Journal of Mathematical Research with Applications
Mar., 2023, Vol. 43, No. 2, pp. 191–203
DOI:10.3770/j.issn:2095-2651.2023.02.007
Http://jmre.dlut.edu.cn

On Gorenstein Homological Dimensions of Groups

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Abstract Let k be a commutative ring with finite weak dimension and let G be a group. In this paper, we explore the criterion that a group G has finite Gorenstein homological dimension. It is shown that the finiteness of the Gorenstein homological dimension of G coincides with the finiteness of the Gorenstein weak dimension of the group ring kG. Furthermore, we give a Gorenstein analogy of the Serre's theorem. Some well-known results for the Gorenstein homological dimension of G over the integer ring are also extended.

Keywords Gorenstein homological dimension; Serre's theorem; group ring

MR(2020) Subject Classification 16E10; 16S34

1. Introduction

The cohomology theory of groups arose from both topological and algebraic sources. There are many (co)homological dimensions assigned to a group. Let $\mathbb{Z}G$ be the integral group ring of a group G. The cohomological dimension $cd_{\mathbb{Z}}G$ of G over $\mathbb{Z}$ is the projective dimension of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. The well-known Serre's theorem says that if Γ is a torsion-free group and Γ' is a subgroup of finite index, then $cd_{\mathbb{Z}}\Gamma' = cd_{\mathbb{Z}}\Gamma$ (see [1, Theorem 8.3.1]). Asadollahi et al studied the Gorenstein cohomological dimension $Gcd_{\mathbb{Z}}G$ (see [2]) of G over $\mathbb{Z}$ which is the Gorenstein projective dimension of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. It was shown that $Gcd_{\mathbb{Z}}G$ is closely related to the spli$\mathbb{Z}G$ (the supremum of the projective dimensions of the injective $\mathbb{Z}G$-modules). For example, $Gcd_{\mathbb{Z}}G < \infty$ if and only if spli$\mathbb{Z}G < \infty$ if and only if any $\mathbb{Z}G$-module has finite Gorenstein projective dimension. Asadollahi et al. [3] considered the Gorenstein homological dimension $Ghd_{\mathbb{Z}}G$ of a group G, i.e., the Gorenstein flat dimension of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, and showed that this invariant is tightly related to the sfli$\mathbb{Z}G$ (the supremum of the flat lengths of injective modules) and reflects several properties of the underlying group G. More recently, Emmanouil [4] generalized many properties of cohomological dimension of G over a commutative ring to Gorenstein cohomological dimension.

Motivated by this, in the present paper, we consider the Gorenstein homological dimension Ghd_kG of a group G (the Gorenstein flat dimension of the trivial kG-module k) over a commutative ring k with finite weak dimension. This paper is organized as follows. Section 3 is devoted to show that $Ghd_kG < \infty$ if and only if sfli$kG < \infty$ if and only if the Gorenstein weak dimension

Received February 21, 2022; Accepted August 19, 2022
Supported by the Natural Science Foundation of Hunan Province (Grant No. 2021JJ30536) and the Scientific Research Foundation of Hunan Provincial Education Department (Grant No. 21A0493).
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Gw.dim$kG < \infty$. Consequently, if k is a commutative ring with finite global dimension, then $Gcd_kG < \infty$ if and only if $Ghd_kG < \infty$ and $\overline{\mathrm{P}}(kG) = \overline{\mathrm{F}}(kG)$ (where $\overline{\mathrm{P}}(kG)$ (resp., $\overline{\mathrm{F}}(kG)$) is the class of modules that have finite projective dimensions (resp., finite flat dimensions)). In Section 4, we prove the following results:

(1) Let R be a commutative ring, and let G be a group. Then $Ghd_RG = 0$ if and only if G is finite.

(2) Let k be a commutative ring with finite weak dimension and (G_α) a directed family of subgroups of a group G such that G is the direct limit of the G_α. If Ghd_kG is finite, then $Ghd_kG = \sup\{Ghd_kG_\alpha\}$.

(3) Let k be a commutative ring with finite weak dimension and let H be a normal subgroup of a group G. Then $Ghd_kG \leq Ghd_kH + Ghd_k(G/H)$.

(4) If k is a commutative ring with finite weak dimension and H is a subgroup of a group G of finite index, then $Ghd_kH = Ghd_kG$.

Furthermore, we give an affirmative answer to Question 4.11 raised in [5].

