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Numerical Range of the Linear Combination of Volterra Operator and Its Adjoint

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Abstract The classical Volterra operator V and its adjoint operator V^* play important roles in the complex space $L^2[0,1]$. As to the properties of linear combination of V and V^*, we present the equivalent condition ensuring $z_1V+z_2V^*(z_1,z_2\in\mathbb{C})$ satisfies the accretive property. Then an accurate representation of the numerical range of $(u+iv)I+mV+nV^*$ $(u,v,m,n\in\mathbb{R},m+n\geq 0)$ is described in this paper.

Keywords Volterra operator; the adjoint operator; numerical range

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1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space equipped with the inner product $(\cdot,\cdot)$, which induces the norm $\|\cdot\|$. Denote by $\mathcal{B}(\mathcal{H})$ the Banach algebra of bounded linear operators acting on $\mathcal{H}$. Let $A\in\mathcal{B}(\mathcal{H})$. The operator norm is defined by

$$\|A\|=\sup_{\|x\|=1}\|Ax\|.$$

Given $A\in\mathcal{B}(\mathcal{H})$, the spectrum of an operator A is defined by

$$\sigma(A)=\{\lambda\in\mathbb{C}:A-\lambda I\ \text{ is not invertible}\},$$

which is a non-empty compact subset of the complex plane. An important method of bounding the spectrum $\sigma(A)$ is by the numerical range of A, which is defined as

$$W(A)=\{(Ax,x),\ x\in\mathcal{H},\ \|x\|=1\}.$$

$W(A)$ has several good properties, such as

$$W(\alpha I+\beta A)=\alpha+\beta W(A)\ \text{for}\ \alpha,\beta\in\mathbb{C},\ \text{where}\ I\ \text{is the identity operator,}\tag{1.1}$$

$$W(A^*)=\{\overline{\lambda},\lambda\in W(A)\}.\tag{1.2}$$

It also holds $W(U^*AU)=W(A)$ for any unitary $U\in\mathcal{B}(\mathcal{H})$. It has been proved that the spectrum of an operator is contained in the closure of its numerical range [1, Theorem 1.2-1]

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and the numerical range of an operator is convex (The Toeplitz-Hausdorff theorem). And an operator A is self-adjoint iff $W(A)$ is real [1, Theorem 1.2-2].

Here we take A as the Volterra operator $V:\ L^2[0,1] \to L^2[0,1]$ defined by

$$(Vf)(x) = \int_0^x f(t)\mathrm{d}t, \quad f \in L^2[0,1],$$

which plays an important role in developing operator theory in Hilbert spaces. It is well-known that the Volterra operator is a compact universal quasinilpotent operator [2, Theorem 1]. Due to its excellent properties, many scholars have studied Volterra operator on various spaces and formulated many interesting results [3–5]. It has been proved that the norm of V is $2/\pi$ (see [6, Problem 188]). After that Lyubich and Tsedenbayar obtained an explicit formula for the norm $\|I + bV\|$ with $b \in \mathbb{C}$ (see [7]). There are also some interesting papers [8–11] that pertain to the characterizations about invariant subspaces of Volterra operator and its dynamical properties.

Recently, Khadkhuu calculated the numerical range and the numerical radius of V and described the envelope of its numerical range in [12]. Thus the numerical range $W(V^*)$ of the adjoint operator of V can be deduced by (1.2), where V^* is

$$(V^*f)(x) = \int_x^1 f(t)\mathrm{d}t, \quad f \in L^2[0,1].$$

Following this line, many authors studied the linear combination of V, V^* and the identity operator I (see [7,13,14]). In this article, we start with the accretive property of $z_1V + z_2V^*$ ($z_1, z_2 \in \mathbb{C}$) in Section 2. And then Section 3 is devoted to characterizing the numerical range of $(u + iv)I + mV + nV^*$ with $u, v, m, n \in \mathbb{R}$ and $m + n \geq 0$.

