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Two New Mock Theta Double Sums

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Abstract Recently, Zhang and Li presented five mock theta functions as q-hypergeometric double sums by using a Bailey pair. In this paper, employing the same Bailey pair, we further establish two new mock theta double sums in terms of Appell-Lerch sums and theta series. Indeed, identities between a new mock theta function and classical mock theta functions are obtained.

Keywords mock theta double sums; Bailey pair; Appell-Lerch sums; theta series

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1. Introduction

The mock theta functions were introduced by Ramanujan in his letter to Hardy [1], in which he gave seventeen mock theta functions of order three, five and seven. Since then, mock theta functions have attracted the attention of lots of mathematicians. For instance, Andrews and Hickerson [2] provided proofs of 11 sixth order mock theta function identities found in Ramanujan's "lost" notebook [1] using the Bailey pairs. These functions include

$$\phi(q)=\sum_{n\geq 0}\frac{(-1)^nq^{n^2}(q;q^2)_n}{(-q)_{2n}},\quad \sigma(q)=\sum_{n\geq 0}\frac{q^{\binom{n+2}{2}}(-q)_n}{(q;q^2)_{n+1}},$$
$$\mu(q)=\sum_{n\geq 0}{}^{*}\frac{(-1)^n(q;q^2)_n}{(-q)_n},\qquad \gamma(q)=\sum_{n\geq 0}\frac{q^{n^2}(q)_n}{(q^3;q^3)_n}.$$

By means of the Bailey pairs, the following two mock theta functions of the sixth order are defined by Berndt and Chan [3].

$$\phi_-(q)=\sum_{n\geq 1}\frac{q^n(-q)_{2n-1}}{(q;q^2)_n},\quad \psi_-(q)=\sum_{n\geq 1}\frac{q^n(-q)_{2n-2}}{(q;q^2)_n}.$$

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In [4], Lovejoy obtained families of q-hypergeometric mock theta multisums by constructing classes of Bailey pairs. Other mock theta functions have been found by many authors using a variety of methods [5–13].

We also note that the Bailey pairs play an important role in the study of mock theta double sums. Lovejoy and Osburn [14, 15] constructed a number of new q-hypergeometric double sums which are mock theta functions. Subsequently, Gu and Liu [16] derived many families of mock theta functions by constructing generalized Bailey pairs with more parameters. Recently, Zhang and Li [17] deduced five mock theta double sums by establishing the following Bailey pair relative to 1,

$$\alpha_{2n} = -(1-q^{4n})q^{4n^2-2n+\frac{1}{2}}\sum_{j=-n}^{n-1} q^{-2j^2-j}, \tag{1.1}$$

$$\alpha_{2n+1} = (1-q^{4n+2})q^{4n^2+2n+\frac{1}{2}}\sum_{j=-n}^{n} q^{-2j^2+j} \tag{1.2}$$

and

$$\beta_n = -\frac{1}{(\sqrt{q})_n}\sum_{j=1}^{n}\frac{(-1)^j q^{\binom{j}{2}+\frac{1}{2}}}{(q)_{n-j}(q)_{j-1}(1+q^{j-\frac{1}{2}})}. \tag{1.3}$$

In [18], Patkowski discussed and offered some double sum q-series, with new relationships among classical mock theta functions.

In this note, we obtain two new mock theta double sums by utilizing the above Bailey pair (1.1)–(1.3).

Theorem 1.1 *The following mock theta functions are true.*

$$\begin{aligned} M_1(q) :=& \sum_{n\geq 1}\sum_{n\geq j\geq 1}\frac{(-1)^{n+j}(-\sqrt{q})_n q^{\binom{j}{2}}}{(q)_{n-j}(q)_{j-1}(1+q^{j-\frac{1}{2}})} \\ =& \frac{(q;q^2)_\infty}{2(q^2;q^2)_\infty} f_{1,3,1}(-q^2,-q,q) \\ =& q^{-1}m(-1,q^8,q) + m(-q^4,q^8,q^{-1}) + \frac{J_{2,4}J_{8,16}J_{2,8}J_{4,16}}{\overline{J}_{1,1}\overline{J}_{1,8}\overline{J}_{3,8}}, \end{aligned} \tag{1.4}$$

