega \in \Omega\})$ consisting of a* $\mathbf{k}$*-module A and a collection of binary operations $*_\omega : A \otimes A \to A, \omega \in \Omega$, that satisfy*

$$x *_\alpha (y *_\beta z) - (x *_\alpha y) *_{\alpha\beta} z = y *_\beta (x *_\alpha z) - (y *_\beta x) *_{\beta\alpha} z$$

for all $x, y \in A, \alpha, \beta \in \Omega$.

3. Rota-Baxter family systems

In this section, we introduce the notion of a Rota-Baxter family system and provide some examples of Rota-Baxter family systems.

Let us first recall the concept of Rota-Baxter system introduced by Brzeziński [6].

Definition 3.1 *A triple (A, R, S) consisting of an associative algebra A and two linear maps $R, S : A \to A$ is called a Rota-Baxter system if*

$$\begin{aligned} R(x)R(y) &= R(R(x)y) + R(xS(y)), \\ S(x)S(y) &= S(R(x)y) + S(xS(y)) \quad \textit{for all} \;\; x, y \in A. \end{aligned}$$

Guo et al. studied the family compatibility of algebraic structures carrying multiple copies of the same operations [1]. Applying it to the case of Rota-Baxter system, we obtain the following concept.

Definition 3.2 *Let Ω be a semigroup. A Rota-Baxter family system is a triple $(A, \{R_\omega|\omega \in \Omega\}, \{S_\omega|\omega \in \Omega\})$ consisting of an associative algebra A and a collection of linear operators $\{R_\omega, S_\omega|\omega \in \Omega\}$ such that*

$$\begin{aligned} R_\alpha(x)R_\beta(y) &= R_{\alpha\beta}(R_\alpha(x)y + xS_\beta(y)), \\ S_\alpha(x)S_\beta(y) &= S_{\alpha\beta}(R_\alpha(x)y + xS_\beta(y)) \quad \textit{for all} \;\; x, y \in A, \alpha, \beta \in \Omega. \end{aligned}$$

Any Rota-Baxter family system can be viewed as a Rota-Baxter system by taking Ω to be a trivial semigroup.

The following result shows the relation between Rota-Baxter family systems and ordinary Rota-Baxter family systems.

Proposition 3.3 *Let $(A, \{R_\omega|\omega \in \Omega\}, \{S_\omega|\omega \in \Omega\})$ be a Rota-Baxter family system. Consider the linear maps*

$$R : A \otimes \mathbf{k}\Omega \to A \otimes \mathbf{k}\Omega, x \otimes \omega \mapsto R_\Omega(x) \otimes \omega,$$

$$S : A \otimes \mathbf{k}\Omega \to A \otimes \mathbf{k}\Omega, x \otimes \omega \mapsto S_\Omega(x) \otimes \omega.$$

Then $(A \otimes \mathbf{k}\Omega, R, S)$ *is a Rota-Baxter system.*

Proof For any $x, y \in A$ and $\alpha, \beta \in \Omega$, we have

$$\begin{aligned} R(x \otimes \alpha)R(y \otimes \beta) &= (R_\alpha(x) \otimes \alpha)(R_\beta(y) \otimes \beta) = R_\alpha(x)R_\beta(y) \otimes \alpha\beta \\ &= R_{\alpha\beta}(R_\alpha(x)y + xS_\beta(y)) \otimes \alpha\beta = R\{(R_\alpha(x)y + xS_\beta(y)) \otimes \alpha\beta\} \\ &= R\{(R_\alpha(x) \otimes \alpha)(y \otimes \beta) + (x \otimes \alpha)(S_\beta(y) \otimes \beta)\} \\ &= R\{R(x \otimes \alpha)(y \otimes \beta) + (x \otimes \alpha)S(y \otimes \beta)\}. \end{aligned}$$

Similarly, we have

$$S(x \otimes \alpha)S(y \otimes \beta) = S\{R(x \otimes \alpha)(y \otimes \beta) + (x \otimes \alpha)S(y \otimes \beta)\}.$$

This completes the proof. □

We have seen that Rota-Baxter family systems generalize Rota-Baxter family algebra. In the following, we show that they also generalize Rota-Baxter family algebras of arbitrary weight.

Proposition 3.4 *Let* $(A, \{R_\omega|\omega \in \Omega\})$ *be a Rota-Baxter family algebra of weight* λ*. Then* $(A, \{R_\omega|\omega \in \Omega\}, \{R_\omega + \lambda id|\omega \in \Omega\})$ *is a Rota-Baxter family system.*

Proof For any $x, y \in A$ and $\alpha, \beta \in \Omega$, we have

$$\begin{aligned} R_\alpha(x)R_\beta(y) &= R_{\alpha\beta}(xR_\beta(y)) + R_{\alpha\beta}(R_\alpha(x)y) + \lambda R_{\alpha\beta}(xy) \\ &= R_{\alpha\beta}(R_\alpha(x)y) + R_{\alpha\beta}(x(R_\beta + \lambda id)(y)) \end{aligned}$$

and

$$\begin{aligned} (R_\alpha + \lambda id)(x)(R_\beta + \lambda_\beta id)(y) &= R_\alpha(x)R_\beta(y) + \lambda R_\alpha(x)y + \lambda xR_\beta(y) + \lambda^2 xy \\ &= R_{\alpha\beta}(xR_\beta(y)) + R_{\alpha\beta}(R_\alpha(x)y) + \lambda R_{\alpha\beta}(xy) + \lambda R_\alpha(x)y + \lambda xR_\beta(y) + \lambda^2 xy \\ &= (R_{\alpha\beta} + \lambda id)(R_\alpha(x)y) + (R_{\alpha\beta} + \lambda id)(x(R_\beta + \lambda id)(y)). \end{aligned}$$

This shows that $(A, \{R_\omega|\omega \in \Omega\}, \{R_\omega + \lambda id|\omega \in \Omega\})$ is a Rota-Baxter family system. □

The following example shows that Rota-Baxter family systems arise from twisted Rota-Baxter family algebras.

Definition 3.5 *Let* A *be an associative algebra and* Ω *be a semigroup. A collection of linear operators* $\{R_\omega : A \to A|\omega \in \Omega\}$ *is said to be a* $\{\sigma_\omega|\omega \in \Omega\}$*-twisted Rota-Baxter family operator if there exist a collection of algebra morphisms* $\{\sigma_\omega : A \to A|\omega \in \Omega\}$ *such that, for all* $x, y \in A, \alpha, \beta \in \Omega$

$$R_\alpha(x)R_\beta(y) = R_{\alpha\beta}(R_\alpha(x)y + x(\sigma \circ R)_\beta(y)),$$

where $\sigma_\beta \circ R_\beta := (\sigma \circ R)_\beta$*. Then the pair* $(A, \{R_\omega|\omega \in \Omega\})$ *is called a* $\{\sigma_\omega|\omega \in \Omega\}$*-twisted Rota-Baxter family algebra.*

Remark 3.6 (1) When $\sigma = id$, a σ-twisted Rota-Baxter family algebra is nothing but a Rota-Baxter family algebra.

(2) When Ω is the trivial monoid with one single element, the σ_Ω-twisted Rota-Baxter family algebra is the σ-twisted Rota-Baxter algebra proposed by Brzeziński [6].