2. Accretive property

In this section, we explore the equivalent condition for accretive operator $z_1V + z_2V^*$ ($z_1, z_2 \in \mathbb{C}$). Recall $A \in \mathcal{B}(\mathcal{H})$ is accretive if $\mathrm{Re}\, A = (A + A^*)/2 \geq 0$. In [12], Khadkhuu proved zV ($z \in \mathbb{C}$) is accretive iff $\mathrm{Re}\ z \geq 0$ and $\mathrm{Im} z = 0$. As an extension, we explore the accretive property of $z_1V + z_2V^*$ ($z_1, z_2 \in \mathbb{C}$) and formulate a scope for the numerical range of $W(z_1V + z_2V^*)$.

Proposition 2.1 *The operator $z_1V + z_2V^*$ ($z_1, z_2 \in \mathbb{C}$) is accretive on $L^2[0,1]$ if and only if $\mathrm{Re}\, z_1 + \mathrm{Re}\, z_2 \geq 0$ and $\mathrm{Im}\, z_1 = \mathrm{Im}\, z_2$. And then*

$$W(z_1V + z_2V^*) \subseteq \{z : 0 \leq \mathrm{Re}\, z \leq \frac{1}{2}(\mathrm{Re}\, z_1 + \mathrm{Re}\, z_2)\}. \tag{2.1}$$

Proof For convenience of writing, we denote $z := z_1 + \overline{z_2}$.

Necessity. Let $z_1V + z_2V^*$ ($z_1, z_2 \in \mathbb{C}$) be accretive. We have

$$\frac{(z_1V + z_2V^*) + (z_1V + z_2V^*)^*}{2} = \frac{(z_1 + \overline{z_2})V + (\overline{z_1} + z_2)V^*}{2} \geq 0,$$

that is,

$$(\frac{zV + \overline{z}V^*}{2} f, f) \geq 0 \ \text{ for all } \ f \in L^2[0,1]. \tag{2.2}$$

Putting $f_k(x)=e^{ik\pi x}$ into (2.2), we obtain

$$(\frac{zV+\overline{z}V^*}{2}f_k,f_k)=\frac{2\operatorname{Im}z}{k\pi}+\frac{2(1-(-1)^k)\operatorname{Re}z}{k^2\pi^2}\geq 0,\quad k\in\mathbb{Z}\setminus\{0\}.$$

So $\operatorname{Re}z\geq 0$ and $\operatorname{Im}z=0$, that is, $\operatorname{Re}z_1+\operatorname{Re}z_2\geq 0$ and $\operatorname{Im}z_1=\operatorname{Im}z_2$.

Sufficiency. Denote

$$(Pf)(x):=((V+V^*)f)(x)=\int_0^1 f(t)\mathrm{d}t.$$

Then it holds that

$$\begin{aligned}((zV+\overline{z}V^*)f,f)&=((zP+(\overline{z}-z)V^*)f,f)=z(Pf,f)+(\overline{z}-z)(V^*f,f)\\&=z\Big|\int_0^1 f(t)\mathrm{d}t\Big|^2-2(\operatorname{Im}z_1-\operatorname{Im}z_2)i(V^*f,f)\\&=z\Big|\int_0^1 f(t)\mathrm{d}t\Big|^2\geq 0,\end{aligned}\tag{2.3}$$

which means $z_1V+z_2V^*$ is accretive. For $f\in L^2[0,1]$ with $\|f\|=1$, (2.3) implies (2.1) is true. The proof is completed. □

3. Main results

In this section, we first characterize the numerical range of $mV+nV^*$ with $m,n\in\mathbb{R},m+n\geq 0$. Recall a theorem about the numerical range of $T\in\mathcal{B}(\mathcal{H})$.

Theorem 3.1 ([15, Theorem 9.3-10]) *If $T\in\mathcal{B}(\mathcal{H})$ and $\theta\in[-\pi,\pi]$, put $\lambda_\theta=\max\sigma(B_\theta)$, where $B_\theta=\frac{1}{2}(e^{-i\theta}T+e^{i\theta}T^*)=B_\theta^*$. Then*

$$\overline{W(T)}=\bigcap_{\theta\in[-\pi,\pi]}H_\theta,$$

where the half-space H_θ is defined by

$$H_\theta=\{z\in\mathbb{C}:\operatorname{Re}(e^{-i\theta}z)\leq\lambda_\theta\}.$$

Using Theorem 3.1, Khadkhuu obtained the numerical range of V.