2. Preliminaries

We set notations and discuss several basic facts which will be useful in the sequel. Unless otherwise stated, R denotes an associative ring with identity and modules are left R-modules. fd_RM denotes the flat dimension of an R-module M. We write w.dim$R$ for the weak dimension of a ring R. More concepts and notations refer to [1,6,7].

A complete flat resolution is an exact sequence of flat R-modules

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots,$$

which remains exact after tensoring by arbitrary injective right R-module. An R-module M is called Gorenstein flat [8] if $M \cong \mathrm{Ker}(F^0 \longrightarrow F^1)$. The Gorenstein flat dimension Gfd_RM is at most n if there is an exact sequence

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

with every G_i Gorenstein flat.

A ring R is called left (right) GF-closed [9] if the class of all Gorenstein flat left (right) R-modules is closed under extensions. Lately, Šaroch et al proved that any ring is left (right) GF-closed [10]. Thus, we can restate Theorems 2.8 and 2.11 in [9] as follows.

Proposition 2.1 *Let R be an arbitrary ring, and let M be an R-module with finite Gorenstein flat dimension. Then the following are equivalent:*

(1) $Gfd_RM \leq n$;

(2) $\mathrm{Tor}_i^R(L, M) = 0$ for all right R-modules L with finite injective dimension, and all $i > n$;

(3) $\mathrm{Tor}_i^R(I, M) = 0$ for all injective right R-modules I, and all $i > n$;

(4) For every exact sequence

$$0 \longrightarrow K_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

where $G_0, \dots, G_{n-1}$ are Gorenstein flat, then so is K_n.

Proposition 2.2 *Let R be an arbitrary ring and consider a short exact sequence of R-modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$. Then the following statements hold.*

(1) If any two of the modules A, B, or C have finite Gorenstein flat dimension, then so has the third.

(2) $Gfd_R A \leq \sup\{Gfd_R B, Gfd_R C - 1\}$ with equality if $Gfd_R B \neq Gfd_R C$.

(3) $Gfd_R B \leq \sup\{Gfd_R A, Gfd_R C\}$ with equality if $Gfd_R C \neq Gfd_R A + 1$.

(4) $Gfd_R C \leq \sup\{Gfd_R B, Gfd_R A + 1\}$ with equality if $Gfd_R B \neq Gfd_R A$.

Proposition 2.3 *Let R be a ring and consider an exact sequence of R-modules*

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F \longrightarrow 0.$$

Then we have $Gfd_R F \leq \max\{i + Gfd_R F_i, i = 0, 1, \dots, n\}$.

Proof Similar to the proof of Lemma 2.8 in [4], it is shown by applying the propositions above. □

Recall that the left Gorenstein weak dimension of a ring R is defined as

$$\text{l.Gw.dim}R = \sup\{Gfd_R M \mid M \text{ is a left } R\text{-module}\}.$$

Similarly, we have the concept of right Gorenstein weak dimension. According to [11, Theorem 6], the left Gorenstein weak dimension of a ring R is equal to its right Gorenstein weak dimension. Thus, we denote the common value by Gw.dim$R$.

Let H be a subgroup of G. Following [12], for an RH-module M, we define the induced module $M \uparrow_H^G := RG \otimes_{RH} M$ with RG acting on the left side and the coinduced module $\text{Hom}_{RH}(RG, M)$. Moreover, every RG-module N can be viewed as an RH-module. It can be verified that the induced functor and the restricted functor also preserve Gorenstein flat modules.

3. Finite Gorenstein homological dimension

An R-module N is called projectively coresolved Gorenstein flat [10] if there exists an exact sequence of projective R-modules

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots,$$

which remains exact after tensoring by any injective right R-module, and $N \cong \text{Ker}(F^0 \longrightarrow F^1)$. By the definition, every projectively coresolved Gorenstein flat module is Gorenstein flat. First of all, we have the following result.

Lemma 3.1 *Let k be a commutative ring with finite weak dimension and let G be a group. Then any projectively coresolved Gorenstein flat kG-module is projective as a k-module.*

Proof By [10, Theorem 4.4], every projectively coresolved Gorenstein flat module is Gorenstein projective. So the result follows from [4, Proposition 1.1]. □

Proposition 3.2 *Let k be a commutative ring with finite weak dimension and let G be a group. Then, for any kG-module M with finite Gorenstein flat dimension, there exists an exact sequence of kG-modules*

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0,$$

where L is projectively coresolved Gorenstein flat and $fd_{kG}N = Gfd_{kG}M$. Moreover, the exact sequence above is k-split.