Definition 3.7 *A differential Rota-Baxter family algebra of weight λ is an associative algebra A together with a collection of linear maps $\{R_\omega, \partial_\omega : A \to A, |\omega \in \Omega\}$ satisfying the following set of identities*

$$\begin{aligned} &\text{(dRF1)} \;\; R_\alpha(x)R_\beta(y) = R_{\alpha\beta}(xR_\beta(y) + R_\alpha(x)y + \lambda xy),\\ &\text{(dRF2)} \;\; \partial_{\alpha\beta}(xy) = \partial_\alpha(x)y + x\partial_\beta(y) + \lambda\partial_\alpha(x)\partial_\beta(y),\\ &\text{(dRF3)} \;\; \partial_\beta \circ R_\beta = id \;\; \text{for all} \;\; \alpha, \beta \in \Omega. \end{aligned}$$

Let $(A, \{R_\omega|\omega \in \Omega\}, \{\partial_\omega|\omega \in \Omega\})$ be a differential Rota-Baxter family algebra of weight λ. It follows from (dRF2) that the map

$$\sigma_\beta : A \to A, \;\; \sigma_\beta(x) = x + \lambda\partial_\beta(x) \;\; \text{for all} \;\; x \in A, \beta \in \Omega$$

is an algebra morphism. Further, (dRF3) implies that

$$(\sigma \circ R)_\beta(x) = R_\beta(x) + \lambda x \;\; \text{for all} \;\; x \in A, \beta \in \Omega.$$

Hence, by (dRF2), we get

$$R_\alpha(x)R_\beta(y) = R_{\alpha\beta}(R_\alpha(x)y + x(\sigma \circ R)_\beta(y)) \;\; \text{for all} \;\; x, y \in A, \alpha, \beta \in \Omega.$$

This shows that $(A, \{R_\omega|\omega \in \Omega\}, \{\sigma_\omega|\omega \in \Omega\})$ is a $\{\sigma_\omega|\omega \in \Omega\}$-twisted Rota-Baxter family algebra.

Proposition 3.8 *Let (A, R_Ω) be a σ_Ω-twisted Rota-Baxter family algebra. Then $(A, \{R_\omega|\omega \in \Omega\}, \{(\sigma \circ R)_\omega|\omega \in \Omega\})$ is a Rota-Baxter family system.*

Proof Note that the condition (dRF1) is same as the first condition of a Rota-Baxter family system. To prove the second one, we observe that

$$\begin{aligned} &(\sigma \circ R)_\alpha(x)(\sigma \circ R)_\beta(y) = (\sigma \circ R)_\alpha(x)(R_\beta(y) + \lambda y)\\ &\quad = (R_\alpha(x) + \lambda_\alpha x)(R_\beta(y) + \lambda y) = R_\alpha(x)R_\beta(y) + \lambda R_\alpha(x)y + \lambda xR_\beta(y) + \lambda^2 xy\\ &\quad = R_{\alpha\beta}(R_\alpha(x)y + x(\sigma \circ R)_\beta(y)) + \lambda R_\alpha(x)y + \lambda xR_\beta(y) + \lambda^2 xy\\ &\quad = R_{\alpha\beta}(R_\alpha(x)y) + \lambda R_\alpha(x)y + R_{\alpha\beta}(x(\sigma \circ R)_\beta(y)) + \lambda xR_\beta(y) + \lambda^2 xy\\ &\quad = R_{\alpha\beta}(R_\alpha(x)y) + \lambda R_\alpha(x)y + R_{\alpha\beta}(x(\sigma \circ R)_\beta(y)) + \lambda x(\sigma \circ R)_\beta(y)\\ &\quad = (\sigma \circ R)_{\alpha\beta}(R_\alpha(x)y) + (\sigma \circ R)_{\alpha\beta}(x(\sigma \circ R)_\beta(y)). \end{aligned}$$

This shows that $(A, \{R_\omega|\omega \in \Omega\}, \{(\sigma \circ R)_\omega|\omega \in \Omega\})$ is a Rota-Baxter family system. □

We now establish the connections between Rota-Baxter family systems and dendriform family algebras. For the classical case of one linear operator [6].

Proposition 3.9 *(1) A Rota-Baxter family system $(A, \{R_\omega|\omega \in \Omega\}, \{S_\omega|\omega \in \Omega\})$ induces a* *dendriform family algebra* $(A, \{\prec_\omega, \succ_\omega | \omega \in \Omega\})$, *where*

$$x \prec_\omega y := xS_\omega(y),\ x \succ_\omega y := R_\omega(x)y \ \text{ for all } \ x, y \in A, \omega \in \Omega. \tag{3.1}$$

(2) If A is a non-degenerate algebra and $(A, \{\prec_\omega, \succ_\omega | \omega \in \Omega\})$ is a dendriform family algebra, then $(A, \{R_\omega|\omega \in \Omega\}, \{S_\omega|\omega \in \Omega\})$ is a Rota-Baxter family system.

Proof (a) We just verify Eqs. (2.2)–(2.4) can be verified in the same way. For any $x, y, z \in A$ and $\alpha, \beta \in \Omega$, we have

$$\begin{aligned}(x \prec_\alpha y) \prec_\beta z &= xS_\alpha(y)S_\beta(z) = xS_{\alpha\beta}(R_\alpha(y)z) + xS_{\alpha\beta}(yS_\beta(z))\\ &= x \prec_{\alpha\beta} (y \prec_\beta z) + x \prec_{\alpha\beta} (y \succ_\alpha z).\end{aligned}$$

(b) In the converse direction, let us assume that $(A, \{\prec_\omega, \succ_\omega | \omega \in \Omega\})$ with $\{\prec_\omega, \succ_\omega | \omega \in \Omega\}$ given by Eq. (3.1), is a dendriform family algebra. Then the dendriform family relation given by Eq. (2.4) comes out as

$$(R_\alpha(x)R_\beta(y) - R_{\alpha\beta}(R_\alpha(x)y) - R_{\alpha\beta}(xS_\beta(y)))z = 0,$$

and hence it gives a Rota-Baxter family system by the non-degeneracy of the product in A. □

According to [4], we have

Corollary 3.10 *Let $(A, \{R_\omega|\omega \in \Omega\}, \{S_\omega|\omega \in \Omega\})$ be a Rota-Baxter family system and Ω be a commutative semigroup. Then $(A, (*_\omega)_{\omega\in\Omega})$ with $*_\omega : A \otimes A \to A$, defined by*

$$x *_\omega y = R_\omega(x)y - yS_\omega(x) \ \text{ for all } \ x, y \in A, \omega \in \Omega,$$

is a pre-Lie family algebra.

Combining [6] and [5], we give the following definition.

Definition 3.11 *Let A be an associative algebra with a unit 1. An associative Yang-Baxter family pair is a collection of elements $\{r_\omega, s_\omega \in A\otimes A|\omega \in \Omega\}$ that satisfy the following equations*

$$r^{13}_{\alpha\beta}r^{12}_\alpha - r^{12}_\alpha r^{23}_\beta + s^{23}_\beta r^{13}_{\alpha\beta} = 0, \quad s^{13}_{\alpha\beta}r^{12}_\alpha - s^{12}_\alpha s^{23}_\beta + s^{23}_\beta s^{13}_{\alpha\beta} = 0,$$

where $r^{12}_\alpha = r^{[1]}_\alpha \otimes r^{[2]}_\alpha \otimes 1, r^{23}_\alpha = 1 \otimes r^{[1]}_\alpha \otimes r^{[2]}_\alpha$, for any $\alpha, \beta \in \Omega$.

Proposition 3.12 *Let $\{r_\omega, s_\omega \in A\otimes A|\omega \in \Omega\}$ be an associative Yang-Baxter family pair. Then the linear operators*

$$R_\omega, S_\omega : A \to A,\ R_\omega(x) := r^{[1]}_\omega x r^{[2]}_\omega,\ S_\omega(x) := s^{[1]}_{\omega,j} x s^{[2]}_\omega$$

determine a Rota-Baxter family system.

