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On Skew Polycyclic Codes over $\mathbb{Z}_4[u]/\langle u^2-2\rangle$

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Abstract In this paper, we investigate some classes of skew polycyclic codes and polycyclic codes over $R=\mathbb{Z}_4[u]/\langle u^2-2\rangle$. We first obtain the generator polynomials of all $(1,2u)$-polycyclic codes over R. Then, by defining some Gray maps, we show that the images of (skew) $(1,2u)$-polycyclic codes over R are cyclic or quasi-cyclic with index 2 over $\mathbb{Z}_4$. Finally, an example of some $(1,2u)$-polycyclic codes over R is given to exhibit the main results of the paper.

Keywords skew polycyclic code; polycyclic code; cyclic code; generator polynomial; Gray map

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1. Introduction

In the early 1990s, Nechaev [1] discovered that the binary nonlinear codes can be regarded as images of linear codes over $\mathbb{Z}_4$ under some Gray maps. Then Hammons et al. [2] proved that some good binary nonlinear codes, such as Kerdock codes, Preparata codes and Goethals codes, can be considered as the Gray images of some cyclic codes over $\mathbb{Z}_4$. These important discoveries made scholars turn to the coding theory on finite rings, especially on $\mathbb{Z}_4$ (see [3–8]). Recently, codes over some ring extensions of $\mathbb{Z}_4$ ($\mathbb{Z}_4[u]/\langle u^2\rangle$, $\mathbb{Z}_4[u]/\langle u^2-1\rangle$ and $\mathbb{Z}_4[u]/\langle u^2-3\rangle$ for examples) have also been widely studied [9–12].

In order to give some more linear codes on engineer, Peterson [13] introduced the notion of pseudo-cyclic codes in 1972. Until 2009, López-Permouth et al. [14] re-defined pseudo-cyclic codes from the viewpoint of linear algebra and called them polycyclic codes. To give a further generalization, Matsuoka [15] put forward the concept of skew polycyclic codes over a finite field in 2011. More studies on polycyclic codes and skew polycyclic codes can be found in [16–21].

Throughout this paper, we denote by $R=\mathbb{Z}_4[u]/\langle u^2-2\rangle$ and R^* the set of all units in R. In the paper, we mainly study skew polycyclic codes and polycyclic codes over the ring R. We first obtain the generator polynomials of all $(1,2u)$-polycyclic codes over R, where we always denote by $(1,2u)=(1,2u,0,\ldots,0)$ in this paper. Then, by defining some Gray maps from R^n to $\mathbb{Z}_4^{2n}$, we show that the images of (skew) $(1,2u)$-polycyclic codes over R are cyclic or quasi-cyclic with index 2 over $\mathbb{Z}_4$. Finally, an example of $(1,2u)$-polycyclic codes over R is given to verify the main results of the paper.

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2. Preliminaries

In this section, we mainly give some basic knowledge on skew polycyclic codes over the finite ring R.

Definition 2.1 *Let* $r=(r_0,r_1,\ldots,r_{n-1})\in R^n$, θ *be a ring-automorphism of* R, *the vector* $\mathbf{a}=(a_0,a_1,\ldots,a_{n-1})\in R^n$, *where* $a_0\in R^*$. *Define the* θ-$\mathbf{a}$-*polycyclic map of* R^n

$$\tau_{\theta,\mathbf{a}}:R^n\to R^n$$

$$\tau_{\theta,\mathbf{a}}(r_0,r_1,\ldots,r_{n-1})=(\theta(r_{n-1})a_0,\theta(r_{n-1})a_1+\theta(r_0),\theta(r_{n-1})a_2+\theta(r_1),\ldots,\theta(r_{n-1})a_{n-1}+\theta(r_{n-2})).$$

Furthermore, for the linear code C *of length* n *over* R, *if* $\tau_{\theta,\mathbf{a}}(C)\subseteq C$, *then* C *is called a* θ-$\mathbf{a}$-*polycyclic code over* R. *Let* $c=(c_0,c_1,\ldots,c_{n-1})\in R^n$,

$$D_{\mathbf{a}}=\begin{pmatrix} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ a_0 & a_1 & \cdots & a_{n-1}\end{pmatrix}$$

and $\theta(c)=(\theta(c_0),\theta(c_1),\ldots,\theta(c_{n-1}))$. *If* $\theta(c)D_{\mathbf{a}}\in C$ *for any* $c\in C$, *then the linear code* C *is called a* θ-$\mathbf{a}$-*polycyclic code over* R. *In particular,*

(1) *If* θ *is an identity map,* C *is called an* $\mathbf{a}$-*polycyclic code over* R;

(2) *If* $\mathbf{a}=(a_0,0,\ldots,0)$, C *is called a* θ-a_0-*constacyclic code over* R;

(3) *If* $\mathbf{a}=(1,0,\ldots,0)$, C *is called a* θ-*cyclic code over* R;

(4) *If* θ *is an identity map and* $\mathbf{a}=(a_0,0,\ldots,0)$, C *is called an* a_0-*constacyclic code over* R;

(5) *If* θ *is an identity map and* $\mathbf{a}=(1,0,\ldots,0)$, C *is called a cyclic code over* R.

In order to study the Euclidean dual codes of skew polycyclic codes, we introduce the skew sequential codes over R.

Definition 2.2 *Let* $c=(r_0,r_1,\ldots,r_{n-1})\in R^n$, θ *be a ring-automorphism of* R, *the vector* $\mathbf{a}=(a_0,a_1,\ldots,a_{n-1})\in R^n$, *where* $a_0\in R^*$. *Define the* θ-$\mathbf{a}$-*sequential map of* R^n

$$\tau'_{\theta,\mathbf{a}}:R^n\to R^n$$

$$\tau'_{\theta,\mathbf{a}}(r_0,r_1,\ldots,r_{n-1})=(\theta(r_1),\theta(r_2),\ldots,\theta(r_{n-1}),\theta(r_0)a_0+\theta(r_1)a_1+\cdots+\theta(r_{n-1})a_{n-1}).$$

Furthermore, for the linear code C *of length* n *over* R, *if* $\tau'_{\theta,\mathbf{a}}(C)\subseteq C$, *then* C *is called a* θ-$\mathbf{a}$-*sequential code over* R. *That is to say, if* $\theta(c)D_{\mathbf{a}}^T\in C$ *for any* $c=(c_0,c_1,\ldots,c_{n-1})\in C$, *then* C *is called a* θ-$\mathbf{a}$-*sequential code over* R.