$$\begin{aligned} M_2(q) :=& \sum_{n\geq 1}\sum_{n\geq j\geq 1}\frac{(-1)^j(-1)_n q^{\binom{n+1}{2}+\binom{j}{2}}}{(\sqrt{q})_n(q)_{n-j}(q)_{j-1}(1+q^{j-\frac{1}{2}})} \\ =& -\frac{2(-q)_\infty}{(q)_\infty} f_{1,2,1}(q^4,q^3,q^2) \\ =& 2m(q^2,q^6,q^{-1}). \end{aligned} \tag{1.5}$$

Meanwhile, identities between the mock theta double sum $M_2(q)$ and the classical mock theta functions are established.

Theorem 1.2 *We have*

$$M_2(q) = \phi(q^2) + \frac{2J_6^3\overline{J}_{1,6}^2}{J_{1,6}^2\overline{J}_{0,6}\overline{J}_{2,6}},$$

$$\begin{aligned}
M_2(q) =& -2\big(\sigma(q) - \frac{J_2^2 J_6^3}{J_{1,6}^3 J_{3,6}}\big),\\
M_2(q) =& \mu(q) + \frac{J_{1,2}\overline{J}_{1,3}}{2\overline{J}_{1,4}} + \frac{2J_6^3\overline{J}_{1,6}^2}{J_{1,6}^2\overline{J}_{0,6}\overline{J}_{2,6}},\\
M_2(q) =& \frac{2}{3}\big(\gamma(q^2) - \frac{J_{2,4}^2}{\overline{J}_{2,6}} + \frac{3J_6^3\overline{J}_{3,6}^2}{J_{1,6}^2\overline{J}_{2,6}^2}\big),\\
M_2(q) =& -2\big(\phi_-(q^2) + q^2\frac{\overline{J}_{6,24}^3}{J_2\overline{J}_{2,8}} - \frac{J_6^3 J_{3,6}^2}{J_2^2 J_{1,6}^2}\big).
\end{aligned}$$

The rest of the paper is arranged as follows. In Section 2, we collect some useful tools and results on q-series and mock theta functions. We prove Theorems 1.1 and 1.2 in Sections 3 and 4, respectively.

2. Preliminaries

Here and throughout the paper, we adopt the standard notation for q-shifted factorials in [19]:

$$(a;q^k)_n = (1-a)(1-aq^k)(1-aq^{2k})\cdots(1-aq^{(n-1)k}) \quad (a;q^k)_\infty = \prod_{m=0}^{\infty}(1-aq^{mk}).$$

When $k=1$ we usually write $(a)_n$ and $(a)_\infty$ instead of $(a;q)_n$ and $(a;q)_\infty$, respectively.

Firstly, we briefly review the Bailey pair and Bailey lemma. Let $\alpha = (\alpha_0, \alpha_1, \ldots)$ and $\beta = (\beta_0, \beta_1, \ldots)$. The pair of sequences (α_n, β_n) is called a Bailey pair with respect to a if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}},$$

for all $n \geq 0$. Bailey's lemma says that if (α_n, β_n) is a Bailey pair relative to a, then

$$\sum_{n=0}^{\infty}(\rho)_n(\sigma)_n(aq/\rho\sigma)^n\beta_n = \frac{(aq/\rho, aq/\sigma)_\infty}{(aq, aq/\rho\sigma)_\infty}\sum_{n=0}^{\infty}\frac{(\rho)_n(\sigma)_n(aq/\rho\sigma)^n}{(aq/\rho)_n(aq/\sigma)_n}\alpha_n. \tag{2.1}$$

The classical theta series is defined by

$$j(x;q) := \sum_{r=-\infty}^{\infty}(-1)^r x^r q^{\binom{r}{2}} = (x, q/x, q)_\infty.$$

Define $J_{a,m} := j(q^a;q^m)$, $\overline{J}_{a,m} := j(-q^a;q^m)$ and $J_m := J_{m,3m}$, where a and m are integers with m positive.