Proposition 3.2 ([12, Proposition 2]) *The numerical range of V is the set lying between the curves*

$$\frac{1-\cos\varphi}{\varphi^2}\pm i\frac{\varphi-\sin\varphi}{\varphi^2},\quad\varphi\in[0,2\pi].$$

Moreover, Khadkhuu further gave the numerical range of some special operators.

Lemma 3.3 ([12, Proposition 3]) *Let V be the Volterra operator on $L^2[0,1]$.*

(i) $W(\operatorname{Re}V)=[0,\frac{1}{2}]$, where $\operatorname{Re}V=(V+V^*)/2$.

(ii) $W(\operatorname{Im}V)=[-\frac{1}{\pi},\frac{1}{\pi}]$, where $\operatorname{Im}V=(V-V^*)/2$.

Based on Proposition 3.2 and the linear properties (1.1) and (1.2), it is easy to obtain the following statements.

Remark 3.4 (i) The numerical range of $(u+iv)I+mV$ $(u,v,m\in\mathbb{R})$ is the set lying between the curves

$$u+\frac{m(1-\cos\varphi)}{\varphi^2}+i(v\pm\frac{m(\varphi-\sin\varphi)}{\varphi^2}),\quad \varphi\in[0,2\pi].$$

(ii) The numerical range of $((u+iv)I+mV)^*$ $(u,v,m\in\mathbb{R})$ is the set lying between the curves

$$u+\frac{m(1-\cos\varphi)}{\varphi^2}+i(-v\pm\frac{m(\varphi-\sin\varphi)}{\varphi^2}),\quad \varphi\in[0,2\pi].$$

Next we consider the numerical range of $mV+nV^*$ for the special cases $m=\pm n$.

Proposition 3.5 *On* $L^2[0,1]$, *we have*

(i) $W(m(V+V^*))=[0,m]$ *with* $m\geq 0$.

(ii) $W(m(V-V^*))=[-\frac{2|m|i}{\pi},\frac{2|m|i}{\pi}]$.

Proof (i) Denote $A:=m(V+V^*)$. By Eq. (2.1), we have

$$W(A)\subseteq\{z:0\leq\operatorname{Re}z\leq m\}.$$

By Lemma 3.3 and linear property (1.1), we obtain

$$(Af,f)=(m(V+V^*)f,f)=2m\cdot\operatorname{Re}(Vf,f)$$

and then $W(A)=[0,m]$.

(ii) Denote $B:=m(V-V^*)$. It follows that

$$(Bf,f)=(m(V-V^*)f,f)=2mi\cdot\operatorname{Im}(Vf,f)$$

and then Lemma 3.3 ensures

$$W(m(V-V^*))=[-\frac{2|m|i}{\pi},\frac{2|m|i}{\pi}].\quad\square$$

In the sequel, we make sure $mV+nV^*$ is accretive. Then (2.1) entails

$$W(mV+nV^*)\subseteq\{z:0\leq\operatorname{Re}z\leq\frac{1}{2}(m+n)\}.$$

Further we calculate $W(mV+nV^*)$ with $m\neq\pm n$ in next theorem.

Theorem 3.6 *Let* $m,n\in\mathbb{R}$ *such that* $m+n\geq 0$ *and* $m\neq\pm n$. *The numerical range of* $mV+nV^*$ *is the set lying between the curves*

$$\frac{m+n}{{\mu_\theta}^2}(1-\cos\mu_\theta)\pm i\frac{n-m}{{\mu_\theta}^2}(\mu_\theta-\sin\mu_\theta),\quad\theta\in[-\pi,\pi],$$

where

$$\mu_\theta=\begin{cases}\arctan\tau_\theta,\quad 2\theta\in(-2\theta_2,-\pi)\cup(-2\theta_1,0)\cup(0,2\theta_1)\cup(\pi,2\theta_2),\\ \arctan\tau_\theta+\pi,\quad 2\theta\in[-2\pi-2\theta_2)\cup[-\pi,-2\theta_1)\cup\{0\}\cup(2\theta_1,\pi]\cup(2\theta_2,2\pi],\\ \frac{\pi}{2},\quad 2\theta=\pm 2\theta_i,\ i=1,2,\end{cases}\tag{3.1}$$