Proof If

$$r^{13}_{\alpha\beta}r^{12}_\alpha - r^{12}_\alpha r^{23}_\beta + s^{23}_\beta r^{13}_{\alpha\beta} = 0, \quad s^{13}_{\alpha\beta}r^{12}_\alpha - s^{12}_\alpha s^{23}_\beta + s^{23}_\beta s^{13}_{\alpha\beta} = 0,$$

then

$$r^{[1]}_{\alpha\beta}r^{[1]}_\alpha \otimes r^{[2]}_\alpha \otimes r^{[2]}_{\alpha\beta} - r^{[1]}_\alpha \otimes r^{[2]}_\alpha r^{[1]}_\beta \otimes r^{[2]}_\beta + r^{[1]}_{\alpha\beta} \otimes s^{[1]}_\beta \otimes s^{[2]}_\beta r^{[2]}_{\alpha\beta} = 0,$$

$$s_{\alpha\beta}^{[1]} r_{\alpha}^{[1]} \otimes r_{\alpha}^{[2]} \otimes s_{\alpha\beta}^{[2]} - s_{\alpha}^{[1]} \otimes s_{\alpha}^{[2]} s_{\beta}^{[1]} \otimes s_{\beta}^{[2]} + s_{\alpha\beta}^{[1]} \otimes s_{\beta}^{[1]} \otimes s_{\beta}^{[2]} s_{\alpha\beta}^{[2]} = 0.$$

Replacing tensor products in above equations by x and y, we have

$$\begin{aligned} &R_\alpha(x)R_\beta(y) = R_{\alpha\beta}(R_\alpha(x)y) + R_{\alpha\beta}(xS_\beta(y)), \\ &S_\alpha(x)S_\beta(y) = S_{\alpha\beta}(R_\alpha(x)y) + S_{\alpha\beta}(xS_\beta(y)) \text{ for all } x, y \in A, \alpha, \beta \in \Omega, \end{aligned}$$

as required. □

4. Averaging family systems

In this section, we introduce a notion of averaging family systems and prove that a averaging family system induces an associative family dialgebra structure.

Definition 4.1 *Let Ω be a semigroup. A left (resp., right) averaging family system is a triple $(A, \{R_\omega | \omega \in \Omega\}, \{S_\omega | \omega \in \Omega\})$ consisting of an associative algebra A and a collection of linear operators $\{R_\omega, S_\omega | \omega \in \Omega\}$ that satisfy*

$$\begin{cases} R_\alpha(x)R_\beta(y) = R_{\alpha\beta}(R_\alpha(x)y), \\ S_\alpha(x)S_\beta(y) = S_{\alpha\beta}(R_\alpha(x)y), \end{cases} \qquad \left(\text{resp.,} \begin{cases} R_\alpha(x)R_\beta(y) = R_{\alpha\beta}(xS_\beta(y)), \\ S_\alpha(x)S_\beta(y) = S_{\alpha\beta}(xS_\beta(y)), \end{cases} \right)$$

for any $x, y \in A, \alpha, \beta \in \Omega$.

An averaging family system is a triple $(A, \{R_\omega | \omega \in \Omega\}, \{S_\omega | \omega \in \Omega\})$ which is both a left averaging family system and a right averaging family system.

Definition 4.2 *Let Ω be a semigroup. An associative family dialgebra is a module D together with a collection of binary operations $\{\dashv_\omega, \vdash_\omega | \omega \in \Omega\}$ such that, for $x, y, z \in D$ and $\alpha, \beta \in \Omega$, there is*

$$\begin{aligned} &x \dashv_\alpha (y \dashv_\beta z) = (x \dashv_\alpha y) \dashv_\beta z = x \dashv_{\alpha\beta} (y \vdash_\alpha z), \\ &(x \vdash_\alpha y) \dashv_\beta z = x \vdash_\alpha (y \dashv_\beta z), \\ &(x \dashv_\beta y) \vdash_\alpha z = (x \vdash_\alpha y) \vdash_{\alpha\beta} z = x \vdash_\alpha (y \vdash_\beta z). \end{aligned}$$

We now establish the connections between averaging family systems and associative family dialgebras.

Proposition 4.3 *An averaging family system $(A, \{R_\omega | \omega \in \Omega\}, \{S_\omega | \omega \in \Omega\})$ induces an associative family dialgebra $(A, \{\dashv_\omega, \vdash_\omega | \omega \in \Omega\})$, where*

$$x \dashv_\omega y := xS_\omega(y), \ \ x \vdash_\omega y := R_\omega(x)y \ \text{ for all } \ x, y \in A, \omega \in \Omega.$$

Proof For any $x, y, z \in A$, we have

$$x \dashv_\alpha (y \dashv_\beta z) = xS_\alpha(yS_\beta(z)) = xS_\alpha(y)S_\beta(z) = \begin{cases} (xS_\alpha(y))S_\beta(z) = (x \dashv_\alpha y) \dashv_\beta z, \\ xS_{\alpha\beta}(R_\alpha(y)z) = x \dashv_{\alpha\beta} (y \vdash_\alpha z). \end{cases}$$

Further,

$$(x \vdash_\alpha y) \dashv_\beta z = (R_\alpha(x)y)S_\beta(z) = R_\alpha(x)(yS_\beta(z)) = x \vdash_\alpha (y \dashv_\beta z).$$

Moreover,

$$(x \dashv_\beta y) \vdash_\alpha z = R_\alpha(xS_\beta(y))z = R_\alpha(x)R_\beta(y)z = \begin{cases} R_\alpha(x)(R_\beta(y)z) = x \vdash_\alpha (y \vdash_\beta z), \\ R_{\alpha\beta}(R_\alpha(x)y)z = (x \vdash_\alpha y) \vdash_{\alpha\beta} z. \end{cases}$$

This completes the proof. $\square$

Proposition 4.4 *Let $\{r_\omega|\omega \in \Omega\}$ and $\{s_\omega|\omega \in \Omega\}$ be two elements in $A \otimes A$ such that $r^{13}_{\alpha\beta}r^{12}_\alpha = r^{12}_\alpha r^{23}_\beta$ and $s^{13}_{\alpha\beta}r^{12}_\alpha = s^{12}_\alpha s^{23}_\beta$. Then the linear operators*

$$R_\omega, S_\omega : A \to A,\ R_\omega(x) = r^{[1]}_\omega x r^{[2]}_\omega,\ S_\omega(x) = s^{[1]}_\omega x s^{[2]}_\omega$$

determine a left averaging family system.

Proof If $r^{13}_{\alpha\beta}r^{12}_\alpha = r^{12}_\alpha r^{23}_\beta$ and $s^{13}_{\alpha\beta}r^{12}_\alpha = s^{12}_\alpha s^{23}_\beta$, then

$$r^{[1]}_{\alpha\beta}r^{[1]}_\alpha \otimes r^{[2]}_\alpha \otimes r^{[2]}_{\alpha\beta} = r^{[1]}_\alpha \otimes r^{[2]}_\alpha r^{[1]}_\beta \otimes r^{[2]}_\beta,$$
$$s^{[1]}_{\alpha\beta}r^{[1]}_\alpha \otimes r^{[2]}_\alpha \otimes s^{[2]}_{\alpha\beta} = s^{[1]}_\alpha \otimes s^{[2]}_\alpha s^{[1]}_\beta \otimes s^{[2]}_\beta.$$

Replacing tensor products in above equations by x and y, we have

$$R_\alpha(x)R_\beta(y) = R_{\alpha\beta}(R_\alpha(x)y), \quad S_\alpha(x)S_\beta(y) = S_{\alpha\beta}(R_\alpha(x)y),$$

as required. This completes the proof. $\square$

Remark 4.5 Similar to the proof of above proposition, if r, s satisfy $r^{12}_\alpha r^{23}_\beta = s^{23}_\beta r^{13}_{\alpha\beta}$ and $s^{12}_\alpha s^{23}_\beta = s^{23}_\beta s^{13}_{\alpha\beta}$, then the above-defined (A, R_Ω, S_Ω) is a right averaging family system. Therefore, if r, s satisfy

$$r^{13}_{\alpha\beta}r^{12}_\alpha = r^{12}_\alpha r^{23}_\beta = s^{23}_\beta r^{13}_{\alpha\beta}, \quad s^{13}_{\alpha\beta}r^{12}_\alpha = s^{12}_\alpha s^{23}_\beta = s^{23}_\beta s^{13}_{\alpha\beta},$$

then $(A, \{R_\omega|\omega \in \Omega\}, \{S_\omega|\omega \in \Omega\})$ is an averaging family system.