From the definition of $j(x;q)$, we have

$$j(x;q) = j(q/x;q) = -xj(x^{-1};q) \tag{2.2}$$

and

$$j(q^n x;q) = (-1)^n q^{-\binom{n}{2}} x^{-n} j(x;q), \tag{2.3}$$

where $n \in \mathbb{Z}$.

Let $x, z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with neither z nor xz an integral power of q. The Appell-Lerch sum is defined by

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_r \frac{(-1)^r z^r q^{\binom{r}{2}}}{1-q^{r-1}xz}.$$

The following property [20, Theorem 3.3] of the Appell-Lerch sums $m(x,q,z)$ will be used: For generic $x, z, z_0 \in \mathbb{C}^*$,

$$m(x,q,z) = m(x,q,z_0) + \frac{z_0 J_1^3 j(z/z_0;q) j(xzz_0;q)}{j(z_0;q) j(z;q) j(xz_0;q) j(xz;q)}. \tag{2.4}$$

The Hecke-type double sums are defined by

$$f_{a,b,c}(x,y,q) := \Big(\sum_{r,s \geq 0} - \sum_{r,s<0} \Big) (-1)^{r+s} x^r y^s q^{a\binom{r}{2} + brs + c\binom{s}{2}},$$

which is an indefinite theta series when $ac < b^2$. Here we assume $a, c > 0$.

In [20], in order to convert from the Hecke-type double sums to Appell-Lerch sums, Hickerson and Mortenson defined

$$\begin{aligned} g_{a,b,c}(x,y,q,z_1,z_0) := & \sum_{t=0}^{a-1} (-y)^t q^{c\binom{t}{2}} j(q^{bt}x;q^a) m(-q^{a\binom{b+1}{2} - c\binom{a+1}{2} - t(b^2-ac)} \frac{(-y)^a}{(-x)^b}, q^{a(b^2-ac)}, z_0) + \\ & \sum_{t=0}^{c-1} (-x)^t q^{a\binom{t}{2}} j(q^{bt}y;q^c) m(-q^{c\binom{b+1}{2} - a\binom{c+1}{2} - t(b^2-ac)} \frac{(-x)^c}{(-y)^b}, q^{c(b^2-ac)}, z_1). \end{aligned} \tag{2.5}$$

The following term "generic" means that the parameters do not cause poles in the Appell-Lerch sums or in the quotients of theta functions.

Lemma 2.1 ([20, Theorem 1.6]) *Let n be a positive integer. For generic $x,\ y \in \mathbb{C}^*$,*

$$f_{n,n+1,n}(x,y,q) = g_{n,n+1,n}(x,y,q,y^n/x^n,x^n/y^n).$$

Lemma 2.2 ([20, Theorem 1.9]) *Let n be a positive odd integer. For generic $x, y \in \mathbb{C}^*$,*

$$f_{n,n+2,n}(x,y,q) = g_{n,n+2,n}(x,y,q,y^n/x^n,x^n/y^n) - \Theta_{n,2}(x,y,q),$$

where

$$\Theta_{n,2}(x,y,q) := \frac{y^{(n+1)/2} J_{2n,4n} J_{4(n+1),8(n+1)} j(y/x;q^{4(n+1)}) j(q^{n+2}xy;q^{4(n+1)}) j(q^{2n}/x^2y^2;q^{8(n+1)})}{q^{(n^2-3)/2} x^{(n-3)/2} j(y^n/x^n;q^{4n(n+1)}) j(-q^{n+2}x^2;q^{4(n+1)}) j(-q^{n+2}y^2;q^{4(n+1)})}.$$

3. Proofs of Theorems 1.1 and 1.2

We first prove Theorem 1.1.