Proof Suppose that $Gfd_{kG}M = n < \infty$. We proceed by induction on n.

(1) The case $n = 0$ follows from [10, Theorem 4.11].

(2) Let $n > 0$. Choose a short exact sequence of kG-modules

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

with F flat and $Gfd_{kG}K = n - 1$ by Proposition 2.2. Applying the induction hypothesis, we have a short exact sequence

$$0 \longrightarrow K \longrightarrow F' \longrightarrow Q \longrightarrow 0$$

with $fd_{kG}F' = n-1$ and Q projectively coresolved Gorenstein flat. Thus, one gets the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & F' \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d] & F'' \arrow[r] \arrow[d] & Q \arrow[r] & 0 \\
 & M \arrow[r, equal] \arrow[d] & M \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Diagram 1 The pushout diagram

where F'' is Gorenstein flat since all rings are GF-closed. Then, in view of [10, Theorem 4.11] again, there is a short exact sequence of kG-modules

$$0 \longrightarrow F'' \longrightarrow P \longrightarrow L \longrightarrow 0$$

with P flat and L projectively coresolved Gorenstein flat. Thus, we have another pushout diagram:

The right column is desired. To see this we must show that $fd_{kG}N = n$. The class of Gorenstein flat modules is projective resolving [9, Theorem 2.3], so if N is flat, then M is

Gorenstein flat, a contradiction. Thus, $fd_{kG}N > 0$, and it implies $fd_{kG}N = fd_{kG}F' + 1 = n$. Moreover, by Lemma 3.1, L is k-projective, and hence the corresponding exact sequence is k-split. □

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F' \arrow[r] \arrow[d, equal] & F'' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F' \arrow[r] & P \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & L \arrow[r, equal] \arrow[d] & L \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Diagram 2 The pushout diagram

The proposition above implies immediately.

Corollary 3.3 *Let k be a commutative ring with finite weak dimension and let G be a group such that $Ghd_kG < \infty$. Then, there exists a k-split exact sequence of kG-modules*

$$0 \longrightarrow k \longrightarrow N \longrightarrow L \longrightarrow 0,$$

and $fd_{kG}N = Ghd_kG$.

Following [13], an exact sequence of R-modules

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

is pure exact if, for any right R-module B,

$$0 \longrightarrow B \otimes_R A' \longrightarrow B \otimes_R A \longrightarrow B \otimes_R A'' \longrightarrow 0$$

is also exact. It is easy to see that every split exact sequence is pure exact. The first exact sequence above is pure exact if and only if A'' is flat.

Let R be a commutative ring and G a group, and let V and W be RG-modules. Then $V \otimes_R W$ becomes an RG-module under the diagonal action $g(v \otimes w) = (gv) \otimes (gw)$ for all $v \in V$, $w \in W$ and $g \in G$. It is trivial that $V \otimes_R W \cong W \otimes_R V$. Let A and B be a right RG-module and a left RG-module, respectively. We set $A \overline{\otimes}_R B$ as a right RG-module with $(a \overline{\otimes} b)g = ag \overline{\otimes} g^{-1}b$, where $a \in A$, $b \in B$, $g \in G$ (see [14]). Now we have the following assertion which is crucial for our considerations.

Proposition 3.4 *Let R be a commutative ring, and let G be a group. If there exists an R-pure exact sequence of RG-modules*

$$0 \longrightarrow R \longrightarrow N \longrightarrow L \longrightarrow 0 \qquad (3.1)$$

with $fd_{RG}N = n < \infty$, then for any R-flat RG-module M, we have $Gfd_{RG}M \leq fd_{RG}N$.

Proof We consider an R-flat RG-module M and its left RG-flat resolution

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with M_i the ith yoke module and $M_0 = M$. To complete the proof, it is enough to prove that the RG-module M_n is Gorenstein flat in the light of Proposition 2.1. Noting that N is R-flat, we have an exact sequence of RG-modules

$$\cdots \longrightarrow F_1 \otimes_R N \longrightarrow F_0 \otimes_R N \longrightarrow M \otimes_R N \longrightarrow 0.$$

Similar to [5, Lemma 3.2], we can prove that the exact sequence above is a flat resolution of the RG-module $M \otimes_R N$ and its ith yoke module is $M_i \otimes_R N$. On the other hand, in view of [5, Lemma 3.3], $fd_{RG}(M \otimes_R N) \leq fd_{RG}N = n$. Thus, the RG-modules $M_i \otimes_R N$ are flat for all $i \geq n$ and hence $M_n \otimes_R A \otimes_R N$ is RG-flat for any R-flat RG-module A.