Let C be a linear code over R. Then we can correspond a codeword $c=(c_0,c_1,\ldots,c_{n-1})\in C$ to a polynomial $c(x)=c_0+c_1x+\cdots+c_{n-1}x^{n-1}\in R[x]$ with $\deg(c(x))\leqslant n-1$. Under this point, a linear code C is a θ-$\mathbf{a}$-polycyclic code over R, if and only if for any $c(x)=c_0+c_1x+\cdots+c_{n-1}x^{n-1}\in C$, we have

$$\theta(c_0)x+\theta(c_1)x^2+\cdots+\theta(c_{n-2})x^{n-1}+\theta(c_{n-1})(a_0+a_1x+\cdots+a_{n-1}x^{n-1})\in C.$$

Similarly, a linear code C is a θ-$\mathbf{a}$-sequential code over R, if and only if for any $c(x)=c_0+c_1x+\cdots+c_{n-1}x^{n-1}\in C$, we have

$$\theta(c_1)+\theta(c_2)x+\cdots+\theta(c_{n-1})x^{n-2}+(a_0\theta(c_0)+a_1\theta(c_1)+\cdots+a_{n-1}\theta(c_{n-1}))x^{n-1}\in C.$$

For a given ring-automorphism θ of R, the set $R[x;\theta]=\{a_0+a_1x+\cdots+a_nx^n \mid a_i\in R, n\geq 0\}$ of formal polynomials forms a ring under the rule $(\sum_i ax^i)(\sum_j bx^j)=\sum_{i,j}a\theta^i(b)x^{i+j}$. The ring $R[x;\theta]$ is called a skew polynomial ring over R. Note that $R[x;\theta]$ is not necessary commutative. We always denote by $\langle x^n-\mathbf{a}(x)\rangle$ the left ideal generated by $x^n-\mathbf{a}(x)$. If $\langle x^n-\mathbf{a}(x)\rangle$ is a two-sided ideal, the quotient $R[x;\theta]/\langle x^n-\mathbf{a}(x)\rangle$ is also a ring. The following proposition shows that a θ-$\mathbf{a}$-polycyclic code over R can be seen as a left ideal of $R[x;\theta]/\langle x^n-\mathbf{a}(x)\rangle$.

Proposition 2.3 *Let $\langle x^n-\mathbf{a}(x)\rangle$ be a two-sided ideal, then C is a θ-$\mathbf{a}$-polycyclic code over R if and only if C is the left ideal of the quotient ring $R[x;\theta]/\langle x^n-\mathbf{a}(x)\rangle$.*

Proof A linear code C is an $\mathbf{a}$-polycyclic code over R,

if and only if for any $c=(c_0,c_1,\ldots,c_{n-1})\in C$, then $\theta(c)D_{\mathbf{a}}\in C$,

if and only if for any $c(x)\in C$, $\theta(c_0)x+\cdots+\theta(c_{n-2})x^{n-1}+\theta(c_{n-1})(a_0+\cdots+a_{n-1}x^{n-1})\in C$,

if and only if for any $c(x)\in C, xc(x)\in C\ (\mathrm{mod}(x^n-\mathbf{a}(x)))$,

if and only if for any $r(x)\in R[x;\theta]/\langle x^n-\mathbf{a}(x)\rangle, r(x)c(x)\in C$ (C is linear),

if and only if C the left ideal of quotient rings $R[x;\theta]/\langle x^n-\mathbf{a}(x)\rangle$. □.

Let C be a linear code of length n over R. The Euclidean dual code $C^{\perp}$ of C is denoted by

$$C^{\perp}=\Big\{(x_0,x_1,\ldots,x_{n-1})\in R^n\Big|\sum_{i=0}^{n-1}x_ic_i=0 \text{ for any } c=(c_0,c_1,\ldots,c_{n-1})\in C\Big\}.$$

The following Theorem gives a relationship of skew polycyclic codes and skew sequential codes.

Theorem 2.4 *Let C be a linear code of length n over R, θ be a ring-automorphism of R, $\langle x^n-\mathbf{a}(x)\rangle$ be a two-sided ideal of $R[x;\theta]$. Then C is a θ-$\mathbf{a}$-polycyclic code over R if and only if $C^{\perp}$ is a θ^{-1}-$\theta^{-1}(\mathbf{a})$-sequential code.*

Proof A linear code C is a θ-$\mathbf{a}$-polycyclic code over R,

if and only if for any $c\in C$, then $\theta(c)D_{\mathbf{a}}\in C$,

if and only if for any $y\in C^{\perp}$, then $0=\langle\theta(c)D_{\mathbf{a}},y\rangle=\theta(c)D_{\mathbf{a}}y^T$,

if and only if

$$\begin{aligned}0&=\theta^{-1}(\theta(c)D_{\mathbf{a}}y^T)=c\theta^{-1}(D_{\mathbf{a}})\theta^{-1}(y^T)=cD_{\theta^{-1}(\mathbf{a})}(\theta^{-1}(y))^T\\&=c(\theta^{-1}(y)D^T_{\theta^{-1}(\mathbf{a})})^T\ (\theta \text{ is a ring-automorphism of } R),\end{aligned}$$

if and only if for any $c\in C$, $\langle c,\theta^{-1}(y)D^T_{\theta^{-1}(\mathbf{a})}\rangle=0$,

if and only if $\theta^{-1}(y)D^T_{\theta^{-1}(\mathbf{a})}\in C^{\perp}$,

if and only if $C^{\perp}$ is a θ^{-1}-$\theta^{-1}(\mathbf{a})$-sequential code. □

3. Skew polycyclic codes over R

It is trivial that R can be seen as a $\mathbb{Z}_4$-module $\mathbb{Z}_4+u\mathbb{Z}_4$ with $u^2=2$. One can verify that R has 8 units $\{1,3,1+u,3+u,1+2u,3+2u,1+3u,3+3u\}$, and 5 ideals: $\langle 0\rangle\subseteq\langle 2u\rangle\subseteq\langle 2\rangle\subseteq\langle u\rangle\subseteq R$. Next, we show all ring-automorphism of R. Construct a $\mathbb{Z}_4$-homomorphism $\theta_1: R\to R$ satisfying

$$\theta_1(0)=0,\ \theta_1(1)=1,\ \theta_1(u)=3u,$$

namely, $\theta_1(a+ub)=a+3ub$, $\forall\, a,b\in\mathbb{Z}_4$. And construct a $\mathbb{Z}_4$-homomorphism θ_2: $R\to R$ satisfying

$$\theta_2(0)=0,\ \theta_2(1)=1,\ \theta_2(u)=2+u,$$

namely, $\theta_2(a+ub)=a+(2+u)b$, $\forall\, a,b\in\mathbb{Z}_4$.

