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A Positive Solution of Mixed Non-Linear Fractional Delay Differential Equations with Integral Boundary Conditions

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Abstract In this paper, we study mixed non-linear fractional delay differential equations with integral boundary conditions. We obtain an equivalence result between the proposed problem and non-linear Fredholm integral equation of the second kind. Further, we establish existence and uniqueness of positive solutions for the problem using Guo-Krasnoseleskii's fixed point theorem and Banach contraction principle.

Keywords Guo-Krasnoseleskii's fixed point theorem; Banach contraction principle; mixed non-linear fractional delay differential equations; integral boundary conditions; existence; uniqueness

MR(2020) Subject Classification 26A33; 34A08; 34A12; 34K20; 37C25

1. Introduction

The concept of fractional differentiation and integration are usually associated with the name of Liouville. However, the creators of differential and integral calculus had already taken into consideration derivatives no longer only of integer order, but of fractional order too. We understand that fractional derivatives were the subjects of Leibniz's study. Euler also took an interest in fractional derivatives. Liouville, Abel, Riemann, Letnikov, Weyl, Hadamard and many different well-known mathematicians of the past and present influenced the development of fractional integro-differentiation, which has now grown to be an extensive subject matter in mathematical evaluation [1].

Fractional Differential Equations (FDEs) have achieved the attention of scientists, due to its applications in applied sciences and engineering problems, such as Viscoelasticity, Food Science, Fractional Diffusion Equations [2]. Fractional order delay differential equations (FDDEs) also have applications in all disciplines which include chemistry, physics and finance [3].

"Riemann-Liouville fractional derivative" or "Caputo fractional derivative" are most popular. However, a new fractional derivative has been added through Khalil et al. in [4] named as "the conformable fractional derivative" that fulfills all of the requirements of the standard derivative.

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Lately, fractional differential equations which include both left and right fractional derivatives are also attracting much attention, and there are more results on boundary value problems concerning mixed fractional derivatives of several types. Recently Ntouyas et al. [5] investigated the existence and uniqueness of solutions of single and multi-valued boundary value problems containing both Riemann-Liouville and Caputo fractional derivatives and nonlocal fractional integro-differential boundary conditions. The authors proved the existence of solutions for a boundary value problem having both left Riemann-Liouville and right Caputo fractional derivatives by using Krasnoselskii's fixed point theorem [6]. Somia and Brahimin [7] proved the existence of positive solutions for integral boundary conditions involving left Caputo fractional derivatives and right conformable fractional derivative by the use of Krasnoselskii's fixed point theorem.

Li, Zhang and Jiang [8] investigated the existence and uniqueness of positive solutions of following integral boundary value problems of fractional differential equations with delay:

$$\begin{aligned}
&{}^{C}D^{\beta}z(t) + g(t, z_t) = 0, \quad t \in [0, 1],\\
&z(t) = \phi(t), \quad t \in [-\tau, 0],\\
&z(0) = z''(0) = z'''(0) = 0,\\
&z(1) = k\int_0^1 z(\theta)\mathrm{d}\theta,
\end{aligned}$$

where $3 < \beta \leq 4$, $0 < k < 2$, ${}^{C}D^{\beta}$ is the caputo fractional derivative.

In this paper, we look on existence, uniqueness of positive solution of mixed non-linear fractional delay differential equations with integral boundary conditions (MFDDEIBC):

$$D^{\nu}_{1^-}({}^{C}D^{\mu}_{0^+}z)(t) = g(t, z_t), \quad t \in [0, 1], \tag{1.1}$$

$$z(t) = \psi(t), \quad t \in [-\tau, 0], \tag{1.2}$$

$$z(0) = \beta\int_0^1 z(t)\mathrm{d}t, \tag{1.3}$$

$${}^{C}D^{\mu}_{0^+}z(1) = 0, \tag{1.4}$$

where $0 < \mu \leq 1$, $0 < \nu \leq 1$, $0 < \beta < 1$, $D^{\nu}_{1^-}$ is the right conformable fractional derivative, ${}^{C}D^{\mu}_{0^+}$ is the left caputo fractional derivative, $g : [0, 1] \times C[-\tau, 0] \to [0, +\infty]$ is a continuous function, $z_t(s) = z(t+s)$, for $t \in [0, 1]$, $s \in [-\tau, 0]$, $\psi \in C[-\tau, 0]$, $C[-\tau, 0]$ is a Banach space with $||\psi||_{[-\tau,0]} = \max_{s\in[-\tau,0]} |\psi(t)|$.

2. Preliminaries

In this section we introduce the some definitions, important lemmas and theorems.

Definition 2.1 ([4, 9]) *The left conformable fractional derivative starting from a of a function $z : [a, +\infty) \to \mathbb{R}$ of order $0 < \nu \leq 1$ is defined as*

$$D^{(\nu)}_{a^+}z(t) = \lim_{\epsilon\to 0} \frac{z(t + \epsilon(t-a)^{1-\nu}) - z(t)}{\epsilon}, \ \textit{for all } t > 0.$$

If $D_{a^+}^{(\nu)}z(t)$ *exists on* (a,b)*, then* $D_{a^+}^{(\nu)}z(a) = \lim_{t\to a^+} D_{a^+}^{(\nu)}z(t)$.

Definition 2.2 ([4,9]) *The right conformable fractional derivative terminating at* b *of a function* z *of order* $0<\nu\leq 1$ z *is defined as*

$$D_{b^-}^{(\nu)}z(t) = -\lim_{\epsilon\to 0}\frac{z(t+\epsilon(b-t)^{1-\nu})-z(t)}{\epsilon}, \text{ for all } t>0.$$

If $D_{b^-}^{(\nu)}z(t)$ *exists on* (a,b)*, then* $D_{b^-}^{(\nu)}z(b) = \lim_{t\to b^-} D_{b^-}^{(\nu)}z(t)$.