Now tensoring the exact sequence (3.1) with the RG-modules $M_n \otimes_R L^{\otimes j}$, where $L^{\otimes j}$ denotes the jth tensor power of L over R, we obtain short exact sequences of RG-modules

$$0 \longrightarrow M_n \longrightarrow M_n \otimes_R N \longrightarrow M_n \otimes_R L \longrightarrow 0,$$

$$0 \longrightarrow M_n \otimes_R L \longrightarrow M_n \otimes_R L \otimes_R N \longrightarrow M_n \otimes_R L^{\otimes 2} \longrightarrow 0,$$

$$\cdots$$

Splicing these exact sequences, one gets an exact sequence

$$0 \longrightarrow M_n \longrightarrow M_n \otimes_R N \longrightarrow M_n \otimes_R L \otimes_R N \longrightarrow M_n \otimes_R L^{\otimes 2} \otimes_R N \longrightarrow \cdots.$$

Furthermore, we obtain a doubly infinite exact sequence of flat RG-modules

$$F^\circ : \cdots \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow M_n \otimes_R N \longrightarrow M_n \otimes_R L \otimes_R N \longrightarrow \cdots.$$

For the moment, it suffices to show that the complex $E \otimes_{RG} F^\circ$ is exact for any injective right RG-module E. Noting that there is a split exact sequence of right RG-modules

$$0 \longrightarrow E \longrightarrow E\overline{\otimes}_R N \longrightarrow E\overline{\otimes}_R L \longrightarrow 0,$$

it is enough to prove that the complex $(E\overline{\otimes}_R N) \otimes_{RG} F^\circ$ is exact. In fact, we have

$$\begin{aligned}
(E\overline{\otimes}_R N) \otimes_{RG} F^\circ &\cong (E\overline{\otimes}_R N) \otimes_{RG} (R \otimes_R F^\circ) \\
&\cong ((E\overline{\otimes}_R N) \otimes_{RG} R) \otimes_R F^\circ \cong (E \otimes_{RG} N) \otimes_R F^\circ \quad (\text{using } [14, \text{Lemma3.1}]) \\
&\cong E \otimes_{RG} (N \otimes_R F^\circ) \cong E \otimes_{RG} (F^\circ \otimes_R N).
\end{aligned}$$

Since $F^\circ \otimes_R N$ is exact and the yoke modules $M_i \otimes_R N$ $(i \geq n)$ and $M_n \otimes_R L^{\otimes j} \otimes_R N$ $(j \geq 1)$ are RG-flat, $E \otimes_{RG} (F^\circ \otimes_R N)$ is exact, as desired. $\square$

Corollary 3.5 *Let R be a commutative ring, and let G be a group. If there exists an R-pure exact sequence of RG-modules*

$$0 \longrightarrow R \longrightarrow N \longrightarrow L \longrightarrow 0$$

with $fd_{RG}N < \infty$, then for any RG-module M, we have $Gfd_{RG}M \leq fd_{RG}N + fd_RM$.

Proof Assume $fd_R(M) = m < \infty$. We proceed by induction on m.

(1) The case $m = 0$ follows from Proposition 3.4.

(2) Let $n > 0$. We consider a short exact sequence of RG-modules

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0,$$

where F is RG-flat, and hence F is R-flat. So $fd_RK \leq m - 1$. By the induction hypothesis,

$$Gfd_{RG}K \leq fd_{RG}N + (m - 1).$$

Therefore, Proposition 2.2 implies $Gfd_{RG}M \leq Gfd_{RG}K + 1 \leq fd_{RG}N + m$. □

Now we establish the main result in this section.

Theorem 3.6 *Let k be a commutative ring with finite weak dimension and let G be a group. Then the following statements are equivalent:*

(1) $\mathrm{Gw.dim}kG < \infty$*;*

(2) $\mathrm{sfli}kG < \infty$*;*

(3) $Ghd_kG < \infty$*;*

(4) There exists a k-split exact sequence of kG-modules

$$0 \longrightarrow k \longrightarrow N \longrightarrow L \longrightarrow 0$$

with $fd_{kG}N < \infty$;

(5) There exists a k-pure exact sequence of kG-modules

$$0 \longrightarrow k \longrightarrow N \longrightarrow L \longrightarrow 0$$

with $fd_{kG}N < \infty$.