Theorem 3.1 *There are exactly 3 ring-automorphisms of R: θ_1, θ_2, the identity map θ_3.*

Proof Let θ be a ring-automorphism of R. Since $u^2=2$ in R, we have $\theta(u^2)=\theta(u)\theta(u)=\theta^2(u)=2$. Set $\theta(u)=a+ub$, $a,b\in\mathbb{Z}_4$, then $(a+ub)^2=2$. It follows that $\theta(u)=3u,2+u$ or u. Set $\theta_1(a+ub)=a+3ub$, $\theta_2(a+ub)=a+(2+u)b$, $\theta_3(a+ub)=a+ub$, $a,b\in\mathbb{Z}_4$. One can verify θ_1,θ_2 and θ_3 are all ring-automorphisms. □

Set $\mathbf{a}(x)=1+2ux$. Then the following proposition gives some equivalent conditions of $\langle x^n-\mathbf{a}(x)\rangle$ to be a two-sided ideal of $R[x;\theta_1]$.

Proposition 3.2 *Let θ_1 be the automorphism of R, where $\theta_1(s+ut)=s+3ut$ with $s,t\in\mathbb{Z}_4$, $\mathbf{a}(x)=1+2ux$. Then the following three statements are equivalent:*

(1) $\langle x^n-\mathbf{a}(x)\rangle$ is a two-sided ideal of $R[x;\theta_1]$;

(2) n is an even number;

(3) $x^n-\mathbf{a}(x)$ is a center element of $R[x;\theta_1]$.

Proof (1)⇒(2). Let $\langle x^n-\mathbf{a}(x)\rangle$ be a two-sided ideal of $R[x;\theta_1]$. Then for any $\alpha\in R$, there exists $\beta=s+ut\in R$ such that

$$\alpha(x^n-\mathbf{a}(x))=(x^n-\mathbf{a}(x))\beta.$$

It follows that $\alpha(x^n-1-2ux)=(x^n-1-2ux)\beta$, i.e.,

$$\alpha x^n-\alpha-2\alpha ux=\theta_1^n(\beta)x^n-\beta-2u\theta_1(\beta)x.$$

By comparing coefficients, we can obtain

$$\alpha=\theta_1^n(\beta),\ \alpha=\beta,\ 2\alpha u=2u\theta_1(\beta).$$

This means

$$\beta=\theta_1^n(\beta),\ 2u(\theta_1(\beta)-\beta)u=0.$$

Since $2u(\theta_1(\beta)-\beta)=2u(\theta_1(s+ut)-(s+ut))=2u(s+3ut-s-ut)=2u2ut=0$, we have $2u(\theta_1(\beta)-\beta)=0$. Now, $s+ut=\beta=\theta_1^n(\beta)=\theta_1^n(s+ut)=s+3^nut$, i.e., $(3^n-1)t=0$. If $t=0$, then $(3^n-1)t=0$; If $t=1$, then $(3^n-1)=0$, so $4|3^n-1$; If $t=2$, then $(3^n-1)2=0$, so $2|3^n-1$; If $t=3$, then $(3^n-1)3=0$, so $4|3^n-1$. From the above, $4|3^n-1$, so n is an even number.

Let $\langle x^n-\mathbf{a}(x)\rangle$ be a two-sided ideal of $R[x;\theta_1]$. Then $x(x^n-\mathbf{a}(x))=(x^n-\mathbf{a}(x))f(x)$, where $f(x)\in R[x;\theta_1]$. Comparing the degree of polynomials on both sides, we might as well set $f(x)=\alpha x+\beta$, where $\alpha,\beta\in R$, then

$$x(x^n-1-2ux)=(x^n-1-2ux)(\alpha x+\beta).$$

Since $x\alpha=\theta(\alpha)x$ and R is a commutative ring,

$$\begin{aligned}x^{n+1}-x-\theta_1(2u)x^2=&x^n\alpha x+x^n\beta-\alpha x\beta-2ux\alpha x-2ux\beta\\=&\theta_1^n(\alpha)x^{n+1}+\theta_1^n(\beta)x^n-\alpha x-\beta-2u\theta_1(\alpha)x^2-2u\theta_1(\beta)x.\end{aligned}$$

Comparing the coefficients on both sides, we have $\theta_1^n(\alpha)=1$, $\theta_1^n(\beta)=0$, $2u\theta_1(\alpha)=2u$, $\alpha+2u\theta_1(\beta)=1$, $\beta=0$. This means $\alpha=1$ and $\beta=0$. Therefore, $x(x^n-1-2ux)=(x^n-1-2ux)x$.

(2)⇒(3). Let $4|3^n-1$. For any $\beta=s+ut\in R$, then $(3^n-1)t=0$. Namely, $3^nt=t$, for any $t\in\mathbb{Z}_4$. Hence

$$\theta_1^n(\beta)=\theta_1^n(s+ut)=s+3^nut=s+ut=\beta.$$

As $\theta_1(s+ut)=s+3ut$, where $s,t\in\mathbb{Z}_4$, and $u^2=2$, it follows that

$$u\theta_1(\beta)=u(s+3ut)=u[(s+ut)+2ut]=u(s+ut)=u\beta.$$

Let $k\in N^+$. Note that $\beta x^k2ux=\beta\theta_1^k(2u)x^{k+1}=\beta3^k2ux^{k+1}=2u\beta x^{k+1}$, we have

$$\begin{aligned}(x^n-\mathbf{a}(x))\beta x^k=&(x^n-1-2ux)\beta x^k=x^n\beta x^k-\beta x^k-2ux\beta x^k\\=&\theta_1^n(\beta)x^{n+k}-\beta x^k-2u\theta_1(\beta)x^{k+1}=\beta x^{n+k}-\beta x^k-2u\beta x^{k+1}\\=&\beta x^k(x^n-\mathbf{a}(x)).\end{aligned}$$

Then $f(x)(x^n-\mathbf{a}(x))=(x^n-\mathbf{a}(x))f(x)$, for any $f(x)\in R[x;\theta_1]$. Thus $x^n-\mathbf{a}(x)$ is a center element of $R[x;\theta_1]$.