Definition 2.3 ([4,9]) *The left and right conformable fractional integrals of order* $0<\nu\leq 1$ *are defined as follows, respectively:*

$$I_{a^+}^{\nu}z(t) = \int_a^t (s-a)^{\nu-1}z(s)\mathrm{d}s, \tag{2.1}$$

$$I_{b^-}^{\nu}z(t) = \int_t^b (b-s)^{\nu-1}z(s)\mathrm{d}s. \tag{2.2}$$

Definition 2.4 ([10]) *The left Caputo fractional derivative of order* $0<\mu\leq 1$ *starting at* a *of an absolutely continuous function* $z:[a,b]\to\mathbb{R}$ *is given by*

$$^{C}D_{a+}^{\mu}z(t) = \frac{1}{\Gamma(1-\mu)}\int_a^t (t-s)^{-\mu}z'(s)\mathrm{d}s. \tag{2.3}$$

Definition 2.5 ([10]) *The right Caputo fractional derivative of order* $0<\mu\leq 1$ *terminating at* b *of an absolutely continuous function* $z:[a,b]\to\mathbb{R}$ *is given by*

$$^{C}D_{b-}^{\mu}z(t) = \frac{1}{\Gamma(1-\mu)}\int_t^b (s-t)^{-\mu}z'(s)\mathrm{d}s. \tag{2.4}$$

Definition 2.6 ([10]) *The left and right Riemann-Liouville fractional integral of order* $0<\mu\leq 1$ *of a function* z *are as follows, respectively:*

$$J_{a^+}^{\mu}z(t) = \frac{1}{\Gamma(\mu)}\int_a^t (t-s)^{\mu-1}z(s)\mathrm{d}s, \tag{2.5}$$

$$J_{b^-}^{\mu}x(t) = \frac{1}{\Gamma(\mu)}\int_t^b (s-t)^{\mu-1}z(s)\mathrm{d}s, \tag{2.6}$$

where $\Gamma(.)$ *is the Euler Gamma function [10, p. 24].*

Lemma 2.7 ([4,9,10]) *(1) If* z *is a continuous function on* (a,b)*, then*

$$D_{b^-}^{\nu}(I_{b^-}^{\nu}z(t)) = D_{a^+}^{\nu}(I_{a^+}^{\nu}z(t)) = {}^{C}D_{b^-}^{\mu}(J_{b^-}^{\mu}z(t)) = {}^{C}D_{a^+}^{\mu}(J_{a^+}^{\mu}z(t)) = z(t). \tag{2.7}$$

(2) If $D_{a^+}^{\nu}$, $D_{b^-}^{\nu}$, ${}^{C}D_{a^+}^{\mu}$, ${}^{C}D_{b^-}^{\mu}$ *are continuous on* (a,b)*, then*

$$I_{a^+}^{\nu}(D_{a^+}^{\nu}z(t)) = J_{a^+}^{\mu}({}^{C}D_{a^+}^{\mu}z(t)) = z(t)-z(a), \tag{2.8}$$

$$I_{b^-}^{\nu}(D_{b^-}^{\nu}z(t)) = J_{b^-}^{\mu}({}^{C}D_{b^-}^{\mu}z(t)) = z(t)-z(b). \tag{2.9}$$

(3) If z *is differentiable on* (a,b)*, then*

$$D_{a^+}^{\nu}z(t) = (t-a)^{1-\nu}z'(s), \tag{2.10}$$

$$D_{b^-}^{\nu} z(t) = -(b-t)^{1-\nu} z'(s). \tag{2.11}$$

Theorem 2.8 (Banach Contraction Principle [11,12]) *Let $(E, ||\cdot||)$ be a Banach space, $P \subset E$ a non-empty closed subset. If $T : P \to P$ is of strict contraction i.e., $\exists L \in (0,1)$, $\forall\, x, y \in P$ such that $||Tx - Ty|| \leq L||x - y||$, then T has unique fixed point in P.*

Theorem 2.9 (Guo-Kransnoselskii's fixed point theorem [13]) *Let $(E, ||\cdot||)$ be a Banach space, $P \subset E$ be a cone and Ω_1, Ω_2 be two bounded open sets in E, such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Let operator $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ be completely continuous. Suppose that one of the two conditions,*

(1) $||Tz|| \leq ||z||, \forall\, z \in P \cap \partial\Omega_1$ and $||Tz|| \geq ||z||, \forall\, z \in P \cap \partial\Omega_2$ and

(2) $||Tz|| \geq ||z||, \forall\, z \in P \cap \partial\Omega_1$ and $||Tz|| \leq ||z||, \forall\, z \in P \cap \partial\Omega_2$

is satisfied, then T has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Existence solution

In this section, we will study the existence of solution and equivalence result between given problems (1.1)–(1.4) and nonlinear integral Fredholm equation of the second kind. For this, we define the Banach space $E = C([-\tau, 1], \mathbb{R})$ with norm $||z||_{[-\tau,1]} = \max\{|z| : z \in [-\tau, 1]\}$.