Proof (1) $\Leftrightarrow$ (2) follow from [15, Theorem 5.3] because the group ring kG is isomorphic with its opposite ring.

(1) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are trivial.

(3) $\Rightarrow$ (4) follows from Corollary 3.3 and (5) $\Rightarrow$ (1) follows from Corollary 3.5. □

Remark 3.7 In the theorem above, $\mathrm{Gw.dim}kG = \mathrm{sfli}kG$ by [15, Theorem 5.3]. If $\mathbb{F}$ is a field, it is shown that $\mathrm{Gw.dim}\mathbb{F}G = Ghd_{\mathbb{F}}G$ (see [5, Proposition 4.2]). However, $\mathrm{Gw.dim}RG$ need not to be equal to Ghd_RG over a commutative ring R. For example, let G be a finite group and $R = \mathbb{Z}$, then $Ghd_{\mathbb{Z}}G = 0$ but $\mathrm{Gw.dim}\mathbb{Z}G = 1$. Moreover, we have the following result which is a Gorenstein state of [16, Proposition 4].

Corollary 3.8 *Let R be a commutative ring, and let G be a group. Then* $\mathrm{Gw.dim}RG \leq Ghd_RG + \mathrm{w.dim}R$.

Proof We assume that $Ghd_RG = m$ and $\mathrm{w.dim}R = n$ are finite. By Corollary 3.3, there is an

R-split short exact sequence of RG-modules

$$0 \longrightarrow R \longrightarrow N \longrightarrow L \longrightarrow 0,$$

and $fd_{RG}N = Ghd_R G = n$. So, in view of Corollary 3.5, we have

$$Gfd_{RG}M \leq fd_{RG}N + fd_R M \leq m + n$$

for any RG-module M. □

Remark 3.9 By Corollary 3.8, if k is a von Numann regular ring, then $\mathrm{Gw.dim}kG = Ghd_k G$.

We do not know whether Gorenstein projective modules are Gorenstein flat modules. The well-known Govorov-Lazard theorem says that a module is flat if and only if it is a direct limit of finitely generated projective modules. Holm [17] constructed an algebra which has no Gorenstein analogue of the Govorov-Lazard Theorem. However, we have the following result over a ring with finite Gorenstein weak dimension, which is an independent interest.

Proposition 3.10 *Let R be a ring (unnecessarily commutative) with* $\mathrm{Gw.dim}R < \infty$*, and let M be an R-module. Then the following are equivalent:*

(1) M is Gorenstein flat;

(2) There is a direct system (M_i) of finitely generated Gorenstein projective R-modules such that $M \cong \varinjlim M_i$.

Proof (2) $\Rightarrow$ (1). By [15, Theorem 5.3] and [18, Proposition 9], every Gorenstein projective R-module is Gorenstein flat. Thus, in view of [19, Lemma 3.1], M is Gorenstein flat because any ring is GF-closed.

(1) $\Rightarrow$ (2). If M is Gorenstein flat, there exists an exact sequence of flat R-modules

$$F^\circ : \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

such that $E \otimes_R F^\circ$ is exact for any injective right R-module E and $M \cong Z(F^\circ)$ $(\mathrm{Ker}(F^0 \longrightarrow F^1)$. By [20, Lemma 8.4], $F^\circ \cong \varinjlim P_i^\circ$, where all P_i° are exact sequences of finitely generated projective R-modules. Set $M_i := Z(P_i^\circ)$. Since $\mathrm{Gw.dim}R$ is finite, every injective R-module has finite flat dimension. It implies that $\mathrm{Tor}_n^R(I, M_i) = 0$ for all $n > 0$ and any injective R-module I. Thus, all M_i are finitely generated, projectively coresolved Gorenstein flat, and hence all M_i are finitely generated Gorenstein projective. Moreover, $M \cong Z(\varinjlim P_i^\circ) \cong \varinjlim M_i$ by [21, Proposition 3.4]. We complete the proof. □

Let $\overline{\mathrm{P}}(R)$ (resp., $\overline{\mathrm{F}}(R)$) be the class consisting of the R-modules with finite projective dimension (resp., finite flat dimension).

Proposition 3.11 *Let k be a commutative ring with finite global dimension and let G be a group. Then the following conditions are equivalent:*

(1) $Gcd_k G < \infty$;

(2) $Ghd_k G < \infty$ and $\overline{\mathrm{P}}(kG) = \overline{\mathrm{F}}(kG)$.