(3)⇒(1). Let $\langle x^n-\mathbf{a}(x)\rangle$ be a center element of $R[x;\theta_1]$ with $\mathbf{a}(x)=1+2ux$. Then

$$f(x)(x^n-\mathbf{a}(x))=(x^n-\mathbf{a}(x))f(x)\in\langle x^n-\mathbf{a}(x)\rangle,\ \text{for any } f(x)\in R[x;\theta_1].$$

It follows that $\langle x^n-\mathbf{a}(x)\rangle$ is a two-sided ideal of $R[x;\theta_1]$. □

By Lemmas 2.3 and 3.2, we obtain the following proposition.

Proposition 3.3 *Let n be an even number. Then C is a θ_1-$\mathbf{a}$-polycyclic code over R if and only if C is a left idea of $R[x;\theta_1]/\langle x^n-\mathbf{a}(x)\rangle$.*

The rest of this section mainly studies some Gray images of skew polycyclic codes over R. We first define a new Gray map as follows:

$$\phi_1:R\to\mathbb{Z}_4^2,\ \phi_1(c)=(s+2t,3s+2t),$$

where $c=s+ut\in R$ with $s,t\in\mathbb{Z}_4$. The Gray map ϕ_1 can be extended to R^n:

$$\phi_1:R^n\to\mathbb{Z}_4^{2n},$$

$$(c_0,c_1,\ldots,c_{n-1})\mapsto(s_0+2t_0,\ldots,s_{n-1}+2t_{n-1},3s_0+2t_0,\ldots,3s_{n-1}+2t_{n-1}),$$

where $c_i=s_i+ut_i\in R$ $(0\leqslant i\leqslant n-1)$.

Then we define another Gray map as follows:

$$\phi_2:R\to\mathbb{Z}_4^2,\ s+ut\mapsto(3s+2t,3s+2t),$$

where $s=r+2q\in\mathbb{Z}_4$, $t=w+2p\in\mathbb{Z}_4$ with $r,q,w,p\in\mathbb{F}_2$. The Gray map can be extended to R^n similarly.

Suppose m,l are positive integers. Let C be a linear code over $\mathbb{Z}_4$. Define a quasi-cyclic map with index l of $\mathbb{Z}_4^{lm}$ as follows: $\eta_l:\mathbb{Z}_4^{lm}\to\mathbb{Z}_4^{lm}$,

$$\begin{aligned}&\eta_l(c_{0,0},\ldots,c_{0,m-1},|c_{1,0},\ldots,c_{1,m-1},|\ldots,|c_{l-1,0},\ldots,c_{l-1,m-1})\\&\quad=(c_{0,m-1},c_{0,0},\ldots,c_{0,m-2},|c_{1,m-1},c_{1,0},\ldots,c_{1,m-2},|\ldots,|c_{l-1,m-1},c_{l-1,0},\ldots,c_{l-1,m-2}).\end{aligned}$$

If $\eta_l(C)\subseteq C$, then C is a quasi-cyclic code with index l over $\mathbb{Z}_4$.

Lemma 3.4 *Let $\tau_{\theta_1,(1,2u)}$ be the θ_1-$(1,2u)$-polycyclic map of R^n, η_2 be the quasi-cyclic map with index 2 of $\mathbb{Z}_4^{2n}$. Let ϕ_1 be defined as above. Then $\phi_1\tau_{\theta_1,(1,2u)}=\eta_2\phi_1$.*

Proof Let $c=(c_0,c_1,\ldots,c_{n-1})\in R^n$, where $c_i=s_i+ut_i$ with $s_i,t_i\in\mathbb{Z}_4$, for $i=0,1,\ldots,n-1$. Since $\theta_1(s+ut)=s+u3t$ is a ring-automorphism and $u^2=2$, we have

$$\begin{aligned}\tau_{\theta_1,(1,2u)}(c)=&(0,\theta_1(c_0),\theta_1(c_1),\ldots,\theta_1(c_{n-2}))+\theta_1(c_{n-1})(1,2u,0,\ldots,0)\\=&(\theta_1(s_{n-1}+ut_{n-1}),\theta_1(s_0+ut_0)+\theta_1(s_{n-1}+ut_{n-1})2u,\theta_1(s_1+ut_1),\ldots,\\&\theta_1(s_{n-2}+ut_{n-2})\\=&(s_{n-1}+3t_{n-1}u,s_0+(3t_0+2s_{n-1})u,s_1+3t_1u,\ldots,s_{n-2}+3t_{n-2}u).\end{aligned}$$

Then

$$\begin{aligned}\phi_1\tau_{\theta_1,(1,2u)}(c)=&(s_{n-1}+2t_{n-1},s_0+2t_0,\ldots,s_{n-2}+2t_{n-2},3s_{n-1}+2t_{n-1},3s_0+2t_0,\ldots,\\&3s_{n-2}+2t_{n-2}).\end{aligned}$$

On the other hand, since $\phi_1(s+ut)=(s+2t,3s+2t)$ and

$$\begin{aligned}\eta_2\phi_1=&(s_{n-1}+2t_{n-1},s_0+2t_0,\ldots,s_{n-2}+2t_{n-2},3s_{n-1}+2t_{n-1},3s_0+2t_0,\ldots,\\&3s_{n-2}+2t_{n-2}),\end{aligned}$$

we have $\phi_1\tau_{\theta_1,(1,2u)}=\eta_2\phi_1$. $\square$

By Lemma 3.4, we can obtain the following theorem 3.5.