Theorem 3.1 *Let $\mu, \nu \in (0,1]$, $\beta \in (0,1)$, $g : [0,1] \times C[-\tau, 0] \to \mathbb{R}^+$ be continuous function. Then $z \in E$ is a solution of MFDDEIBC given by (1.1)–(1.4) if and only if $z \in E$ satisfies the non-linear and homogeneous Fredholm integral equation of second kind,*

$$z(t) = \begin{cases} \int_0^1 G(t,s) g(s, z_s) \mathrm{d}s, & t \in [0,1], \\ \psi(t), & t \in [-\tau, 0], \end{cases} \tag{3.1}$$

where G is the Green's function given by

$$G(t,s) = \begin{cases} [\frac{\beta}{(1-\beta)(\mu+1)}(1-(1-s)^{\mu+1}) + t^{\mu} - (t-s)^{\mu}]\frac{(1-s)^{\nu-1}}{\beta(\mu+1)}, & \text{if } 0 \leq s \leq t \leq 1, \\ [\frac{\beta}{(1-\beta)(\mu+1)}(1-(1-s)^{\mu+1}) + t^{\mu}]\frac{(1-s)^{\nu-1}}{\beta(\mu+1)}, & \text{if } 0 \leq t \leq s \leq 1. \end{cases} \tag{3.2}$$

Proof In the first part we show the necessity:

Applying the right fractional integral $I_{1^-}^{\nu}$ in (2.2) on the Eq. (1.1) and using (2.9) and (1.4), we get

$${}^{C}D_{0^+}^{\mu} z(t) = I_{1^-}^{b} g(t, z_t). \tag{3.3}$$

Applying the left fractional integral $J_{0^+}^{\mu}$ in (2.5) on the Eq. (3.3) and using (2.8) and (1.3), we get

$$z(t) = \beta \int_0^1 z(t) \mathrm{d}t + J_{0^+}^{\mu} I_{1^-}^{\nu} g(t, z_t). \tag{3.4}$$

Next, using (2.2) and (2.5), we get

$$J_{0^+}^{\mu} I_{1^-}^{\nu} g(t, z_t) = \frac{1}{\Gamma\mu} \int_o^t (t-m)^{\mu-1} I_{1^-}^{\nu} g(m, z_m) \mathrm{d}m$$

$$
\begin{aligned}
=&\frac{1}{\Gamma\mu}\int_o^t (t-m)^{\mu-1}\Big[\int_m^1 (1-s)^{\nu-1} g(s,z_s)\mathrm{d}s\Big]\mathrm{d}m \\
=&\frac{1}{\Gamma\mu}\int_o^t (t-m)^{\mu-1}\Big[\int_m^t (1-s)^{\nu-1} g(s,z_s)\mathrm{d}s+\int_t^1 (1-s)^{\nu-1} g(s,z_s)\mathrm{d}s\Big]\mathrm{d}m.
\end{aligned}
$$

Using Fubini theorem, we obtain

$$
\begin{aligned}
J_{0+}^{\mu} I_{1-}^{\nu} g(t,z_t) =&\frac{1}{\Gamma\mu}\int_o^t \Big(\int_0^s (t-m)^{\mu-1}\mathrm{d}m\Big)(1-s)^{\nu-1} g(s,z_s)\mathrm{d}s+ \\
&\frac{1}{\Gamma\mu}\int_t^1 \Big(\int_0^t (t-m)^{\mu-1}\mathrm{d}m\Big)(1-s)^{\nu-1} g(s,z_s)\mathrm{d}s \\
=&\frac{1}{\Gamma(\mu+1)}\int_o^t (t^{\mu}-(t-s)^{\mu})(1-s)^{\nu-1} g(s,z_s)\mathrm{d}s+ \\
&\frac{1}{\Gamma(\mu+1)}\int_t^1 t^{\mu}(1-s)^{\nu-1} g(s,z_s)\mathrm{d}s \\
=&\frac{1}{\Gamma(\mu+1)}\Big[\int_o^1 t^{\mu}(1-s)^{\nu-1} g(s,z_s)\mathrm{d}s-\int_0^t (t-s)^{\mu}(1-s)^{\nu-1} g(s,z_s)\mathrm{d}s\Big].
\end{aligned}
$$

Now, Eq. (3.4) becomes,

$$
\begin{aligned}
z(t) =&\beta\int_0^1 z(t)\mathrm{d}t+\frac{1}{\Gamma(\mu+1)}\Big[\int_o^1 t^{\mu}(1-s)^{\nu-1} g(s,z_s)\mathrm{d}s- \\
&\int_0^t (t-s)^{\mu}(1-s)^{\nu-1} g(s,z_s)\mathrm{d}s\Big].
\end{aligned}
\tag{3.5}
$$

Integrating (3.5) on $[0,1]$ in both sides gives,

$$
\begin{aligned}
(1-\beta)\int_0^1 z(t)\mathrm{d}t =&\frac{1}{\Gamma(\mu+1)}\int_0^1 \Big[\int_o^1 t^{\mu}(1-s)^{\nu-1} g(s,z_s)\mathrm{d}s\Big]\mathrm{d}t- \\
&\frac{1}{\Gamma(\mu+1)}\int_0^1 \Big[\int_0^t (t-s)^{\mu}(1-s)^{\nu-1} g(s,z_s)\mathrm{d}s\Big]\mathrm{d}t.
\end{aligned}
$$

Using Fubini theorem, we obtain

$$
\begin{aligned}
(1-\beta)\int_0^1 z(t)\mathrm{d}t =&\frac{1}{\Gamma(\mu+1)}\int_0^1 t^{\mu}\mathrm{d}t\int_o^1 (1-s)^{\nu-1} g(s,z_s)\mathrm{d}s- \\
&\frac{1}{\Gamma(\mu+1)}\int_0^1 \Big[\int_s^1 (t-s)^{\mu}\mathrm{d}t\Big](1-s)^{\nu-1} g(s,z_s)\mathrm{d}s \\
\int_0^1 z(t)\mathrm{d}t =&\frac{1}{(1-\beta)\Gamma(\mu+2)}\int_0^1 [(1-s)^{\nu-1}-(1-s)^{\mu+\nu}] g(s,z_s)\mathrm{d}s.
\end{aligned}
\tag{3.6}
$$

Substituting Eq. (3.6) in (3.5), we get

$$
\begin{aligned}
z(t) =& \int_0^t [\frac{\beta}{(1-\beta)(\mu+1)}(1-(1-s)^{\mu+1})+t^{\mu}-(t-s)^{\mu}]\frac{(1-s)^{\nu-1}}{\Gamma(\mu+1)} g(s,z_s)\mathrm{d}s+ \\
&\int_t^1 [\frac{\beta}{(1-\beta)(\mu+1)}(1-(1-s)^{\mu+1})+t^{\mu}]\frac{(1-s)^{\nu-1}}{\Gamma(\mu+1)} g(s,z_s)\mathrm{d}s \\
=& \int_0^1 G(t,s) g(s,z_s)\mathrm{d}s
\end{aligned}
$$

and also, we have

$$z(t)=\psi(t),\quad t\in[-\tau,0].$$

Thus the first part is proved.