5. Commuting Rota-Baxter systems and quadri family algebras

In this section, we consider Rota-Baxter family systems on a dendriform algebra and commuting Rota-Baxter systems on an associative algebra. We introduce the notion of quadri family algebras and show how these structures induce quadri family algebras.

Definition 5.1 *Let Ω be a semigroup. A Rota-Baxter family system on a dendriform algebra is a quintuple $(D, \prec, \succ, \{R_\omega|\omega \in \Omega\}, \{S_\omega|\omega \in \Omega\})$ consisting of a dendriform algebra $(D, \prec, \succ)$ and a collection of linear operators $\{R_\omega, S_\omega|\omega \in \Omega\}$ that satisfy*

$$\begin{cases} R_\alpha(x) \prec R_\beta(y) = R_{\alpha\beta}(R_\alpha(x) \prec y) + R_\alpha(x \prec S_\beta(y)), \\ S_\alpha(x) \prec S_\beta(y) = S_{\alpha\beta}(R_\alpha(x) \prec y) + S_\alpha(x \prec S_\beta(y)), \end{cases}$$

and

$$\begin{cases} R_\alpha(x) \succ R_\beta(y) = R_{\alpha\beta}(R_\alpha(x) \succ y) + R_\alpha(x \succ S_\beta(y)), \\ S_\alpha(x) \succ S_\beta(y) = S_{\alpha\beta}(R_\alpha(x) \succ y) + S_\alpha(x \succ S_\beta(y)), \end{cases}$$

for any $x, y \in D, \alpha, \beta \in \Omega$.

Definition 5.2 *A quadri family algebra is a* **k***-module D equipped with a collection of binary operations $\{\nwarrow_\omega, \nearrow_\omega, \swarrow_\omega, \searrow_\omega |\omega \in \Omega\}$ such that*

$$(x \nwarrow_\alpha y) \nwarrow_\beta z = x \nwarrow_{\alpha\beta} (y \nwarrow_\beta z) + x \nwarrow_{\alpha\beta} (y \nearrow_\beta z) + x \nwarrow_{\alpha\beta} (y \swarrow_\alpha z) + x \nwarrow_{\alpha\beta} (y \searrow_\alpha z),$$
$$(x \nearrow_\alpha y) \nwarrow_\beta z = x \nearrow_{\alpha\beta} (y \nwarrow_\beta z) + x \nearrow_{\alpha\beta} (y \swarrow_\alpha z),$$
$$(x \nwarrow_\alpha y) \nearrow_\beta z + (x \nearrow_\alpha y) \nearrow_\beta z = x \nearrow_\alpha (y \nearrow_\beta z) + x \nearrow_\beta (y \searrow_\alpha z),$$
$$(x \swarrow_\alpha y) \nwarrow_\beta z = x \swarrow_{\alpha\beta} (y \nwarrow_\beta z) + x \swarrow_{\alpha\beta} (y \nearrow_\alpha z),$$
$$(x \searrow_\alpha y) \nwarrow_\beta z = x \searrow_\alpha (y \nwarrow_\beta z),$$
$$(x \swarrow_\beta y) \nearrow_{\alpha\beta} z + (x \searrow_\alpha y) \nearrow_{\alpha\beta} z = x \searrow_\alpha (y \nearrow_\beta z),$$
$$(x \nwarrow_\beta y) \swarrow_{\alpha\beta} z + (x \swarrow_\alpha y) \swarrow_{\alpha\beta} z = x \swarrow_{\alpha\beta} (y \swarrow_\beta z) + x \swarrow_{\alpha\beta} (y \searrow_\alpha z),$$
$$(x \nearrow_\beta y) \swarrow_{\alpha\beta} z + (x \searrow_\alpha y) \swarrow_{\alpha\beta} z = x \searrow_\alpha (y \swarrow_\beta z),$$
$$(x \nwarrow_\beta y) \searrow_{\alpha\beta} z + (x \nearrow_\beta y) \searrow_{\alpha\beta} z + (x \swarrow_\beta y) \searrow_{\alpha\beta} z + (x \searrow_\alpha y) \searrow_{\alpha\beta} z = x \searrow_\alpha (y \searrow_\beta z),$$

for any $x, y, z \in D$ and $\alpha, \beta \in \Omega$.

It turns out that in a quadri family algebra, $(D, \{\nwarrow_\omega + \swarrow_\omega, \nearrow_\omega + \searrow_\omega |\omega \in \Omega\})$ and $(D, \{\nwarrow_\omega + \nearrow_\omega, \swarrow_\omega + \searrow_\omega |\omega \in \Omega\})$ are both dendriform family algebras.

Proposition 5.3 *Let $(D, \{R_\omega|\omega \in \Omega\}, \{S_\omega|\omega \in \Omega\})$ be a Rota-Baxter family system. Then $(D, \{\nwarrow_\omega, \nearrow_\omega, \swarrow_\omega, \searrow_\omega |\omega \in \Omega\})$ is a quadri family algebra, where*

$$x \nwarrow_\omega y := x \prec S_\omega(y),\ x \nearrow_\omega y := x \succ S_\omega(y),\ x \swarrow_\omega y := R_\omega(x) \prec y,\ x \searrow_\omega y := R_\omega(x) \succ y$$

for any $x, y \in D$ and $\omega \in \Omega$.

Proof For any $x, y, z \in D$ and $\alpha, \beta \in \Omega$, we have

$$\begin{aligned}
(x \nwarrow_\alpha y) \nwarrow_\beta z &= (x \prec S_\alpha(y)) \prec S_\beta(z) = x \prec (S_\alpha(y) \prec S_\beta(z)) + x \prec (S_\alpha(y) \succ S_\beta(z))\\
&= x \prec (S_{\alpha\beta}(R_\alpha(y) \prec z) + S_{\alpha\beta}(y \prec S_\beta(z))) + x \prec (S_{\alpha\beta}(R_\alpha(y) \succ z) + S_{\alpha\beta}(y \succ S_\beta(z)))\\
&= x \prec S_{\alpha\beta}(R_\alpha(y) \prec z) + x \prec S_{\alpha\beta}(y \prec S_\beta(z)) + x \prec S_{\alpha\beta}(R_\alpha(y) \succ z) + x \prec S_{\alpha\beta}(y \succ S_\beta(z))\\
&= x \nwarrow_{\alpha\beta} (y \nwarrow_\beta z) + x \nwarrow_{\alpha\beta} (y \nearrow_\beta z) + x \nwarrow_{\alpha\beta} (y \swarrow_\alpha z) + x \nwarrow_{\alpha\beta} (y \searrow_\alpha z).
\end{aligned}$$