Theorem 3.5 *Let C be a θ_1-$(1,2u)$-polycyclic code of length n over R. Then $\phi_1(C)$ is a quasi-cyclic code with index 2 of length $2n$ over $\mathbb{Z}_4$.*

Proof Assume that C is a θ_1-$(1,2u)$-polycyclic code of length n over R. By Lemma 3.4, we see that $\eta_2\phi_1(C)=\phi_1\tau_{\theta_1,(1,2u)}(C)=\phi_1(C)$, which means $\phi_1(C)$ is a quasi-cyclic code with index 2 of length $2n$ over $\mathbb{Z}_4$. $\square$

Lemma 3.6 *Let* $\tau_{\theta_1,(1,2u)}$ *be the* θ_1*-*$(1,2u)$*-polycyclic map of* R^n*,* σ *be the cyclic map of* $\mathbb{Z}_4^{2n}$*. Let* ϕ_2 *be defined as above. Then* $\phi_2\tau_{\theta_1,(1,2u)}=\sigma\phi_2$*.*

Proof Since $\phi_2(s+ut)=(3s+2t,3s+2t)$, we have

$$\sigma\phi_2=(3s_{n-1}+2t_{n-1},3s_0+2t_0,\dots,3s_{n-1}+2t_{n-1},3s_0+2t_0,\dots,3s_{n-2}+2t_{n-2}).$$

It is easy to verify that $\phi_2\tau_{\theta_1,(1,2u)}=\sigma\phi_2$. □

Theorem 3.7 *Let* C *be a* θ_1*-*$(1,2u)$*-polycyclic code of length* n *over* R*. Then* $\phi_2(C)$ *is a cyclic code of length* $2n$ *over* $\mathbb{Z}_4$*.*

Proof Let C be a θ_1-$(1,2u)$-polycyclic code of length n over R. By Lemma 3.6, we have

$$\sigma\phi_2(C)=\phi_2\tau_{\theta_1,(1,2u)}(C)=\phi_2(C).$$

It is easy to verify $\phi_2(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. □

4. Polycyclic codes over R

In this section, we mainly study polycyclic codes over R, which is a special case of skew polycyclic codes. In the rest of this paper, we always denote $\mathbf{a}=(1,2u)$ to be $(1,2u,0,\dots,0)$ for short. We first consider the Gray images of **a**-polycyclic code.

Lemma 4.1 *Let* $\tau_{(1,2u)}$ *be the* $(1,2u)$*-polycyclic map of* R^n*,* η_2 *be the quasi-cyclic map with index* 2 *of* $\mathbb{Z}_4^{2n}$*. Let* ϕ_1 *be defined as the previous section. Then* $\phi_1\tau_{(1,2u)}=\eta_2\phi_1$*.*

Proof Let $c=(c_0,c_1,\dots,c_{n-1})\in R^n$, where $c_i=s_i+ut_i$ with $s_i,t_i\in\mathbb{Z}_4$ for $i=0,1,\dots,n-1$. Since $u^2=2$, we have

$$\begin{aligned}\tau_{(1,2u)}(c)=&(0,c_0,c_1,\dots,c_{n-2})+c_{n-1}(1,2u,0,\dots,0)\\ =&(s_{n-1}+ut_{n-1},s_0+u(t_0+2s_{n-1}),s_1+ut_1,\dots,s_{n-2}+ut_{n-2}).\end{aligned}$$

So

$$\begin{aligned}\phi_1\tau_{(1,2u)}(c)=&(s_{n-1}+2t_{n-1},s_0+2t_0,\dots,s_{n-2}+2t_{n-2},3s_{n-1}+2t_{n-1},3s_0+2t_0,\dots,\\ &3s_{n-2}+2t_{n-2}).\end{aligned}$$

On the other hand, as $\phi_1(s+ut)=(s+2t,3s+2t)$, we can obtain

$$\begin{aligned}\eta_2\phi_1=&(s_{n-1}+2t_{n-1},s_0+2t_0,\dots,s_{n-2}+2t_{n-2},3s_{n-1}+2t_{n-1},3s_0+2t_0,\dots,\\ &3s_{n-2}+2t_{n-2}).\end{aligned}$$

Consequently, $\phi_1\tau_{(1,2u)}=\eta_2\phi_1$. □

By Lemma 4.1, we can obtain the following result.

Theorem 4.2 *Let* C *be a* $(1,2u)$*-polycyclic code of length* n *over* R*. Then* $\phi_1(C)$ *is a quasi-cyclic code with index* 2 *of length* $2n$ *over* $\mathbb{Z}_4$*.*

Proof Assume that C is a $(1,2u)$-polycyclic code of length n over R. By Lemma 4.1, we have

$$\eta_2\phi_1(C)=\phi_1\tau_{(1,2u)}(C)=\phi_1(C).$$

It is easy to verify $\phi_1(C)$ is a quasi-cyclic code with index 2 of length $2n$ over $\mathbb{Z}_4$. □

Lemma 4.3 *Let $\tau_{(1,2u)}$ be the $(1,2u)$-polycyclic map of R^n, σ be the cyclic map of $\mathbb{Z}_4^{2n}$. Let ϕ_2 be defined as the previous section. Then $\phi_2\tau_{(1,2u)}=\sigma\phi_2$.*

Proof Since $\phi_2(s+ut)=(3s+2t,3s+2t)$, we have

$$\sigma\phi_2=(3s_{n-1}+2t_{n-1},3s_0+2t_0,\ldots,3s_{n-1}+2t_{n-1},3s_0+2t_0,\ldots,3s_{n-2}+2t_{n-2}).$$

It is easy to verify that $\phi_2\tau_{(1,2u)}=\sigma\phi_2$. □

By Lemma 4.3, the following theorem can be obtained.

Theorem 4.4 *Let C be a $(1,2u)$-polycyclic code of length n over R. Then $\phi_2(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof Let C be a $(1,2u)$-polycyclic code of length n over R. By Lemma 4.3, we get

$$\sigma\phi_2(C)=\phi_2\tau_{(1,2u)}(C)=\phi_2(C).$$

It is easy to verify $\phi_2(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. □

To study the Gray images over $\mathbb{F}_2$ of polycyclic codes, we define a Gray map as follows:

$$\phi_3:R\to\mathbb{F}_2^4,\ \ s+ut\mapsto(r,r+w,q+w,r+q+w),$$

where $s=r+2q\in\mathbb{Z}_4,\ t=w+2p\in\mathbb{Z}_4$ with $r,q,w,p\in\mathbb{F}_2$. The Gray map ϕ_3 can be extended to R^n:

$$\phi_3:R^n\to\mathbb{F}_2^{4n},$$

$$\begin{aligned}(c_0,c_1,\ldots,c_{n-1})\mapsto(&r_0,\ldots,r_{n-1},r_0+w_0,\ldots,r_{n-1}+w_{n-1},q_0+w_0,\ldots,\\&q_{n-1}+w_{n-1},r_0+q_0+w_0,\ldots,r_n+q_{n-1}+w_{n-1}),\end{aligned}$$

where $c_i=(r_i+2q_i)+u(w_i+2p_i)\in R$ with $r_i,q_i,w_i,p_i\in\mathbb{F}_2$ $(0\leqslant i\leqslant n-1)$.