Now, next part is sufficient part:

Let $z\in E$ be the solution of Eq. (3.1). We first prove that z satisfies (1.3). Using (3.1) and (3.2), we get

$$\begin{aligned}z(0)-\beta\int_0^1 z(t)\mathrm{d}t=&\int_0^1 G(0,s)g(s,z_s)\mathrm{d}s+\\&\beta\int_0^1\Big[\int_0^t G(t,s)g(s,z_s)\mathrm{d}s+\int_t^1 G(t,s)g(s,z_s)\mathrm{d}s\Big]\mathrm{d}t.\end{aligned}$$

Using Fubini theorem, we get

$$\begin{aligned}z(0)-\beta\int_0^1 z(t)\mathrm{d}t=&\int_0^1 G(0,s)g(s,z_s)\mathrm{d}s-\\&\beta\int_0^1\Big[\int_s^1 G(t,s)\mathrm{d}t+\int_0^s G(t,s)\mathrm{d}t\Big]g(s,z_s)\mathrm{d}s.\end{aligned}\tag{3.7}$$

Now, from (3.2) it follows

$$\begin{aligned}\int_s^1 G(t,s)\mathrm{d}t=&\frac{\beta(1-s)}{(1-\beta)\Gamma(\mu+2)}[(1-s)^{\nu-1}-(1-s)^{\mu+\nu}]+\\&\frac{1}{\Gamma(\mu+2)}[1-s^{\mu+1}-(1-s)^{\mu+1}](1-s)^{\nu-1}\end{aligned}\tag{3.8}$$

and

$$\int_0^s G(t,s)\mathrm{d}t=\frac{\beta s}{(1-\beta)\Gamma(\mu+2)}[(1-s)^{\nu-1}-(1-s)^{\mu+\nu}]+\frac{1}{\Gamma(\mu+2)}s^{\mu+1}(1-s)^{\nu-1}.\tag{3.9}$$

Substituting (3.8) and (3.9) in (3.7) and using (3.2), we obtain

$$\begin{aligned}z(0)-\beta\int_0^1 z(t)\mathrm{d}t=&\int_0^1 G(0,s)g(s,z_s)\mathrm{d}s-\\&\frac{\beta}{(1-\beta)\Gamma(\mu+2)}\int_0^1[1-(1-s)^{\mu+1}](1-s)^{\nu-1}g(s,z_s)\mathrm{d}s\\=&\int_0^1 G(0,s)g(s,z_s)\mathrm{d}s-\int_0^1 G(0,s)g(s,z_s)\mathrm{d}s=0,\end{aligned}$$

$$z(0)=\beta\int_0^1 z(t)\mathrm{d}t.$$

Now, we have to prove $z\in E$ satisfies boundary condition (1.4).

Using (3.1), (3.2) and (2.3) gives

$$\begin{aligned}{}^{C}D^{\mu}_{0+}z(1)=&\frac{1}{\Gamma(1-\mu)}\int_0^1(1-s)^{-\mu}z'(s)\mathrm{d}s\\=&\frac{1}{\Gamma(1-\mu)}\int_0^1(1-s)^{-\mu}\int_0^s\frac{\partial G}{\partial s}(s,m)g(m,z_m)\mathrm{d}m\mathrm{d}s+\\&\frac{1}{\Gamma(1-\mu)}\int_0^1(1-s)^{-\mu}\int_s^1\frac{\partial G}{\partial s}(s,m)g(m,z_m)\mathrm{d}m\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\mu)\Gamma(1-\mu)} \int_0^1 \int_0^s (1-s)^{-\mu}[s^{\mu-1} - (s-m)^{\mu-1}](1-m)^{\nu-1} g(m, z_m) \mathrm{d}m \mathrm{d}s + \\
&\quad \frac{1}{\Gamma(\mu)\Gamma(1-\mu)} \int_0^1 \int_s^1 (1-s)^{-\mu} s^{\mu-1}(1-m)^{\nu-1} g(m, z_m) \mathrm{d}m \mathrm{d}s \\
&= \frac{1}{\Gamma(\mu)\Gamma(1-\mu)} \int_0^1 \int_0^1 (1-s)^{-\mu} s^{\mu-1}(1-m)^{\nu-1} g(m, z_m) \mathrm{d}m \mathrm{d}s - \\
&\quad \frac{1}{\Gamma(\mu)\Gamma(1-\mu)} \int_0^1 \int_0^s (1-s)^{-\mu}(s-m)^{\mu-1}(1-m)^{\nu-1} g(m, z_m) \mathrm{d}m \mathrm{d}s.
\end{aligned}
$$

By Fubini theorem,

$$
\begin{aligned}
{}^C D_{0+}^{\mu} z(1) =& \frac{1}{\Gamma(\mu)\Gamma(1-\mu)} \Big[\int_0^1 (1-s)^{-\mu} s^{\mu-1} \mathrm{d}s \Big] \Big[\int_0^1 (1-m)^{\nu-1} g(m, z_m) \mathrm{d}m \Big] - \\
& \frac{1}{\Gamma(\mu)\Gamma(1-\mu)} \int_0^1 \Big[\int_m^1 (1-s)^{-\mu}(s-m)^{\mu-1} \mathrm{d}s \Big] (1-m)^{\nu-1} g(m, z_m) \mathrm{d}m. \qquad (3.10)
\end{aligned}
$$