Further,

$$\begin{aligned}
(x \nearrow_\alpha y) \nwarrow_\beta z &= (x \succ S_\alpha(y)) \prec S_\beta(z) = x \succ (S_\alpha(y) \prec S_\beta(z))\\
&= x \succ (S_{\alpha\beta}(R_\alpha(y) \prec z) + S_{\alpha\beta}(y \prec S_\beta(z)))\\
&= x \succ S_{\alpha\beta}(R_\alpha(y) \prec z) + x \succ S_{\alpha\beta}(y \prec S_\beta(z))\\
&= x \nearrow_{\alpha\beta} (y \nwarrow_\beta z) + x \nearrow_{\alpha\beta} (y \swarrow_\alpha z).
\end{aligned}$$

Moreover,

$$\begin{aligned}
(x \nwarrow_\alpha y) \nearrow_\beta z + (x \nearrow_\alpha y) \nearrow_\beta z &= (x \prec S_\alpha(y)) \succ S_\beta(z) + (x \succ S_\alpha(y)) \succ S_\beta(z)\\
&= x \succ (S_\alpha(y) \succ S_\beta(z)) = x \succ (S_{\alpha\beta}(R_\alpha(x) \succ y) + S_{\alpha\beta}(x \succ S_\beta(y)))
\end{aligned}$$

$$= x \succ S_{\alpha\beta}(R_\alpha(x) \succ y) + x \succ S_{\alpha\beta}(x \succ S_\beta(y))$$
$$= x \nearrow_{\alpha\beta} (y \nearrow_\beta z) + x \nearrow_{\alpha\beta} (y \searrow_\alpha z).$$

By the same argument, the other identities are similar to verify. This completes the proof. □

Proposition 5.4 *Let $(D, \{R_\omega|\omega \in \Omega\}, \{S_\omega|\omega \in \Omega\})$ be a Rota-Baxter family system and (A, P, Q) a Rota-Baxter system, if*

$$P \circ R_\omega = R_\omega \circ P,\ P \circ S_\omega = S_\omega \circ P,\ Q \circ R_\omega = R_\omega \circ Q,\ Q \circ S_\omega = S_\omega \circ Q,\quad \omega \in \Omega,$$

then they are said to commute. With the notations as above, then $(D, \{R_\omega|\omega \in \Omega\}, \{S_\omega|\omega \in \Omega\})$ is a Rota-Baxter family system on the dendriform algebra $(D, \prec, \succ)$ induced from (P, Q), i.e., $x \prec y := xQ(y)$ and $x \succ y := P(x)y$, for $x, y \in D$.

Proof For any $x, y \in D$ and $\alpha, \beta \in \Omega$, we have

$$R_\alpha(x) \prec R_\beta(y) = R_\alpha(x)(QR_\beta(y)) = R_\alpha(x)(R_\beta Q(y))$$
$$= R_{\alpha\beta}(R_\alpha(x)Q(y)) + R_{\alpha\beta}(xS_\beta Q(y))$$
$$= R_{\alpha\beta}(R_\alpha(x) \prec y) + R_{\alpha\beta}(x \prec S_\beta(y))$$

and

$$S_\alpha(x) \prec S_\beta(y) = S_\alpha(x)(QS_\beta(y)) = S_\alpha(x)(S_\beta Q(y))$$
$$= S_{\alpha\beta}(R_\alpha(x)Q(y)) + S_{\alpha\beta}(xS_\beta Q(y))$$
$$= S_{\alpha\beta}(R_\alpha(x) \prec y) + S_{\alpha\beta}(x \prec S_\beta(y)).$$

Here we have verified the first two identities. The other two identities can be verified similarly. This completes the proof. □

Combining Propositions 5.3 and 5.4, we get the following.

Proposition 5.5 *With the notations as above, then $(D, \{\nwarrow_\omega, \nearrow_\omega, \swarrow_\omega, \searrow_\omega |\omega \in \Omega\})$ is a quadri family algebra, where*

$$x \nwarrow_\omega y := xQS_\omega(y),\ x \nearrow_\omega y := P(x)S_\omega(y),\ x \swarrow_\omega y := R_\omega(x)Q(y),\ x \searrow_\omega y := PR_\omega(x)y$$

for any $x, y \in D$ and $\omega \in \Omega$.

6. Free Rota-Baxter family systems and Gröbner-Shirshov bases

In this section, we construct free objects in the category of Rota-Baxter family systems, via a method of Gröbner-Shirshov bases. For the rest of this paper, we will use the infix notation Ω to denote the set of multiple operators.

First, we recall the construct of the free monoid with multiple operators and the free associative algebra with multiple linear operators [10, 18], which generalise simultaneously the free associative algebra with unary operators constructed by Guo [1] and the free associative algebra with one operator captured by Ebrahimi-Fard and Guo [20].

Definition 6.1 ([10]) *An associative algebra with multiple linear operators is an associative* $\mathbf{k}$*-algebra* R *with a set* Ω *of multilinear operators (operations).*

Let X *be a set and*

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n,$$

where Ω_n *is a set of* n*-ary operations. For any set* Y*, denote by*

$$\Omega(Y) := \bigcup_{n=1}^{\infty} \{\omega^{(n)}(y_1, \ldots, y_n) \,|\, \omega^{(n)} \in \Omega_n\}.$$

Define

$$\langle X; \Omega\rangle_0 := M(X).$$

Assume that we have defined $\langle X; \Omega\rangle_{n-1}$*, and define*

$$\langle X; \Omega\rangle_n := M(X \cup \langle X; \Omega\rangle_{n-1}).$$

Then we have $\langle X; \Omega\rangle_n \subseteq \langle X; \Omega\rangle_{n+1}$ *for any* $n \geq 0$*. Denote by*

$$\langle X; \Omega\rangle := \varinjlim \langle X; \Omega\rangle_n = \bigcup_{n\geq 0} \langle X; \Omega\rangle_n.$$

We now collect some basic definitions and facts on $\langle X; \Omega\rangle$ from [10], which will be used in the rest of the paper.

(1) For any $u \in \langle X; \Omega\rangle$, define $\mathrm{dep}(u) := \min\{n \,|\, u \in \langle X; \Omega\rangle_n\}$ to be the depth of u.

(2) Let $\mathbf{k}\langle X; \Omega\rangle$ be the linear space spanned by $\langle X; \Omega\rangle$ over $\mathbf{k}$. Then the element in $\langle X; \Omega\rangle$ (resp., $\mathbf{k}\langle X; \Omega\rangle$) is called an Ω-word (resp., Ω-polynomial).

(3) For any $u \in X \cup \Omega(\langle X; \Omega\rangle)$, u is called prime. Then for any $\langle X; \Omega\rangle$, u has a unique canonical form $u = u_1 u_2 \cdots u_n$, $n \geq 0$, where each u_i is prime.

(4) If $u = u_1 u_2 \cdots u_n \in \langle X; \Omega\rangle$, where u_i is prime Ω-word for all i, then the breath of u is defined to be the number n, denoted by $\mathrm{bre}(u)$.

(5) Each

$$\omega^{(n)} : \langle X; \Omega\rangle^n \to \langle X; \Omega\rangle,\ (x_1, x_2, \ldots, x_n) \mapsto \omega^{(n)}(x_1, x_2, \ldots, x_n)$$

can be extended linearly to $\mathbf{k}\langle X; \Omega\rangle$.

Proposition 6.2 ([10]) *The* $\mathbf{k}\langle X; \Omega\rangle$ *is a free associative algebra with multiple linear operators* Ω *on the set* X*.*

We first recall the Composition-Diamond lemma for the free associative algebra with multiple linear operators $\mathbf{k}\langle X; \Omega\rangle$, with an eye toward constructing a linear basis of a free Rota-Baxter family system on a set X. See [10, 21, 23, 24] for more details.