Proof It follows from Theorem 3.6, [4, Theorem 1.7] and [4, Corollary 4.11]. □

4. Analogy of Serre's theorem

In this section, we generalize some well-known results about Gorenstein homological dimensions of groups over ordinary integer rings onto commutative coefficient rings.

Lemma 4.1 *Let R be a commutative ring, and let G be a group. Then $Ghd_R G \leq Ghd_{\mathbb{Z}} G$.*

Proof If $Ghd_{\mathbb{Z}} G$ is infinite, there is nothing to prove. Now let $Ghd_{\mathbb{Z}} G = n < \infty$. By Theorem 3.6, there exists a $\mathbb{Z}$-pure exact sequence of $\mathbb{Z}G$-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow N \longrightarrow L \longrightarrow 0$$

with $fd_{\mathbb{Z}G} N \leq n$. Then we have an exact sequence of RG-modules

$$0 \longrightarrow R \longrightarrow R \otimes_{\mathbb{Z}} N \longrightarrow R \otimes_{\mathbb{Z}} L \longrightarrow 0.$$

Noting that $R \otimes_{\mathbb{Z}} L$ is also R-flat, the exact sequence above is R-pure. Choose a left $\mathbb{Z}G$-flat resolution of N

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0.$$

Since N is $\mathbb{Z}$-flat, we obtain an exact sequence of RG-modules

$$0 \longrightarrow R \otimes_{\mathbb{Z}} F_n \longrightarrow \cdots \longrightarrow R \otimes_{\mathbb{Z}} F_1 \longrightarrow R \otimes_{\mathbb{Z}} F_0 \longrightarrow R \otimes_{\mathbb{Z}} N \longrightarrow 0$$

with all RG-modules $R \otimes_{\mathbb{Z}} F_i$ ($i = 0, 1, 2, \ldots, n$) being flat, and hence $fd_{RG}(R \otimes_{\mathbb{Z}} N) \leq n$. Thus, in view of Proposition 3.4, $Ghd_R G = Gfd_{RG} R \leq fd_{RG}(R \otimes_{\mathbb{Z}} N) \leq n$. □

Proposition 4.2 *Let R be a commutative ring, and let G be a group. Then $Ghd_R G = 0$ if and only if G is finite.*

Proof If G is finite, in view of [3, Proposition 4.12], $Ghd_{\mathbb{Z}} G = 0$, and hence $Ghd_R G = 0$ by Lemma 4.1. Conversely, if $Ghd_R G = 0$, the trivial RG-module R is Gorenstein flat, and hence $\mathrm{Hom}_{RG}(R, F) \neq 0$ for some flat RG-module F. By the Govorov-Lazard theorem, $F \cong \varinjlim P_i$, where P_i is finitely generated projective. Thus,

$$\varinjlim \mathrm{Hom}_{RG}(R, P_i) \cong \mathrm{Hom}_{RG}(R, \varinjlim P_i) \cong \mathrm{Hom}_{RG}(R, F) \neq 0.$$

Then $\mathrm{Hom}_{RG}(R, P_i) \neq 0$ for some i, and hence G is finite. □

Remark 4.3 (1) There is another way to partially prove the proposition above. If G is a finite group, there is an R-split exact sequence of RG-modules

$$0 \longrightarrow R \longrightarrow RG \longrightarrow \overline{B} \longrightarrow 0,$$

where $\overline{B} = RG/Ru$, $u = \sum_{g \in G} g$. Thus, in view of Proposition 3.4,

$$Ghd_R G = Gfd_{RG} R \leq fd_{RG} RG = 0.$$

(2) Let R be a commutative ring and G be a infinite cyclic group. Then

$$\mathrm{Gw.dim} RG = Ghd_R G = 1.$$

Proposition 4.4 *Let k be a commutative ring with finite weak dimension and let H be a subgroup of a group G. Then $Ghd_kH \leq Ghd_kG$.*

Proof Assume $Ghd_kG = n < \infty$. By Corollary 3.3, there exists a k-split exact sequence of kG-modules

$$0 \longrightarrow k \longrightarrow N \longrightarrow L \longrightarrow 0$$

with $fd_{kG}N = n$. Noting that it is also a k-split exact sequence of kH-modules with $fd_{kH}N \leq fd_{kG}N = n$ by [14, Proposition 2.2 (2)]. Thus, in view of Proposition 3.4, $Ghd_kH = Gfd_{kH}k \leq fd_{kH}N \leq n$. □