Using relation formula of Euler gamma function and beta [10, p. 26], we get

$$
B(\mu, \nu) = \int_0^1 (1-s)^{\mu-1} s^{\nu-1} \mathrm{d}s = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \qquad (3.11)
$$

and by change of variable $\alpha = \frac{s-m}{1-m}$, we have

$$
\int_m^1 (1-s)^{-\mu}(s-m)^{\mu-1} \mathrm{d}s = B(\mu, 1-\mu). \qquad (3.12)
$$

So, from (3.10)–(3.12), we get

$$
{}^C D_{0+}^{\mu} z(1) = 0.
$$

Next, it remains to prove that $z \in E$ satisfies (1.1). By using (2.3), (3.1) and (3.2),

$$
\begin{aligned}
{}^C D_{0+}^{\mu} z(t) =& \frac{1}{\Gamma(1-\mu)} \int_0^t (t-s)^{-\mu} z'(s) \mathrm{d}s \\
=& \frac{1}{\Gamma(\mu)\Gamma(1-\mu)} \int_0^t \int_0^s (t-s)^{-\mu}[s^{\mu-1} - (s-m)^{\mu-1}](1-m)^{\nu-1} g(m, z_m) \mathrm{d}m \mathrm{d}s + \\
& \frac{1}{\Gamma(\mu)\Gamma(1-\mu)} \int_0^t \int_s^1 (t-s)^{-\mu} s^{\mu-1}(1-m)^{\nu-1} g(m, z_m) \mathrm{d}m \mathrm{d}s \\
=& \frac{1}{\Gamma(\mu)\Gamma(1-\mu)} \int_0^t \int_0^1 (t-s)^{-\mu} s^{\mu-1}(1-m)^{\nu-1} g(m, z_m) \mathrm{d}m \mathrm{d}s - \\
& \frac{1}{\Gamma(\mu)\Gamma(1-\mu)} \int_0^t \int_0^s (t-s)^{-\mu}(s-m)^{\mu-1}(1-m)^{\nu-1} g(m, z_m) \mathrm{d}m \mathrm{d}s.
\end{aligned}
$$

Now, we use Fubini theorem to get

$$
\begin{aligned}
{}^C D_{0+}^{\mu} z(t) =& \frac{1}{\Gamma(\mu)\Gamma(1-\mu)} \Big[\int_0^t (t-s)^{-\mu} s^{\mu-1} \mathrm{d}s \Big] \Big[\int_0^1 (1-m)^{\nu-1} g(m, z_m) \mathrm{d}m \Big] - \\
& \frac{1}{\Gamma(\mu)\Gamma(1-\mu)} \int_0^t \Big[\int_m^t (t-s)^{-\mu}(s-m)^{\mu-1} \mathrm{d}s \Big] (1-m)^{\nu-1} g(m, z_m) \mathrm{d}m. \qquad (3.13)
\end{aligned}
$$

By change of variable $\alpha = \frac{s}{t}$, we obtain

$$
\int_0^t (t-s)^{-\mu} s^{\mu-1} \mathrm{d}s = B(\mu, 1-\mu) \qquad (3.14)
$$

and change of variable $\alpha = \frac{s-m}{t-m}$, we obtain

$$\int_m^t (t-s)^{-\mu}(s-m)^{\mu-1}\mathrm{d}s = B(\mu, 1-\mu). \tag{3.15}$$

So, from (3.13), (3.14), (3.15) and (3.11), we have

$${}^{C}D^{\mu}_{0+}z(t) = \int_0^1 (1-m)^{\nu-1}g(m, z_m)\mathrm{d}m - \int_0^t (1-m)^{\nu-1}g(m, z_m)\mathrm{d}m. \tag{3.16}$$

Now, applying right conformable derivative defined in (2.11) in both sides of (3.16), we get

$$D^{\nu}_{1-}(\,{}^{C}D^{\mu}_{0+}z)(t) = -\,(1-t)^{1-\nu}\frac{\partial}{\partial t}\Big[\int_0^1 (1-m)^{\nu-1}g(m, z_m)\mathrm{d}m - \int_0^t (1-m)^{\nu-1}g(m, z_m)\mathrm{d}m\Big].$$

Using Leibniz integral rule, we get

$$D^{\nu}_{1-}(\,{}^{C}D^{\mu}_{0+}z)(t) = (1-t)^{1-\nu}(1-t)^{\nu-1}g(t, z_t) = g(t, z_t).$$

Hence, the proof is completed. □

Lemma 3.2 *For all $s \in [0,1]$ and $t \in [0,1]$*

(1) $G(t,s) > 0$.

(2) $t^{\mu}G(1,s) \le G(t,s) \le G(1,s)$.

Proof (1) For all $t \in [0,1]$, using (3.2), we get

$$\frac{\partial G}{\partial t}(t,s) = \frac{1}{\Gamma\mu}\begin{cases}(t^{\mu-1} - (t-s)^{\mu-1})(1-s)^{\nu-1}, & \text{if } 0 \le s \le t \le 1,\\ (t^{\mu-1}(1-s)^{\nu-1}), & \text{if } 0 \le t \le s \le 1.\end{cases} \tag{3.17}$$

Clearly, it is seen that $\frac{\partial G}{\partial t}(t,s) \ge 0$ for all $s \in [0,1]$ and $t \in [0,1]$, which implies $G(t,s)$ is increasing with respect to $t \in [0,1]$. Therefore, for all $s \in [0,1]$ and $t \in [0,1]$, we have

$$G(t,s) \ge G(0,s) = \big[\frac{\beta}{(1-\beta)\Gamma(\mu+2)}(1-(1-s)^{\mu+1})\big](1-s)^{\nu-1} > 0.$$