Proof For (1.4), substituting the Bailey pair (1.1)–(1.3) into Bailey's lemma (2.1) with $\rho = \sqrt{q}$, $\sigma = -\sqrt{q}$, we have

$$-2q^{\frac{1}{2}} M_1(q) = 2\sum_{n\geq 0} (-1)^n (q;q^2)_n \beta_n = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n \geq 0} (-1)^n \alpha_n$$

$$
\begin{aligned}
&=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\Big(\sum_{n\geq 0}\alpha_{2n}-\sum_{n\geq 0}\alpha_{2n+1}\Big)\\
&=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\Big(-\sum_{n\geq 0}q^{4n^2-2n+\frac{1}{2}}\sum_{j=-n}^{n-1}q^{-2j^2-j}+\sum_{n\geq 0}q^{4n^2+2n+\frac{1}{2}}\sum_{j=-n}^{n-1}q^{-2j^2-j}-\\
&\quad\sum_{n\geq 0}q^{4n^2+2n+\frac{1}{2}}\sum_{j=-n}^{n}q^{-2j^2+j}+\sum_{n\geq 0}q^{4n^2+6n+\frac{5}{2}}\sum_{j=-n}^{n}q^{-2j^2+j}\Big).
\end{aligned}
$$

On the right-hand side of the above identity, replacing n with $-n$ in the second sum and n with $-n-1$ in the fourth sum and letting $n=(r+s+1)/2$, $j=(r-s-1)/2$ in the first two sums and $n=(r+s)/2$, $j=(r-s)/2$ in the latter two sums, we arrive at

$$
\begin{aligned}
-2q^{\frac{1}{2}}M_1(q)=&\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\Big\{-\Big(\sum_{\substack{r,\,s\geq 0\\ r\not\equiv s \,(\mathrm{mod}\, 2)}}-\sum_{\substack{r,\,s<0\\ r\not\equiv s\,(\mathrm{mod}\,2)}}\Big)q^{\frac{1}{2}r^2+3rs+\frac{1}{2}s^2+\frac{3}{2}r+\frac{1}{2}s+\frac{1}{2}}-\\
&\Big(\sum_{\substack{r,\,s\geq 0\\ r\equiv s\,(\mathrm{mod}\,2)}}-\sum_{\substack{r,\,s<0\\ r\equiv s\,(\mathrm{mod}\,2)}}\Big)q^{\frac{1}{2}r^2+3rs+\frac{1}{2}s^2+\frac{3}{2}r+\frac{1}{2}s+\frac{1}{2}}\Big\}\\
=&-\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\Big(\sum_{r,s\geq 0}-\sum_{r,s<0}\Big)q^{\frac{1}{2}r^2+3rs+\frac{1}{2}s^2+\frac{3}{2}r+\frac{1}{2}s+\frac{1}{2}}\\
=&-\frac{q^{\frac{1}{2}}(q;q^2)_\infty}{(q^2;q^2)_\infty}f_{1,3,1}(-q^2,-q,q).
\end{aligned}
$$

With the help of Lemma 2.2, (2.2), (2.3) and (2.5), we deduce

$$f_{1,3,1}(-q^2,-q,q)=q^{-1}j(-q;q)m(-1,q^8,q)+j(-q;q)m(-q^4,q^8,q^{-1})-\Theta_{1,2}(-q^2,-q,q),$$

so we have

$$M_1(q)=q^{-1}m(-1,q^8,q)+m(-q^4,q^8,q^{-1})+\frac{J_{2,4}J_{8,16}J_{2,8}J_{4,16}}{\overline{J}_{1,1}\overline{J}_{1,8}\overline{J}_{3,8}}.$$

For (1.5), we take the Bailey pair (1.1)–(1.3) into Bailey's lemma (2.1) with $\rho=-1$, $\sigma\to\infty$ to give