Definition 6.3 ([10]) *Let* $\mathbf{k}\langle X; \Omega\rangle$ *be a free associative algebra with multiple linear operators* Ω *on* X *and* $\star \notin X$*.*

(1) By a $\star$*-*Ω*-word, we mean any expression in* $\langle X \cup \{\star\}; \Omega\rangle$ *with exactly one occurrence of* $\star$*, counting multiplicities. The set of all* $\star$*-*Ω*-words on* X *is defined by* $\langle X; \Omega\rangle^{\star}$*.*

(2) Let u be a $\star$-Ω-word and $s \in \mathbf{k}\langle X;\Omega\rangle$. Then

$$u|_s := u|_{\star \mapsto s}$$

is called an s-Ω-word.

(3) For $q \in \langle X;\Omega\rangle^\star$ and $s = \Sigma_i c_i q|_{u_i} \in \mathbf{k}\langle X;\Omega\rangle$, where $c_i \in \mathbf{k}$ and $u_i \in \langle X;\Omega\rangle$, we define

$$q|_s := \Sigma_i c_i q|_{u_i}$$

and give this notation an expansion to any $q \in \mathbf{k}\langle X;\Omega\rangle^\star$ by linearity.

Definition 6.4 ([10]) *(1) A monomial order on $\langle X;\Omega\rangle$ is a well order $\leq$ on $\langle X;\Omega\rangle$ such that for any $v, w \in \langle X;\Omega\rangle$, $u \in \langle X;\Omega\rangle^\star$,*

$$w < v \Rightarrow u|_w < u|_v.$$

(2) For every Ω-polynomial $f \in \mathbf{k}\langle X;\Omega\rangle$, let $\bar{f}$ be the leading Ω-word of f. If the coefficient of $\bar{f}$ is 1, then we call f is monic with respect to a monomial order $\leq$.

The following is the concept of compositions.

Definition 6.5 ([10]) *Let f and g be two monic Ω-polynomials. Then there are two kinds of compositions.*

(1) If there exists an Ω-word $w = \bar{f}a = b\bar{g}$ for some $a, b \in \langle X;\Omega\rangle$ such that $\mathrm{bre}(w) < \mathrm{bre}(\bar{f}) + \mathrm{bre}(\bar{g})$, then we call $(f,g)_w = fa - bg$ the intersection composition of f and g with respect to w.

(2) If there exists an Ω-word $w = \bar{f} = u|_{\bar{g}}$ for some $u \in \langle X;\Omega\rangle^\star$, then we call $(f,g)_w = f - u|_g$ the including composition of f and g with respect to w. In this case transformation $f \mapsto (f,g)_w$ is called the Elimination of the Leading Word (ELW) of g in f.

Definition 6.6 ([10]) *Let S be a set of monic Ω-polynomials.*

(1) The composition $(f,g)_w$ is called trivial modulo (S,w) if

$$(f,g)_w = \sum \alpha_i u_i|_{s_i},$$

where each $\alpha_i \in \mathbf{k}$, $u_i \in \langle X;\Omega\rangle^\star$, $s_i \in S$ and $u_i|_{\overline{s_i}} < w$. In this case, we may write

$$(f,g)_w \equiv 0 \bmod (S,w).$$

(2) In general, for any two Ω-polynomials p and q, $p \equiv q \bmod (S,w)$ means that $p - q = \sum \alpha_i u_i|_{s_i}$, where each $\alpha_i \in \mathbf{k}$, $u_i \in \langle X;\Omega\rangle^\star$, $s_i \in S$ and $u_i|_{\overline{s_i}} < w$.

(3) S is called a Gröbner-Shirshov basis in $\mathbf{k}\langle X;\Omega\rangle$ if any composition $(f,g)_w$ of $f, g \in S$ is trivial modulo (S,w).

The following theorem is the Composition-Diamond Lemma for Ω-(unitary) algebras, adapting from the case for Ω-nonunitary algebras in [10,22].

Theorem 6.7 (Composition-Diamond lemma) *Let S be a set of monic Ω-polynomials in $\mathbf{k}\langle X;\Omega\rangle$, $\leq$ a monomial ordering on $\langle X;\Omega\rangle$ and $Id(S)$ the Ω-ideal of $\mathbf{k}\langle X;\Omega\rangle$ generated by S. Then the following statements are equivalent:*

(1) S *is a Gröbner-Shirshov basis in* $\mathbf{k}\langle X;\Omega\rangle$.

(2) $f \in Id(S) \Rightarrow \bar{f} = u|_{\bar{s}}$ *for some* $u \in \langle X;\Omega\rangle^\star$ *and* $s \in S$.

(3) $f \in Id(S) \Rightarrow f = \alpha_1 u_1|_{s_1} + \alpha_2 u_2|_{s_2} + \cdots + \alpha_n u_n|_{s_n}$ *where each* $\alpha_i \in \mathbf{k}$, $s_i \in S$, $u_i \in \langle X;\Omega\rangle^\star$ *and* $u_1|_{\overline{s_1}} > u_2|_{\overline{s_2}} > \cdots > u_n|_{\overline{s_n}}$.

(4) $\mathrm{Irr}(S) = \{w \in \langle X;\Omega\rangle | w \neq u|_{\bar{s}}$ *for any* $u \in \langle X;\Omega\rangle^\star$ *and* $s \in S\}$ *is a* $\mathbf{k}$*-basis of* $\mathbf{k}\langle X;\Omega|S\rangle = \mathbf{k}\langle X;\Omega\rangle/Id(S)$.

We first give a Gröbner-Shirshov basis of $\mathbf{k}\langle X;\Omega\rangle$ and then a linear basis of the free Rota-Baxter family system is obtained by the Composition-Diamond lemma.

We first recall the concept of a degree lexicographical order $\leq_{\mathrm{dl}}$, the reader is refereed to [10,18] for more details on degree lexicographical orders. Denote by $R_\Omega \sqcup S_\Omega = (R_\omega)_{\omega\in\Omega} \sqcup (S_\omega)_{\omega\in\Omega}$ a set of unary linear operators. Without loss of confusion, we still use the set Ω to identify them.

Definition 6.8 *Let* X *be a well-ordered set. Let* $R_\Omega \sqcup S_\Omega$ *be a well-ordered set of unary linear operators and suppose that* $S_\alpha < R_\beta$ *for any* $\alpha, \beta \in \Omega$.

(1) For any $u \in \langle X;\Omega\rangle$, *we define* $\deg(u)$ *to be the number of all occurrence of* $x \in X$ *and* R_ω, S_ω *with* $\omega \in \Omega$. *For instance, if* $u = x_1x_2x_3S_\alpha(x_4R_\beta(x_5))$, *where* $x_i \in X, 1 \leq i \leq 5$, *then* $\deg(u) = 7$.