(2) Now using increasing of Green's function $G(t,s)$ with respect to $t \in [0,1]$, we get for all $s \in [0,1]$ and $t \in [0,1]$,

$$G(t,s) \le G(1,s) = \frac{1}{\Gamma(\mu+1)}\big[\frac{\beta}{(1-\beta)\Gamma(\mu+1)}(1-(1-s)^{\mu+1}) + 1 - (1-s)^{\mu}\big](1-s)^{\nu-1}.$$

Using (3.2) yields

$$G(t,s) - t^{\mu}G(1,s) = \begin{cases}\frac{\beta(1-t^{\mu})(1-s)^{\nu-1}}{(1-\beta)\Gamma(\mu+2)}[1-(1-s)^{\mu+1}] + \frac{t^{\mu}(1-s)^{\nu-1}}{\Gamma(\mu+1)}[(1-s)^{\mu} - (1-\frac{s}{t})^{\mu}], & \text{if } s \le t,\\ \frac{\beta(1-t^{\mu})(1-s)^{\nu-1}}{(1-\beta)\Gamma(\mu+2)}[1-(1-s)^{\mu+1}] + \frac{t^{\mu}(1-s)^{\mu}(1-s)^{\nu-1}}{\Gamma(\mu+1)}, & \text{if } s \ge t,\end{cases}$$

$$G(t,s) - t^{\mu}G(1,s) \ge 0,$$

this implies that,

$$t^{\mu}G(1,s) \le G(t,s) \le G(1,s). \quad \square$$

4. Existence of positive solution

In this section, we show the existence of positive solution of problem (1.1)–(1.4).

Let $\Omega_q \subset E$ be bounded open set defined as

$$\Omega_q = \{z \in E; \|z\| < q, q > 0\}$$

and P be defined as

$$P = \{z \in E; z(t) \geq t^{\mu}\|z\|, t \in [0,1]\}.$$

Now, we define the operator $T : E \to E$ such that

$$Tz(t) = \begin{cases} \int_0^1 G(t,s)g(s,z_s)\mathrm{d}s, & t \in [0,1], \\ \psi(t), & t \in [-\tau, 0]. \end{cases} \tag{4.1}$$

Lemma 4.1 *There hold the following conclusions*

(1) $T(P) \subset P$.

(2) The operator $T : P \to P$ *is completely continuous.*

Proof (1) Let $z \in P$. Using Lemma 3.2, we get

$$\begin{aligned} Tz(t) = \int_0^1 G(t,s)g(s,z_s)\mathrm{d}s &\geq t^{\mu}\int_0^1 G(1,s)g(s,z_s)\mathrm{d}s \\ &\geq t^{\mu}\int_0^1 G(t,s)g(s,z_s)\mathrm{d}s, \quad t \in [0,1], \end{aligned}$$

and

$$Tz(t) \geq t^{\mu}\max_{t\in[0,1]}\int_0^1 G(t,s)g(s,z_s)\mathrm{d}s \geq t^{\mu}\|Tz\|.$$

Thus, $Tz \in P$.

(2) Let $\Omega_n \subset P$ be defined as $\Omega_n = \{z \in P; \|z\| < n, n > 0\}$ and define

$$L_n = \max_{t\in[0,1], z\in\Omega_n} g(s,z_s).$$

Then, for all $z \in \Omega_n$, we get

$$\|Tz(t)\| = \max_{t\in[0,1]}\int_0^1 G(t,s)g(s,z_s)\mathrm{d}s \leq L_n\int_0^1 G(1,s)\mathrm{d}s.$$

Therefore, $T(\Omega_n)$ is bounded in P. For every $z \in \Omega_n$, we get

$$\begin{aligned} |(Tz)'(t)| &= \Big|\int_0^1 \frac{\partial G}{\partial t}(t,s)g(s,z_s)\mathrm{d}s\Big| \\ &= \Big|\frac{1}{\Gamma\mu}\int_0^t (t^{\mu-1} - (t-s)^{\mu-1})(1-s)^{\nu-1}g(s,z_s)\mathrm{d}s + \\ &\quad \frac{1}{\Gamma\mu}\int_t^1 t^{\mu-1}(1-s)^{\nu-1}g(s,z_s)\mathrm{d}s\Big| \\ &= \Big|\frac{1}{\Gamma\mu}\int_0^1 t^{\mu-1}(1-s)^{\nu-1}g(s,z_s)\mathrm{d}s - \\ &\quad \frac{1}{\Gamma\mu}\int_0^t (t-s)^{\mu-1}(1-s)^{\nu-1}g(s,z_s)\mathrm{d}s\Big| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{L_n}{\Gamma\mu}\int_0^1 (1-s)^{\nu-1}\mathrm{d}s + \frac{L_n}{\Gamma\mu}\int_0^t (t-s)^{\mu-1}\mathrm{d}s \\
&\leq \frac{L_n}{\Gamma\mu}[\frac{1}{\nu}+\frac{1}{\mu}] \leq \frac{(\mu+\nu)L_n}{\nu\Gamma(\mu+1)}.
\end{aligned}
$$

Let $z \in \Omega_n$ and $t_1 < t_2$, $t_1, t_2 \in [-\tau, 1]$.

If $0 \leq t_1 < t_2 \leq 1$, then

$$
|(Tz)(t_2) - (Tz)(t_1)| = \Big|\int_{t_1}^{t_2} (Tz)'(s)\mathrm{d}s\Big| \leq \int_{t_1}^{t_2} |(Tz)'(s)|\mathrm{d}s \leq \frac{(\mu+\nu)L_n}{\nu\Gamma(\mu+1)}|t_2 - t_1|.
$$

If $-\tau \leq t_1 < t_2 \leq 0$, then

$$
|(Tz)(t_2) - (Tz)(t_1)| = |\psi(t_2) - \psi(t_1)|.
$$

If $-\tau \leq t_1 < 0 < t_2 \leq 1$, then

$$
\begin{aligned}
|(Tz)(t_2) - (Tz)(t_1)| &= |(Tz)(t_2) - (Tz)(0)| + |(Tz)(0) - (Tz)(t_1)| \\
&\leq \int_0^1 |G(t_2, s) - G(0, s)||g(s, z_s)|\mathrm{d}s + |\psi(0) - \psi(t_2)| \\
&\leq L_n||t_2 - t_1| + |\psi(0) - \psi(t_2)|.
\end{aligned}
$$

Therefore, $T(\Omega_n)$ is equicontinuous.