$$
\begin{aligned}
-q^{\frac{1}{2}}M_2(q)=&\sum_{n\geq 0}(-1)_n q^{\binom{n+1}{2}}\beta_n=\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 0}\frac{2q^{\binom{n+1}{2}}}{1+q^n}\alpha_n\\
=&\frac{2(-q)_\infty}{(q)_\infty}\Big(\sum_{n\geq 0}\frac{q^{2n^2+n}}{1+q^{2n}}\alpha_{2n}+\sum_{n\geq 0}\frac{q^{2n^2+3n+1}}{1+q^{2n+1}}\alpha_{2n+1}\Big)\\
=&\frac{2(-q)_\infty}{(q)_\infty}\Bigg(-\sum_{n\geq 0}q^{6n^2-n+\frac{1}{2}}\sum_{j=-n}^{n-1}q^{-2j^2-j}+\sum_{n\geq 0}q^{6n^2+n+\frac{1}{2}}\sum_{j=-n}^{n-1}q^{-2j^2-j}+\\
&\sum_{n\geq 0}q^{6n^2+5n+\frac{3}{2}}\sum_{j=-n}^{n}q^{-2j^2+j}-\sum_{n\geq 0}q^{6n^2+7n+\frac{5}{2}}\sum_{j=-n}^{n}q^{-2j^2+j}\Big).
\end{aligned}
$$

On the right-hand side of the above equation, replacing n with $-n$ in the second sum and n with $-n-1$ in the fourth sum and setting $n=(r+s+1)/2$, $j=(r-s-1)/2$ in the first two sums and $n=(r+s)/2$, $j=(r-s)/2$ in the latter two sums, we get

$$\begin{aligned}-q^{\frac{1}{2}}M_2(q)=&\frac{2(-q)_\infty}{(q)_\infty}\Big\{-\Big(\sum_{\substack{r,\,s\geq 0\\ r\not\equiv s\,(\mathrm{mod}\,2)}}-\sum_{\substack{r,\,s<0\\ r\not\equiv s\,(\mathrm{mod}\,2)}}\Big)q^{r^2+4rs+s^2+3r+2s+\frac{3}{2}}+\\
&\Big(\sum_{\substack{r,\,s\geq 0\\ r\equiv s\,(\mathrm{mod}\,2)}}-\sum_{\substack{r,\,s<0\\ r\equiv s\,(\mathrm{mod}\,2)}}\Big)q^{r^2+4rs+s^2+3r+2s+\frac{3}{2}}\Big\}\\
=&\frac{2(-q)_\infty}{(q)_\infty}\Big(\sum_{r,s\geq 0}-\sum_{r,s<0}\Big)(-1)^{r+s}q^{r^2+4rs+s^2+3r+2s+\frac{3}{2}}\\
=&\frac{2q^{\frac{3}{2}}(-q)_\infty}{(q)_\infty}f_{1,2,1}(q^4,q^3,q^2).\end{aligned}$$

In view of Lemma 2.1, (2.3) and (2.5), we have

$$f_{1,2,1}(q^4,q^3,q^2)=-q^{-1}j(q;q^2)m(q^2,q^6,q^{-1}),$$

so we arrive at

$$M_2(q)=2m(q^2,q^6,q^{-1}).$$

This completes the proof. □

Next, we prove Theorem 1.2.

Proof In [20], Hickerson and Mortenson expressed classical mock theta functions in terms of the Appell–Lerch sums and theta series. Namely,

$$\begin{aligned}\phi(q)&=2m(q,q^3,-1),\\
\sigma(q)&=-m(q^2,q^6,q),\\
\mu(q)&=2m(q^2,q^6,-1)-\frac{J_{1,2}\overline{J}_{1,3}}{2\overline{J}_{1,4}},\\
\gamma(q)&=3m(q,q^3,-q)+\frac{J_{1,2}^2}{\overline{J}_{1,3}},\\
\phi_-(q)&=-m(q,q^3,q)-q\frac{\overline{J}_{3,12}^3}{J_1\overline{J}_{1,4}}.\end{aligned}\tag{3.1}$$

In view of (2.2) and (2.4), we get

$$m(q^2,q^6,q^{-1})=m(q^2,q^6,-1)+\frac{J_6^3\overline{J}_{1,6}^2}{\overline{J}_{0,6}\overline{J}_{2,6}J_{1,6}^2}.$$

Combining (1.5) with (3.1) yields the first equality of Theorem 1.2. The proofs of the remaining identities are similar, so we omit them. □

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