(2) For any $u = u_1u_2\cdots u_n \in \langle X;\Omega\rangle$ *with* $n \geq 1$, *where each* u_i *is prime [18], in other words, for each* $1 \leq i \leq n-1$, *either* u_i *or* u_{i+1} *is in* X, *we set*

$$\mathrm{wt}(u) = (\deg(u), \mathrm{bre}(u), u_1, u_2, \ldots, u_n).$$

(3) For any $u = u_1u_2\cdots u_n \in \langle X;\Omega\rangle$ *with* $n \geq 1$, $v = v_1v_2\cdots v_m \in \langle X;\Omega\rangle$ *with* $m \geq 1$, *where* u_i, v_j *are prime, define*

$$u \leq_{\mathrm{dl}} v \ \text{ if } \ \mathrm{wt}(u) < \mathrm{wt}(v) \ \textit{ lexicographically.}$$

Here $u_i < v_i$ *if* $\deg(u_i) < \deg(v_i)$ *or* $\deg(u_i) = \deg(v_i)$ *such that one of the following conditions holds:*

(a) $u_i, v_i \in X$ *and* $u_i < v_i$;

(b) $u_i \in X$ *and* $v_i = R_\omega(v_i')$ *for some* $\omega \in \Omega$;

(c) $u_i \in X$ *and* $v_i = S_\omega(v_i')$ *for some* $\omega \in \Omega$;

(d) $u_i = R_\alpha(u_i')$ *and* $v_i = R_\beta(v_i')$ *for some* $\alpha, \beta \in \Omega$ *such that*

$$(R_\alpha, u_i') < (R_\beta, v_i') \ \textit{ lexicographically;}$$

(e) $u_i = S_\alpha(u_i')$ *and* $v_i = S_\beta(v_i')$ *for some* $\alpha, \beta \in \Omega$ *such that*

$$(S_\alpha, u_i') < (S_\beta, v_i') \ \textit{ lexicographically;}$$

(f) $u_i = S_\alpha(u_i')$ *and* $v_i = R_\beta(v_i')$ *for some* $\alpha, \beta \in \Omega$ *such that*

$$(S_\alpha, u_i') < (R_\beta, v_i') \ \textit{ lexicographically.}$$

With a similar argument to the case of $\leq_{\mathrm{dl}}$ on $\langle X;\{R,S\}\rangle$ (see [18]), the above defined $\leq_{\mathrm{dl}}$ is a monomial order on $\langle X;\Omega\rangle$. In fact when Ω is a singleton, the above defined $\leq_{\mathrm{dl}}$ is exactly the one given in [18] on $\langle X;\{R,S\}\rangle$.

We now show that the defining relations of the Rota-Baxter family system is a Gröbner-Shirshov basis in $\mathbf{k}\langle X;\Omega\rangle$.

Theorem 6.9 *With the order $\leq_{\rm dl}$ on $\langle X;\Omega\rangle$, the defining relations of Rota-Baxter family system*

$$S:=\left\{\begin{array}{l} R_\alpha(x)R_\beta(y)-R_{\alpha\beta}(xS_\beta(y))-R_{\alpha\beta}(R_\alpha(x)y)\\ S_\alpha(x)S_\beta(y)-S_{\alpha\beta}(xS_\beta(y))-S_{\alpha\beta}(R_\alpha(x)y)\end{array}\middle| x,y\in\langle X;\Omega\rangle\right\}\tag{6.1}$$

is a Gröbner-Shirshov basis in $\mathbf{k}\langle X;\Omega\rangle$.

Proof For any $x,y,z\in\langle X;\Omega\rangle$, we write

$$f:=f_{\alpha,\beta}(x,y)=R_\alpha(x)R_\beta(y)-R_{\alpha\beta}(xS_\beta(y))-R_{\alpha\beta}(R_\alpha(x)y),$$
$$g:=g_{\alpha,\beta}(y,z)=S_\alpha(x)S_\beta(y)-S_{\alpha\beta}(xS_\beta(y))-S_{\alpha\beta}(R_\alpha(x)y).$$

Then the ambiguities of all possible compositions of Ω-polynomials in S are the following cases.

(1) $(f_{\alpha,\beta}(x,y),f_{\beta,\gamma}(y,z))_{w_1}$, $w_1=R_\alpha(x)R_\beta(y)R_\gamma(z)$,

(2) $(f_{\alpha\beta,\gamma}(u\,|_{R_\alpha(x)R_\beta(y)},y),f_{\alpha,\beta}(x,y))_{w_2}$, $w_2=R_{\alpha\beta}(u\,|_{R_\alpha(x)R_\beta(y)})R_\gamma(z)$,

(3) $(f_{\alpha,\beta\gamma}(x,u\,|_{R_\beta(y)R_\gamma(z)}),f_{\beta,\gamma}(y,z))_{w_3}$, $w_3=R_\alpha(x)R_{\beta\gamma}(u\,|_{R_\beta(y)R_\gamma(z)})$,

(4) $(f_{\alpha\beta,\gamma}(u\,|_{S_\alpha(x)S_\beta(y)},y),g_{\alpha,\beta}(x,y))_{w_4}$, $w_4=R_{\alpha\beta}(u\,|_{S_\alpha(x)S_\beta(y)})R_\gamma(z)$,

(5) $(f_{\alpha,\beta\gamma}(x,u\,|_{S_\beta(y)S_\gamma(z)}),g_{\beta,\gamma}(y,z))_{w_5}$, $w_5=R_\alpha(x)R_{\beta\gamma}(u\,|_{S_\beta(y)S_\gamma(z)})$,

(6) $(g_{\alpha,\beta\gamma}(x,y),g_{\beta,\gamma}(y,z))_{w_6}$, $w_6=S_\alpha(x)S_\beta(y)S_\gamma(z)$,

(7) $(g_{\alpha\beta,\gamma}(u\,|_{S_\alpha(x)S_\beta(y)},y),g_{\alpha,\beta}(x,y))_{w_7}$, $w_7=S_{\alpha\beta}(u\,|_{S_\alpha(x)S_\beta(y)})S_\gamma(z)$,

(8) $(g_{\alpha,\beta\gamma}(x,u\,|_{S_\beta(y)S_\gamma(z)}),g_{\beta,\gamma}(y,z))_{w_8}$, $w_8=S_\alpha(x)S_{\beta\gamma}(u\,|_{S_\beta(y)S_\gamma(z)})$,

(9) $(g_{\alpha\beta,\gamma}(u\,|_{R_\alpha(x)R_\beta(y)},y),f_{\alpha,\beta}(x,y))_{w_9}$, $w_9=S_{\alpha\beta}(u\,|_{R_\alpha(x)R_\beta(y)})S_\gamma(z)$,

(10) $(g_{\alpha,\beta\gamma}(x,u\,|_{R_\beta(y)R_\gamma(z)}),f_{\beta,\gamma}(y,z))_{w_{10}}$, $w_{10}=S_\alpha(x)S_{\beta\gamma}(u\,|_{R_\beta(y)R_\gamma(z)})$,

where $x,y,z\in\langle X;\Omega\rangle$, $R_\alpha,R_\beta,R_\gamma,S_\alpha,S_\beta,S_\gamma\in\Omega$, $u\in\langle X;\Omega\rangle^\star$. It remains to prove that all the compositions are trivial. Without loss of generality, we just show the case (a), as the other cases can be verified similarly. In the case (a), we have