Now, using Arzelà-Ascoli theorem [11], we say that $T(\Omega_n)$ is relatively compact. Thus, $T : P \to P$ is completely continuous. $\square$

Next, we study the positive solution for problem (1.1)–(1.4). For this, some notations are presented below

$$
g^0 = \lim_{\|z\|\to 0} \max_{t\in[0,1]} \frac{g(t, z_t)}{\|z\|_{[-\tau,0]}}, \quad g^\infty = \lim_{||z||\to +\infty} \max_{t\in[0,1]} \frac{g(t, z_t)}{\|z\|_{[-\tau,0]}},
$$

$$
g_0 = \lim_{\|z\|\to 0} \min_{t\in[0,1]} \frac{g(t, z_t)}{\|z\|_{[-\tau,0]}}, \quad g_\infty = \lim_{||z||\to +\infty} \min_{t\in[0,1]} \frac{g(t, z_t)}{\|z\|_{[-\tau,0]}},
$$

$$
\lambda_1 = \int_0^1 G(1, s)\mathrm{d}s, \quad \lambda_2 = \sigma^\mu \int_\sigma^{1-\sigma} G(1, s)\mathrm{d}s,
$$

where $\sigma \in (0, \frac{1}{2})$ and the function G is defined in (3.2).

Theorem 4.2 *Suppose that any one of the three following conditions is satisfied, then problem (1.1)–(1.4) has at least one positive solution.*

(1) There exists $m_2 > m_1 > 0$, such that $\forall z \in [m_1, m_2]$, $\forall t \in [0, 1] : \frac{m_1}{\lambda_2} \leq g(t, z_t) \leq \frac{m_2}{\lambda_1}$.

(2) $\lambda_1 g^0 \leq \frac{2}{3}$ and $\lambda_2 g_\infty \geq 3$.

(3) $\lambda_2 g_0 \geq 3$ and $\lambda_1 g^\infty \leq \frac{3}{4}$.

Proof (1) Let $z \in P \cap \partial\Omega_{m_1}$, which implies that, $z \in P$ and $\|z\| = m_1$. Using Lemma 3.2, we get

$$
\|Tz(t)\| = \begin{cases} \max_{t\in[0,1]} \int_0^1 G(t, s)g(s, z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}
$$

$$
\begin{aligned}
&\geq \begin{cases} \int_0^1 G(t,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\geq \begin{cases} t^\mu \int_0^1 G(1,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\geq \begin{cases} \frac{m_1}{\lambda_2} t^\mu \int_0^1 G(1,s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\geq \begin{cases} \frac{m_1}{\lambda_2} t^\mu \int_\sigma^{1-\sigma} G(1,s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\geq \begin{cases} \frac{m_1}{\lambda_2} \left(\frac{t}{\sigma}\right)^\mu \sigma^\mu \int_\sigma^{1-\sigma} G(1,s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\geq \begin{cases} \frac{m_1}{\lambda_2} \lambda_2 \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\geq \begin{cases} m_1 \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
\therefore\ & ||Tz(t)|| \geq ||z||_{[-\tau,1]}
\end{aligned}
$$

For $z \in P \cap \partial\Omega_{m_2}$, which implies that, $z \in P$ and $\|z\| = m_2$, using Lemma 3.2, we get

$$
\begin{aligned}
\|Tz(t)\| &= \begin{cases} \max_{t\in[0,1]} \int_0^1 G(t,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\leq \begin{cases} \int_0^1 G(1,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\leq \begin{cases} \frac{m_2}{\lambda_1} \int_0^1 G(1,s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\leq \begin{cases} \frac{m_2}{\lambda_1} \lambda_1 \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\leq \begin{cases} m_2 \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
\therefore\ \|Tz(t)\| &\leq \|z\|_{[-\tau,1]},
\end{aligned}
$$

then by Theorem 2.9, T has an at least one fixed point in $z \in P \cap \overline{\Omega}_{m_2} \backslash \Omega_{m_1}$ with $m_1 \leq ||z|| \leq m_2$. This implies that problem (1.1)–(1.4) has at least one positive solution.

(2) Using definition of f^0, we may choose $\|\psi(t)\|_{[-\tau,0]} \leq m_1$ such that $g(t,z_t) \leq (f^0 + \delta)\|z\|$ for $0 < \|z\| \leq m_1$, where $\delta > 0$ satisfies $\lambda_1 \delta \leq \frac{1}{3}$. Let $z \in P \cap \partial\Omega_{m_1}$, which implies that, $z \in P$ and $\|z\| = m_1$. Using Lemma 3.2, we get

$$
\|Tz(t)\| = \begin{cases} \max_{t\in[0,1]} \int_0^1 G(t,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}
$$

$$
\begin{aligned}
&\le \begin{cases} \int_0^1 G(1,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\le \begin{cases} (f^0+\delta)\int_0^1 G(1,s)\|z\|\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\le \begin{cases} \lambda_1(f^0+\delta)\|z\| \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
\therefore \|Tz(t)\| &\le \|z\|_{[-\tau,1]}.
\end{aligned}
$$