Lemma 4.5 *Let $\tau_{(1,2u)}$ be the $(1,2u)$-polycyclic map of R^n, η_4 be the quasi-cyclic map with index 4 of $\mathbb{F}_2^{4n}$. Let ϕ_3 be defined as above. Then $\phi_3\tau_{(1,2u)}=\eta_4\phi_3$.*

Proof Since $\phi_3(s+ut)=(r,r+w,q+w,r+q+w)$, where $s=r+2q\in\mathbb{Z}_4$, $t=w+2p\in\mathbb{Z}_4$ with $r,q,w,p\in\mathbb{F}_2$, we have

$$\begin{aligned}\tau_{(1,2u)}(c)=&(0,c_0,c_1,\ldots,c_{n-2})+c_{n-1}(1,2u,0,\ldots,0)\\=&(r_{n-1}+2q_{n-1}+u(w_{n-1}+2p_{n-1}),r_0+2q_0+u(w_0+2(p_0+r_{n-1})),\\&r_1+2q_1+u(w_1+2p_1),\ldots,r_{n-2}+2q_{n-2}+u(w_{n-2}+2p_{n-2})).\end{aligned}$$

Then

$$\sigma\phi_3\tau_{(1,2u)}=(r_{n-1},r_0,\ldots,r_{n-2},r_{n-1}+w_{n-1},r_0+w_0,\ldots,r_{n-2}+w_{n-2},$$

$$q_{n-1}+w_{n-1},q_0+w_0,\dots,q_{n-2}+w_{n-2},r_{n-1}+q_{n-1}+w_{n-1},$$
$$r_0+q_0+w_0,\dots,r_{n-2}+q_{n-2}+w_{n-2}).$$

By the definition of ϕ_3, we obtain $\phi_3\tau_{(1,2u)}=\sigma\phi_3$. □

By Lemma 4.5, we can obtain the following result.

Theorem 4.6 *Let* C *be* $(1,2u)$*-polycyclic code of length* n *over* R*. Then* $\phi_3(C)$ *is a quasi-cyclic code with index* 4 *of length* $4n$ *over* $\mathbb{F}_2$*.*

Proof Let C be a $(1,2u)$-polycyclic code of length n over R. By Lemma 4.5, we have

$$\phi_3\tau_{(1,2u)}(C)=\eta_4\phi_3(C).$$

It is easy to verify $\phi_3(C)$ is a quasi-cyclic code of length $4n$ over $\mathbb{F}_2$. □

The rest of this section will study the generator polynomials of $(1,2u)$-polycyclic codes over R.

Lemma 4.7 ([22]) *Let* C *be a cyclic code of length* n *over* $\mathbb{Z}_4$*.*

(1) If n *is odd, then* $C=\langle g(x),2r(x)\rangle=\langle g(x)+2r(x)\rangle$*, where* $g(x),r(x)$ *are polynomials with* $r(x)|g(x)|(x^n-1)$ mod 4.

(2) Assume that n *is even, then either:*

(a) C *is a free module of generator* $C=\langle g(x)+2p(x)\rangle$*, where* $g(x)|(x^n-1)$ mod 2 *and* $(g(x)+2p(x))|(x^n-1)$ mod 4*, or,*

(b) $C=\langle g(x)+2p(x),2r(x)\rangle$*, where* $g(x),r(x)$ *and* $p(x)$ *are polynomials with* $g(x)|(x^n-1)$ mod 2, $r(x)|g(x)$ mod 2, $r(x)|(p(x)\frac{x^n-1}{g(x)})$ mod 2, *and* $\deg(r(x))>\deg(p(x))$*.*

We can associate a linear code C over R with two linear codes over $\mathbb{Z}_4$ of length n. The residue code $\mathrm{Res}(C)=\{x\in\mathbb{Z}_4^n|\exists\, y\in\mathbb{Z}_4^n : x+uy\in C\}$ and the torsion code $\mathrm{Tor}(C)=\{y\in\mathbb{Z}_4^n|uy\in C\}$. Let

$$\mu: R^n\to\mathbb{Z}_4^n,(c_0,c_1,\dots,c_{n-1})\mapsto(s_0,s_1,\dots,s_{n-1}),$$

where $c_i=s_i+ut_i$ for $i=0,1,\dots,n-1$. Clearly, the map μ is a $\mathbb{Z}_4$-homomorphism with $\mathrm{Ker}\,\mu\cong\mathrm{Tor}(C)$ and $\mu(C)=\mathrm{Res}(C)$. In the following result, we give the generator polynomials of all $(1+2u)$-polycyclic codes over R.

Theorem 4.8 *Let* C *be a* $(1,2u)$*-polycyclic code of length* n *over* R*.*

(1) If n *is odd, then* $C=\langle g_1(x)+2r_1(x)+ub(x),u(g_2(x)+2r_2(x))\rangle$*, where* $b(x)$ *is a polynomial over* $\mathbb{Z}_4$*, and* $g_i(x),r_i(x)$ *are polynomials with* $r_i(x)|g_i(x)|(x^n-1)$ mod 4, $i=1,2$.

(2) Assume that n *is even, then either:*

(a) $C=\langle g_1(x)+2p_1(x)+ud(x),u(g_2(x)+2p_2(x))\rangle$*, where* $d(x)$ *is a polynomial over* $\mathbb{Z}_4$*, and* $g_i(x),p_i(x)$ *are polynomials with* $g_i(x)|(x^n-1)$ mod 2, $(g_i(x)+2p_i(x))|(x^n-1)$ mod 4, $i=1,2$.