with $(m^2+n^2)\cos(2\theta_i)+2mn=0$, $\theta_i\in(0,\pi)$ *and*

$$\tau_\theta:=\frac{(n^2-m^2)\sin(2\theta)}{(m^2+n^2)\cos(2\theta)+2mn},\quad\theta\neq\theta_i\tag{3.2}$$

for $i = 1, 2$.

Proof Denote $A := mV + nV^*$ with $m + n \geq 0$, $m \neq \pm n$. We apply Theorem 3.1 to calculate the envelope of $W(mV + nV^*)$. Then we proceed from

$$B_\theta f = \lambda f \tag{3.3}$$

to a differential equation by applying the operator $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Thus

$$\lambda f'(x) = \frac{1}{2}(e^{-i\theta} f(x) - e^{i\theta} f(x)) = (n - m) i \sin(\theta) f(x).$$

Therefore, $\lambda \neq 0$ and $f = e^{i\mu x}$ for all $x \in [0, 1]$, where

$$\mu := \frac{(n - m)\sin\theta}{\lambda}.$$

The actual eigenvalues λ are obtained by putting $f = e^{i\mu x}$ into (3.3). This yields

$$e^{i\mu} = \frac{(m^2 + n^2)\cos(2\theta) + 2mn + i(n^2 - m^2)\sin(2\theta)}{m^2 + n^2 + 2mn\cos(2\theta)},$$

which implies

$$\cos\mu = \frac{(m^2 + n^2)\cos(2\theta) + 2mn}{m^2 + n^2 + 2mn\cos(2\theta)} \quad \text{and} \quad \sin\mu = \frac{(n^2 - m^2)\sin(2\theta)}{m^2 + n^2 + 2mn\cos(2\theta)}. \tag{3.4}$$

To find $\lambda_\theta = \max\sigma(B_\theta)$, we will prove there is μ_θ satisfying (3.4) such that

$$\lambda_\theta = \frac{(n - m)\sin\theta}{\mu_\theta}, \quad \theta \in [-\pi, \pi].$$

For $\theta \neq \pm\theta_i$ $(i = 1, 2)$, we get that $\tan\mu = \tau_\theta$. This means

$$\lambda = \frac{(n - m)\sin\theta}{\arctan\tau_\theta + k\pi}, \quad k \in \mathbb{Z}. \tag{3.5}$$

Here we deal with the case $n > m$. Because λ is odd with respect to θ, we only need to discuss $\theta \in [0, \pi]$. At this time,

$$\tau'_\theta = \frac{2(n^2 - m^2)(m^2 + n^2 + 2mn\cos 2\theta)}{((m^2 + n^2)\cos(2\theta) + 2mn)^2} \geq 0.$$

We further suppose $n > 0 > m$ and deduce that $\cos 2\theta = -2mn/(m^2 + n^2) \in (0, 1)$ and $2\theta_1 \in (0, \pi/2)$ and $2\theta_2 \in (3\pi/2, 2\pi)$.

For the case $2\theta \in (0, 2\theta_1) \cup (\pi, 2\theta_2)$, it follows $\tau_\theta > 0$ and $\arctan\tau_\theta > 0$, so let $k = 0$ in (3.5) to get λ_θ. For $2\theta \in \{0\} \cup (2\theta_1, \pi] \cup (2\theta_2, 2\pi]$, it yields that $\tau_\theta \leq 0$ and $\arctan\tau_\theta \leq 0$, so let $k = 1$ in (3.5) to obtain λ_θ. Moreover, it holds that

$$\lim_{2\theta \to 2\theta_i^+} \mu_\theta = \lim_{2\theta \to 2\theta_i^-} \mu_\theta = \frac{\pi}{2},$$