Proposition 4.5 *Let k be a commutative ring with finite weak dimension and (G_α) a directed family of subgroups of a group G such that G is the direct limit of the G_α. If Ghd_kG is finite, then $Ghd_kG = \sup\{Ghd_kG_\alpha\}$.*

Proof By Proposition 4.4, we also have $Ghd_kG_\alpha \leq Ghd_kG$. Conversely, since G is the direct limit of G_α, it follows that kG is the direct limit of the kG_α. Thus, in view of [6, Chapter VI, Exercise 17],

$$\mathrm{Tor}_*^{kG}(I,k) = \varinjlim \mathrm{Tor}_*^{kG_\alpha}(I,k)$$

for any injective right kG-module I. Therefore, the result follows from Proposition 2.1. □

Remark 4.6 Let k be a commutative ring with finite weak dimension and G a group.

(1) If $Ghd_kG = n$, then there exists a finitely generated subgroup H such that $Ghd_kH = n$.

(2) If G is locally finite such that Ghd_kG is finite, then $Ghd_kG = 0$, and hence G is finite.

Let H be a normal subgroup of a group G. We now establish the estimate for the Gorenstein homological dimension of G by the corresponding values of H and G/H. The following lemma is needed.

Lemma 4.7 *Let R be a commutative ring and let H be a normal subgroup of a group G. Then, for any flat $R(G/H)$-module F, we have $Gfd_{RG}F \leq Ghd_RH$.*

Proof Similar to the proof of [14, Proposition 2.5 (1)], we can prove that if M is Gorenstein flat as an RH-module then $M\uparrow_H^G$ is Gorenstein flat as an RG-module. Now let $Ghd_RH = n < \infty$. Then there is an exact sequence of RH-modules

$$0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow R \longrightarrow 0,$$

where G_i is Gorenstein flat for all i. Furthermore, there is an exact sequence of RG-modules

$$0 \longrightarrow G_n\uparrow_H^G \longrightarrow \cdots \longrightarrow G_1\uparrow_H^G \longrightarrow G_0\uparrow_H^G \longrightarrow R\uparrow_H^G \longrightarrow 0$$

and $G_i\uparrow_H^G$ is Gorenstein flat for all i. So $Gfd_{RG}(R\uparrow_H^G) \leq n$. It is known that $R\uparrow_H^G \cong R(G/H)$. Thus, in view of [9, Proposition 2.10], $Gfd_{RG}Q \leq n$ for any free $R(G/H)$-module Q. For any flat $R(G/H)$-module F, by the Govorov-Lazard theorem, $F \cong \varinjlim Q_i$ with Q_i finitely generated free. Since the direct limit is an exact functor and Gorenstein flat modules are closed under direct limits, $Gfd_{RG}F$ is finite. Therefore,in view of Proposition 2.1 and [13, Theorem 8.11], $Gfd_{RG}F \leq n$. □

Theorem 4.8 *Let k be a commutative ring with finite weak dimension and let H be a normal subgroup of a group G. Then $Ghd_kG \leq Ghd_kH + Ghd_k(G/H)$.*

Proof We assume that $Ghd_kH = m$ and $Ghd_k(G/H) = n$. By Theorem 3.6, there exists a k-pure exact sequence of $k(G/H)$-modules

$$0 \longrightarrow k \longrightarrow N \longrightarrow L \longrightarrow 0,$$

where N is k-flat and $fd_{k(G/H)}N \leq n$. Thus, there is a left $k(G/H)$-flat resolution of N,

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0.$$

By Lemma 4.7, $Gfd_{kG}F_i \leq Ghd_kH = m$ for all i. So, $Gfd_{kG}N \leq m + n$ by Proposition 2.3. Moreover, Proposition 3.2 infers a k-split exact sequence of kG-modules

$$0 \longrightarrow N \longrightarrow A \longrightarrow B \longrightarrow 0,$$

where $fd_{kG}A = Gfd_{kG}N$. Thus, [22, Example 4.84 (e)] implies a k-pure exact sequence of kG-modules

$$0 \longrightarrow k \longrightarrow A \longrightarrow C \longrightarrow 0.$$

Therefore, $Ghd_kG = Gfd_{kG}k \leq fd_{kG}A \leq m + n$ by applying Proposition 3.4. □

By Propositions 4.2, 4.4 and Theorem 4.8, we have the next result immediately.

Corollary 4.9 *Let k be a commutative ring with finite weak dimension, and let H be a normal subgroup of a group G. If G/H is finite, then $Ghd_kH = Ghd_kG$.*

Remark 4.10 Unfortunately, we do not know whether $Ghd_kG = Ghd_k(G/H)$ provided the normal subgroup H is finite.