(b) $C=\langle g_1(x)+2p_1(x)+ue_1(x),2r_1(x)+ue_2(x),u(g_2(x)+2p_2(x)),2ur_2(x)\rangle$*, where* $e_i(x)$ *is a polynomial over* $\mathbb{Z}_4$*, and* $g_i(x),r_i(x),p_i(x)$ *are polynomials with* $g_i(x)|(x^n-1)$ mod 2, $r_i(x)|g_i(x)$ mod 2, $r_i(x)|(p_i(x)\frac{x^n-1}{g_i(x)})$ mod 2, $\deg(r_i(x))>\deg(p_i(x))$, $i=1,2$.

Proof (1) Suppose n is an odd integer. Let $c=(c_0,c_1,\dots,c_{n-1})\in C$, where $c_i=s_i+ut_i$ $(0\leqslant$

$i \leqslant n-1$), then $(s_0, s_1, \dots, s_{n-1}) \in \mathrm{Res}(C)$. Since C is a $(1, 2u)$-polycyclic code of length n over R, we have $(0, c_0, \dots, c_{n-2}) + c_{n-1}(1, 2u, 0, \dots, 0) \in C$. Note that $2uc_{n-1} = 2us_{n-1}$. So we obtain that

$$(s_{n-1}, s_0, \dots, s_{n-2}) \in \mathrm{Res}(C).$$

Hence $\mathrm{Res}(C)$ is a cyclic code over $\mathbb{Z}_4$, which means $\mu(C)$ is a cyclic code of length n over $\mathbb{Z}_4$. By (1) of Lemma 4.7, we obtain that $\mu(C) = \langle g_1(x) + 2r_1(x)\rangle$, where $g_1(x), r_1(x)$ are polynomials with $r_1(x)|g_1(x)|(x^n-1)$ mod 4. Note that $\mu(C) = \mathrm{Res}(C)$. By the definition of $\mathrm{Res}(C)$, there exists a polynomial $b(x) \in \mathbb{Z}_4[x]$ such that $g_1(x) + 2r_1(x) + ub(x) \in C$.

Also, let $(ut_0, ut_1, \dots, ut_{n-1}) \in C$. Obviously, $(ut_0, ut_1, \dots, ut_{n-1}) \in \mathrm{Ker}\mu$. Since C is a $(1, 2u)$-polycyclic code of length n over R, we get

$$(0, ut_0, \dots, ut_{n-2}) + ut_{n-1}(1, 2u, 0, \dots, 0) \in C.$$

Since $u^2 = 2$, $(ut_{n-1}, ut_0, \dots, ut_{n-2}) \in C$. Obviously, $\mu(ut_{n-1}, ut_0, \dots, ut_{n-2}) = 0$, then $(ut_{n-1}, ut_0, \dots, ut_{n-2}) \in C \cap \mathrm{Ker}\mu$. Therefore, $C \cap \mathrm{Ker}\mu$ is a cyclic code of length n over $\mathbb{Z}_4 + u\mathbb{Z}_4$. By Lemma 4.7 again, $C \cap \mathrm{Ker}\mu = u\langle g_2(x) + 2r_2(x)\rangle$, where $g_2(x), r_2(x)$ are polynomials with $r_2(x)|g_2(x)|(x^n-1)$ mod 4. Hence

$$\langle g_1(x) + 2r_1(x) + ub(x), u(g_2(x) + 2r_2(x))\rangle \subseteq C.$$

On the other hand, for any $f(x) = f_1(x) + uf_2(x) \in C$, where $f_i(x) \in \mathbb{Z}_4[x]$, $i = 1, 2$, then $f_1(x) \in \mu(C)$. So there exists $m(x) \in \mathbb{Z}_4[x]$ such that

$$\begin{aligned} f(x) =& f_1(x) + uf_2(x) = m(x)(g_1(x) + 2r_1(x)) + uf_2(x) \\ =& m(x)(g_1(x) + 2r_1(x) + ub(x)) + u(f_2(x) - m(x)b(x)). \end{aligned}$$

Since $u(f_2(x) - m(x)b(x)) \in C \cap \mathrm{Ker}\mu$, $f(x) \in \langle g_1(x) + 2r_1(x) + ub(x), u(g_2(x) + 2r_2(x))\rangle$. That is to say $C \subseteq \langle g_1(x) + 2r_1(x) + ub(x), u(g_2(x) + 2r_2(x))\rangle$. Then $C = \langle g_1(x) + 2r_1(x) + ub(x), u(g_2(x) + 2r_2(x))\rangle$.

(2) Assume that n is even, we only prove (b) since (a) is similar. By (b) of Lemma 4.7 and similar proof of Theorem 4.8 (1), we can get $\mu(C) = \langle g_1(x) + 2p_1(x), 2r_1(x)\rangle$, where $g_1(x), r_1(x), p_1(x)$ are polynomials with $g_1(x)|(x^n-1)$ mod 2, $r_1(x)|g_1(x)$ mod 2, $r_1(x)|(p_1(x)\frac{x^n-1}{g_1(x)})$ mod 2, and $\deg(r_1(x)) > \deg(p_1(x))$. Note that $\mu(C) = \mathrm{Res}(C)$. By the definition of $\mathrm{Res}(C)$, there exist $e_1(x), e_1(x) \in \mathbb{Z}_4[x]$ such that $g_1(x) + 2p_1(x) + ue_1(x) \in C$ with $2r_1(x) + ue_2(x) \in C$. By (b) of Lemma 4.7 and similar proof of Theorem 4.8 (1), we also have

$$C \cap \mathrm{Ker}\mu = u\langle g_2(x) + 2p_2(x), 2r_2(x)\rangle,$$

where $g_2(x), r_2(x), p_2(x)$ are polynomials with

$$g_2(x)|(x^n-1) \bmod 2,\ \ r_2(x)|g_2(x) \bmod 2, r_2(x)|(p_2(x)\frac{x^n-1}{g_2(x)}) \bmod 2$$

and $\deg(r_2(x)) > \deg(p_2(x))$. Then

$$\langle g_1(x) + 2p_1(x) + ue_1(x), 2r_1(x) + ue_2(x), u(g_2(x) + 2p_2(x)), 2ur_2(x)\rangle \subseteq C.$$