By definition of f_∞, we may choose $m_2 \geq ||\psi(t)||_{[-\tau,0]}$ such that $g(t,z_t) \geq (f_\infty - \delta)||z||$ for $\|z\| \geq m_2$, where $\delta > 0$ satisfies $\lambda_2\delta \leq 2$. Let $z \in P \cap \partial\Omega_{m_2}$, which implies that, $z \in P$ and $\|z\| = m_2$. Using Lemma 3.2, we get

$$
\begin{aligned}
\|Tz(t)\| &= \begin{cases} \max_{t\in[0,1]}\int_0^1 G(t,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\ge \begin{cases} \int_0^1 G(t,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\ge \begin{cases} t^\mu\int_0^1 G(1,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\ge \begin{cases} t^\mu\int_\sigma^{1-\sigma} G(1,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\ge \begin{cases} t^\mu(f_\infty-\delta)\int_\sigma^{1-\sigma} G(1,s)\|z\|\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\ge \begin{cases} (\frac{t}{\sigma})^\mu(f_\infty-\delta)\|z\|\sigma^\mu\int_\sigma^{1-\sigma} G(1,s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\ge \begin{cases} \lambda_2(f_\infty-\delta)\|z\| \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
\therefore \|Tz(t)\| &\ge \|z\|_{[-\tau,1]},
\end{aligned}
$$

then by Theorem 2.9, operator T has at least one fixed point in $z \in P \cap \overline{\Omega}_{m_2} \backslash \Omega_{m_1}$ with $m_1 \leq \|z\| \leq m_2$. This implies that, the problem (1.1)–(1.4) has at least one positive solution.

(3) By definition of f_0, we may choose $m_2 \geq \|\psi(t)\|_{[-\tau,0]} \leq m_1$ such that $g(t,z_t) \geq (f_0-\delta)\|z\|$ for $0 < \|z\| \leq m_1$, where $\delta > 0$ satisfies $\lambda_2\delta \leq 2$. Let $z \in P \cap \partial\Omega_{m_1}$, which implies that $z \in P$ and $\|z\| = m_1$. Using Lemma 3.2, we get

$$
\begin{aligned}
\|Tz(t)\| &= \begin{cases} \max_{t\in[0,1]}\int_0^1 G(t,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}\\
&\ge \begin{cases} \int_0^1 G(t,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}
\end{aligned}
$$

$$\geq \begin{cases} t^{\mu}\int_0^1 G(1,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}$$

$$\geq \begin{cases} t^{\mu}\int_{\sigma}^{1-\sigma} G(1,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}$$

$$\geq \begin{cases} t^{\mu}(f_0-\delta)\int_{\sigma}^{1-\sigma} G(1,s)\|z\|\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}$$

$$\geq \begin{cases} (\frac{t}{\sigma})^{\mu}(f_0-\delta)\|z\|\sigma^{\mu}\int_{\sigma}^{1-\sigma} G(1,s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}$$

$$\geq \begin{cases} \lambda_2(f_0-\delta)\|z\| \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}$$

$$\therefore \|Tz(t)\| \geq \|z\|_{[-\tau,1]}.$$

Using definition of f^{∞}, we may choose $||\psi(t)||_{[-\tau,0]} \geq m_2$ such that $g(t,z_t) \leq (f^{\infty}+\delta)||z||$ for $\|z\| \geq m_2$, where $\delta > 0$ satisfies $\lambda_1\delta \leq \frac{1}{4}$. Let $z \in P \cap \partial\Omega_{m_2}$. This implies that $z \in P$ and $\|z\| = m_2$. Using Lemma 3.2, we get

$$\|Tz(t)\| = \begin{cases} \max_{t\in[0,1]}\int_0^1 G(t,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}$$

$$\leq \begin{cases} \int_0^1 G(1,s)g(s,z_s)\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}$$

$$\leq \begin{cases} (f^{\infty}+\delta)\int_0^1 G(1,s)\|z\|\mathrm{d}s \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}$$

$$\leq \begin{cases} \lambda_1(f^{\infty}+\delta)\|z\| \\ \|\psi(t)\|_{[-\tau,0]} \end{cases}$$

$$\therefore \|Tz(t)\| \leq \|z\|_{[-\tau,1]},$$

then by Theorem 2.9, operator T has at least one fixed point in $z \in P \cap \overline{\Omega}_{m_2} \backslash \Omega_{m_1}$ with $m_1 \leq \|z\| \leq m_2$. This implies that problem (1.1)–(1.4) has at least one positive solution. □

5. Uniqueness of positive solution

In this section, we will discuss the uniqueness of positive solution of problem (1.1)–(1.4).

Theorem 5.1 *Suppose there exists $K > 0$ such that*

$$\|g(s,x_s) - g(s,y_s)\| \leq K\|x_s - y_s\| \quad \textit{for all} \quad t \in [0,1] \quad \textit{and every} \quad x, y \in E. \tag{5.1}$$

If

$$0 < K\lambda < 1, \tag{5.2}$$

then, the problem (1.1)–(1.4) has exactly one positive solution in E.

Proof Let $x, y : [-\tau, 1] \to \mathbb{R}_+$ and $x \neq y$ be two solutions of problem (1.1)–(1.4). Using (5.1) and Lemma 3.1, we have

$$\begin{aligned}\|Tx - Ty\| &\leq \int_0^1 G(1,s)\|g(s,x_s) - g(s,y_s)\|\mathrm{d}s \\ &\leq K\int_0^1 G(1,s)\|x_s - y_s)\|\mathrm{d}s \leq K\lambda\|x_s - y_s)\| \\ \therefore \|Tx - Ty\| &\leq K\lambda\|x_s - y_s)\|.\end{aligned}$$

As $0 < K\lambda < 1$, this implies that operator T is a strict contraction. Then, from Theorem 2.8, the given problem (1.1)–(1.4) has exactly one positive solution in E. □

6. Conclusion

In this paper, we obtained existence of positive solution of mixed non-linear fractional delay differential equations with integral boundary conditions using Guo-Krasnoseleskii's fixed point theorem and uniqueness of positive solution by using Banach contraction principle. This work may provide a new way for the research to get positive solution of Fractional differential equations with delay.

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