$$\begin{aligned}
&(f_{\alpha,\beta}(x,y),f_{\beta,\gamma}(y,z))_{w_1}=f_{\alpha,\beta}(x,y)R_\gamma(z)-R_\alpha(x)f_{\beta,\gamma}(y,z)\\
&\quad=R_\alpha(x)R_\beta(y)R_\gamma(z)-R_{\alpha\beta}(xS_\beta(y))R_\gamma(z)-R_{\alpha\beta}(R_\alpha(x)y)R_\gamma(z)-\\
&\qquad R_\alpha(x)R_\beta(y)R_\gamma(z)+R_\alpha(x)R_{\beta\gamma}(yS_\gamma(z))+R_\alpha(x)R_{\beta\gamma}(R_\beta(y)z)\\
&\quad=-R_{\alpha\beta}(xS_\beta(y))R_\gamma(z)-R_{\alpha\beta}(R_\alpha(x)y)R_\gamma(z)+R_\alpha(x)R_{\beta\gamma}(yS_\gamma(z))+R_\alpha(x)R_{\beta\gamma}(R_\beta(y)z)\\
&\quad=-f_{\alpha\beta,\gamma}(xS_\beta(y),z)-R_{\alpha\beta\gamma}(xS_\beta(y)S_\gamma(z))-R_{\alpha\beta\gamma}(R_\alpha(xS_\beta(y))z)-\\
&\qquad f_{\alpha\beta,\gamma}(R_\alpha(x)y,z)-R_{\alpha\beta\gamma}(R_\alpha(x)yS_\gamma(z))-R_{\alpha\beta\gamma}(R_{\alpha\beta}(R_\alpha(x)y)z)+\\
&\qquad f_{\alpha,\beta\gamma}(x,yS_\gamma(z))+R_{\alpha\beta\gamma}(xS_\beta(yS_\gamma(z)))+R_{\alpha\beta\gamma}(R_\alpha(x)yS_\gamma(z))+\\
&\qquad f_{\alpha,\beta\gamma}(x,R_\beta(y)z)+R_{\alpha\beta\gamma}(xS_{\beta\gamma}(R_\beta(y)z))+R_{\alpha\beta\gamma}(R_\alpha(x)R_\beta(y)z)*\\
&\quad=-f_{\alpha\beta,\gamma}(xS_\beta(y),z)-R_{\alpha\beta\gamma}(xS_\beta(y)S_\gamma(z))-R_{\alpha\beta\gamma}(R_{\alpha\beta}(xS_\beta(y))z)-\\
&\qquad f_{\alpha\beta,\gamma}(R_\alpha(x)y,z)-R_{\alpha\beta\gamma}(R_{\alpha\beta}(R_\alpha(x)y)z)+f_{\alpha,\beta\gamma}(x,yS_\gamma(z))+\\
&\qquad R_{\alpha\beta\gamma}(xS_{\beta\gamma}(yS_\gamma(z)))+f_{\alpha,\gamma}(x,R_\beta(y)z)+R_{\alpha\beta\gamma}(xS_{\beta\gamma}(R_\beta(y)z))+R_{\alpha\beta\gamma}(R_\alpha(x)R_\beta(y)z)\\
&\quad=-f_{\alpha\beta,\gamma}(xS_\beta(y),z)-R_\alpha(xg_{\beta,\gamma}(y,z))-R_{\alpha\beta\gamma}(xS_{\beta\gamma}(yS_\gamma(z)))-R_{\alpha\beta\gamma}(xS_{\beta\gamma}(R_\beta(y)z))-
\end{aligned}$$

$$
\begin{aligned}
& R_{\alpha\beta\gamma}(R_{\alpha\beta}(xS_\beta(y))z) - f_{\alpha\beta,\,\gamma}(R_\alpha(x)y,z) - R_{\alpha\beta\gamma}(R_{\alpha\beta}(R_\alpha(x)y)z) + f_{\alpha,\,\beta\gamma}(x, yS_\gamma(z)) + \\
& R_{\alpha\beta\gamma}(xS_{\beta\gamma}(yS_\gamma(z))) + f_{\alpha,\,\beta\gamma}(x, R_\beta(y)z) + R_{\alpha\beta\gamma}(xS_{\beta\gamma}(R_\beta(y)z)) + R_{\alpha\beta\gamma}(f_{\alpha,\,\beta}(x,y)z) + \\
& R_{\alpha\beta\gamma}(R_{\alpha\beta}(xS_\beta(y))z) + R_{\alpha\beta\gamma}(R_{\alpha\beta}(R_\alpha(x)y)z) \\
= & -R_{\alpha\beta\gamma}(x\, g_{\beta,\,\gamma}(y,z)) + R_{\alpha\beta\gamma}(f_{\alpha,\,\beta}(x,y)z) - f_{\alpha\beta,\,\gamma}(xS_\beta(y),z) - f_{\alpha\beta,\,\gamma}(R_\alpha(x)y,z) + \\
& f_{\alpha,\,\beta\gamma}(x, yS_\gamma(z)) + f_{\alpha,\,\beta\gamma}(x, R_\beta(y)z) \\
= & -R_{\alpha\beta\gamma}(x \star|_{g_{\beta,\,\gamma}(y,z)}) + R_{\alpha\beta\gamma}(\star|_{f_{\alpha,\,\beta}(x,y)}\, z) - \star|_{f_{\alpha\beta,\,\gamma}(xS_\beta(y),z)} - \star|_{f_{\alpha\beta,\,\gamma}(R_\alpha(x)y,z)} + \\
& \star|_{f_{\alpha,\,\beta\gamma}(x,yS_\gamma(z))} + \star|_{f_{\alpha,\,\beta\gamma}(x,R_\beta(y)z)},
\end{aligned}
$$

where

$$
\begin{aligned}
& R_{\alpha\beta\gamma}(x \star|_{g_{\beta,\,\gamma}(\bar{y},z)}) = R_{\alpha\beta\gamma}(x \star|_{R_\beta(y)R_\gamma(z)}) = R_{\alpha\beta\gamma}(xR_\beta(y)R_\gamma(z)) dl R_\alpha(x)R_\beta(y)R_\gamma(z) = w_1, \\
& R_{\alpha\beta\gamma}(\star|_{f_{\alpha,\,\beta}(\bar{x},y)}\, z) = R_{\alpha\beta\gamma}(\star|_{R_\alpha(x)R_\beta(y)} z) = R_{\alpha\beta\gamma}(R_\alpha(x)R_\beta(y)z) dl R_\alpha(x)R_\beta(y)R_\gamma(z) = w_1, \\
& \star|_{f_{\alpha\beta,\,\gamma}(x\bar{S}_\beta(y),z)} = \star|_{R_{\alpha\beta}(xS_\beta(y))R_\gamma(z)} = R_{\alpha\beta}(xS_\beta(y))R_\gamma(z) dl R_\alpha(x)R_\beta(y)R_\gamma(z) = w_1, \\
& \star|_{f_{\alpha\beta,\,\gamma}(\bar{R}_\alpha(x)y,z)} = \star|_{R_{\alpha\beta}(R_\alpha(x)y)R_\gamma(z)} = R_{\alpha\beta}(R_\alpha(x)y)R_\gamma(z) dl R_\alpha(x)R_\beta(y)R_\gamma(z) = w_1, \\
& \star|_{f_{\alpha,\,\beta\gamma}(\bar{x},yS_\gamma(z))} = \star|_{R_\alpha(x)R_{\beta\gamma}(yS_\gamma(z))} = R_\alpha(x)R_{\beta\gamma}(yS_\gamma(z)) dl R_\alpha(x)R_\beta(y)R_\gamma(z) = w_1, \\
& \star|_{f_{\alpha,\,\beta\gamma}(\bar{x},R_\beta(y)z)} = \star|_{R_\alpha(x)R_{\beta\gamma}(R_\beta(y)z)} = R_\alpha(x)R_{\beta\gamma}(R_\beta(y)z) dl R_\alpha(x)R_\beta(y)R_\gamma(z) = w_1.
\end{aligned}
$$

Hence

$$(f_{\alpha,\,\beta}(x,y), f_{\beta,\,\gamma}(y,z))_{w_1} \equiv 0 \bmod (S, R_\alpha(x)R_\beta(y)R_\gamma(z)).$$

This completes the proof. □

Theorem 6.10 *With the order $\leq_{\rm dl}$ on $\langle X;\Omega\rangle$ and the S given in Eq. (6.1), the set*

$$\operatorname{Irr}(S) = \{w \in \langle X;\Omega\rangle \mid w \neq q\,|_{\bar{s}}, q \in \langle X;\Omega\rangle^\star, s \in S\}$$

is a linear basis of the free Rota-Baxter family system $\mathbf{k}\langle X;\Omega|S\rangle$ on the set X.

Proof It follows from Theorem 6.9. □

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