then we can supplement the definition at $2\theta_i$ $(i = 1, 2)$. To sum up, μ_θ is same as shown in Eq. (3.1) for $\theta \in [0, \pi]$. Since λ is odd with respect to θ, (3.1) holds for μ_θ. The other case $n > m > 0$ can be similarly proved. Besides, we can consider μ_θ on $[-\pi, 0]$ for $n < m$. It is easy to check the same result (3.1) is true. Next, we calculate the boundary of $W(A)$. The envelope curve is

$$x\cos\theta + y\sin\theta = \lambda_\theta \tag{3.6}$$

for $\theta \in [-\pi, \pi]$. (3.6) implies that the boundary of numerical range is

$$\begin{cases} x\cos\theta + y\sin\theta = \lambda_\theta, \\ -x\sin\theta + y\cos\theta = \lambda'_\theta. \end{cases}$$

This entails that

$$\begin{cases} x = \lambda_\theta\cos\theta - \lambda'_\theta\sin\theta = \frac{m+n}{{\mu_\theta}^2}(1-\cos\mu_\theta), \\ y = \lambda_\theta\sin\theta + \lambda'_\theta\cos\theta = \frac{n-m}{{\mu_\theta}^2}(\mu_\theta - \sin\mu_\theta). \end{cases}$$

If $z = (Af, f) \in W(A)$, then $\overline{z} = (A\overline{f}, \overline{f}) \in W(A)$, so we have

$$\begin{cases} x = \frac{m+n}{{\mu_\theta}^2}(1-\cos\mu_\theta), \\ y = \pm\frac{n-m}{{\mu_\theta}^2}(\mu_\theta - \sin\mu_\theta). \end{cases}$$

This means the boundary of $W(A)$ is

$$\frac{m+n}{{\mu_\theta}^2}(1-\cos\mu_\theta) \pm i\frac{n-m}{{\mu_\theta}^2}(\mu_\theta - \sin\mu_\theta).$$

When we choose $f_{\mu_\theta}(t) = e^{it\mu_\theta}$, it reaches the boundary of $W(A)$ and satisfies

$$\begin{cases} x = \mathrm{Re}(Af_{\mu_\theta}, f_{\mu_\theta}) = \frac{m+n}{{\mu_\theta}^2}(1-\cos\mu_\theta), \\ y = \mathrm{Im}(Af_{\mu_\theta}, f_{\mu_\theta}) = \frac{n-m}{{\mu_\theta}^2}(\mu_\theta - \sin\mu_\theta). \end{cases}$$

The proof is completed. □

Theorem 3.6 together with Proposition 3.5 imply a corollary for $W((u+iv)I + mV + nV^*)$ with $u, v, m, n \in \mathbb{R}$ and $m + n \geq 0$.

Corollary 3.7 *Let $u, v, m, n \in \mathbb{R}$ such that $m + n \geq 0$.*

(i) For $m \neq \pm n$, $W((u+iv)I + mV + nV^*)$ is the set lying between the curves

$$u + \frac{m+n}{{\mu_\theta}^2}(1-\cos\mu_\theta) + i(v \pm \frac{n-m}{{\mu_\theta}^2}(\mu_\theta - \sin\mu_\theta)),$$

where μ_θ is given in (3.1).

(ii) For $m = n\,(m \geq 0)$, $W((u+iv)I + m(V+V^*)) = [u+iv, u+m+iv]$.

(iii) For $m = -n$, $W((u+iv)I + m(V-V^*)) = [u+i(v-\frac{2|m|}{\pi}), u+i(v+\frac{2|m|}{\pi})]$.

Proof of Corollary 3.7 (ii) Proposition 3.5 implies that $W(m(V+V^*)) = [0, m]$ with $m \geq 0$. From the property of Eq. (1.1), we can get that

$$W((u+iv)I + m(V+V^*)) = [u+iv, u+m+iv]. \quad \square$$

At the end of this section, we present an interesting remark.

Remark 3.8 Proposition 3.2 can be deduced from Theorem 3.6 with $m = 1$, $n = 0$. In this case, $\tau_\theta = -\tan(2\theta)$ and $\mu_\theta = -2\theta \in [0, 2\pi]$.

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