A group G is called Polycyclic-by-finite if there is a finite subnormal series for G,

$$\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G,$$

where G_{i+1}/G_i is either infinite cyclic or finite.

Corollary 4.11 *If k is a commutative ring with finite weak dimension and G is Polycyclic-by-finite, then Ghd_kG is finite.*

Proof We proceed by induction on i.

(1) The case $i = 0$ is trivial.

(2) Suppose that $i > 0$ and $Ghd_kG_i = m$ is finite. If G_{i+1}/G_i is finite, then $Ghd_kG_{i+1} = Ghd_kG_i = m$ is finite by Corollary 4.9. If G_{i+1}/G_i is infinite cyclic, Remark 4.3 involves $Ghd_kG_{i+1}/G_i = 1$. Thus, $Ghd_kG_{i+1} \leq m + 1$ is finite by Theorem 4.8.

As $G_n = G$, Ghd_kG is finite from the principle of induction. □

We conclude with the following theorem which is a Gorenstein analogy of the Serre's theorem.

Theorem 4.12 *If k is a commutative ring with finite weak dimension and H is a subgroup of a group G of finite index, then $Ghd_k H = Ghd_k G$.*

Proof By Proposition 4.4, it suffices to prove $Ghd_k G \leq Ghd_k H$. Now let $Ghd_k H = n < \infty$ and consider a left kG-flat resolution of k

$$0 \longrightarrow K_n \longrightarrow F_{n-1} \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow k \longrightarrow 0,$$

where K_n is the nth yoke module. To complete the proof, it suffices to show that K_n is Gorenstein flat as a kG-module. The exact sequence above is also an exact sequence of kH-modules. Proposition 2.1 implies that K_n is Gorenstein flat as a kH-module. By [10, Theorem 4.1], there exists a short exact sequence of kH-modules

$$0 \longrightarrow K_n \longrightarrow F \longrightarrow L \longrightarrow 0,$$

where F is flat and L is projectively coresolved Gorenstein flat. It is known that the kG-monomorphism $F \longrightarrow \mathrm{Hom}_{kH}(kG, F)$ is kH-split, and $\mathrm{Hom}_{kH}(kG, F) \cong kG \otimes_{kH} F$ is a flat kG-module by [23, Lemma 9.2]. Thus, there is a flat kH-module U such that $\mathrm{Hom}_{kH}(kG, F) \cong F \oplus U$. Consequently, one has the short exact sequence of kG-modules

$$0 \longrightarrow K_n \longrightarrow \mathrm{Hom}_{kH}(kG, F) \longrightarrow L' \longrightarrow 0,$$

where $L' \cong L \oplus U$ is Gorenstein flat as a kH-module. We repeat the argument with L' replacing K_n and in this way we get a right kG-flat resolution of K_n

$$0 \longrightarrow K_n \longrightarrow F'_0 \longrightarrow F'_1 \longrightarrow F'_2 \longrightarrow \cdots.$$

Splicing the left kG-flat resolution of K_n, we obtain a doubly infinite exact sequence of flat kG-modules

$$F^\circ : \cdots \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow F'_0 \longrightarrow F'_1 \longrightarrow F'_2 \longrightarrow \cdots$$

and

$$K_n \cong \mathrm{Ker}(F'_0 \longrightarrow F'_1).$$

Now it is enough to show that $E \otimes_{kG} F^\circ$ is exact for any injective right kG-module E. Noting that E is a direct summand of $\mathrm{Hom}_{kH}(kG, E)$, it is sufficient to prove $\mathrm{Hom}_{kH}(kG, E) \otimes_{kG} F^\circ$ is exact. Indeed, we have

$$\mathrm{Hom}_{kH}(kG, E) \otimes_{kG} F^\circ \cong (E \otimes_{kH} kG) \otimes_{kG} F^\circ \cong E \otimes_{kH} F^\circ.$$

Since E is injective as a kH-module and every yoke module of F° is Gorenstein flat as a kH-module, $E \otimes_{kH} F^\circ$ is exact, as desired. $\square$

Remark 4.13 In [5, Proposition 4.9], the conditions that $K[G]$ is right coherent and $GwD(K[G])$ is finite can be omitted. In [3, Theorem 4.18], the condition that silfΓ is finite can be omitted.

Acknowledgements We thank the referees for their time and comments.

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