On the other hand, for any $f(x)=f_1(x)+uf_2(x)\in C$, where $f_i(x)\in\mathbb{Z}_4[x]$ for $i=1,2$, we have $f_1(x)\in\mu(C)$. Hence there exist $m_1(x),m_2(x)\in\mathbb{Z}_4[x]$ such that

$$\begin{aligned}f(x)=&f_1(x)+uf_2(x)=m_1(x)(g_1(x)+2p_1(x))+2m_2(x)r_1(x)+uf_2(x)\\=&m_1(x)(g_1(x)+2p_1(x)+ue_1(x))+m_2(x)(2r_1(x)+ue_2(x))+\\&u(f_2(x)-m_1(x)e_1(x)-m_2(x)e_2(x)).\end{aligned}$$

Since $u(f_2(x)-m_1(x)e_1(x)-m_2(x)e_2(x))\in C\cap\mathrm{Ker}\mu$, $f(x)\in\langle g_1(x)+2p_1(x)+ue_1(x),2r_1(x)+ue_2(x),u(g_2(x)+2p_2(x)),2ur_2(x)\rangle$. That is to say

$$C\subseteq\langle g_1(x)+2p_1(x)+ue_1(x),2r_1(x)+ue_2(x),u(g_2(x)+2p_2(x)),2ur_2(x)\rangle.$$

Hence,

$$C=\langle g_1(x)+2p_1(x)+ue_1(x),2r_1(x)+ue_2(x),u(g_2(x)+2p_2(x)),2ur_2(x)\rangle.\quad\square$$

5. An Example

This section will verify some main results of this paper through an example.

First, we recall some weights of linear codes over $\mathbb{Z}_4$. Define the Hamming weight of elements $0,1,2,3$ in $\mathbb{Z}_4$ as $0,1,1,1$, respectively; the Lee weight of elements $0,1,2,3$ in $\mathbb{Z}_4$ as $0,1,2,1$, respectively, and the Eulidean weight of elements $0,1,2,3$ in $\mathbb{Z}_4$ as $0,1,4,1$, respectively. Let $x=(x_0,\dots,x_{n-1})\in\mathbb{Z}_4^n$. Define the Hamming (resp., Lee, Eulidean) weight for x, $w_H(x)=\sum_{i=0}^{n-1}w_H(x_i)$ (resp., $w_L(x)=\sum_{i=0}^{n-1}w_L(x_i),w_E(x)=\sum_{i=0}^{n-1}w_E(x_i)$). Assume that C is a linear code of length n over $\mathbb{Z}_4$, the Hamming (resp., Lee, Eulidean) distance of C is defined as the minimum value of Hamming (resp., Lee, Eulidean) weights of non-zero codewords in C.

n	$g_1'(x)$	$g_2'(x)$	$\mathbb{Z}_4$	d_H	d_L	d_E
3	0	$x+1$	4^02^2	4	8^*	16
3	$3x+1$	0	4^22^0	4	4^*	4
3	$3x^2+3x+3$	0	4^12^0	6	6^*	6
4	$x+3$	0	4^32^0	4	4^*	4
4	x^3+3x^2+x+3	0	4^12^0	8	8^*	8
5	0	$3x+1$	4^02^4	4	8^*	16
5	$3x^4+3x^3+3x^2+3x+3$	0	4^12^0	10	10^*	10
6	x^3+2x^2+2x+3	0	4^32^0	4	8^*	8
7	$3x^6+3x^5+3x^4+3x^3$	0	4^12^0	14	14^*	14
9	0	$x^7+3x^6+x^4+3x^3+x+1$	4^02^2	12	24^*	48
9	$3x^7+x^6+3x^4+x^3$	0	4^22^0	12	12^*	12
9	$x^8+x^7+x^6+x^5+x^4$	0	4^12^0	18	18^*	18

Table 1 Some $(1,2u)$-polycyclic good codes of length at most 9 over R

Example 5.1 Let C be a $(1,2u)$-polycyclic code of length n over R. By (1) and (a) of Theorem 4.8, we can set $C=\langle g_1'(x),ug_2'(x)\rangle$. By Theorem 4.2, we get $\phi_1(C)$ is a linear code of length $2n$

over $\mathbb{Z}_4$. Therefore, it is of type $4^{k_1}2^{k_2}$ (see [6, Proposition 1.1]). Using MATLAB, we can give Table 1. In the table, the column of $\mathbb{Z}_4$ represents the type of $\phi_1(C)$ in $\mathbb{Z}_4$. The column of d_H, d_L and d_E represent the distance of Hamming, the distance of Lee and the distance of Eulidean, respectively. The Gray images marked $*$ in the column d_L are good codes over $\mathbb{Z}_4$ (see [23]).

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References

[1] A. A. NECHAEV. *Kerdock codes in cyclic form*. Discrete Math. Appl., 1991, **1**(4): 365–384.

[2] A. R. HAMMONS, P. V. KUMAR, A. R. CALDERBANK, et al. *The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals, and related codes*. IEEE Trans. Inform. Theory, 1994, **40**(2): 301–319.

[3] N. AYDIN, D. K. RAY-CHAUDHURI. *Quasi-cyclic codes over $\mathbb{Z}_4$ and some new binary codes*. IEEE Trans. Inform. Theory, 2001, **47**(7): 2065–2069.

[4] P. LANGEVIN. *Duadic $\mathbb{Z}_4$-codes*. Finite Fields Appl., 2000, **6**(3): 309–326.

[5] E. RAINS. *Bounds for self-dual codes over $\mathbb{Z}_4$*. Finite Fields Appl., 2000, **6**(2): 146–163.

[6] Zhexian WAN. *Quaternary Codes*. World Scientific, Singapore, 1997.

[7] J. WOLFMAN. *Binary images of cyclic codes over $\mathbb{Z}_4$*. IEEE Trans. Inform. Theory, 2001, **47**(5): 1773–1777.

[8] J. WOLFMAN. *Negacyclic and cyclic codes over $\mathbb{Z}_4$*. IEEE Trans. Inform. Theory, 1999, **45**(7): 2527–2532.

[9] N. AYDIN, Y. CENGELLENMIS, A. DERTLI. *On some constacyclic codes over $\mathbb{Z}_4[u]/\langle u^2-1\rangle$, their $\mathbb{Z}_4$ images, and new codes*. Des. Codes Cryptogr., 2017, **86**(6): 1–7.

[10] T. BAG, H. ISLAM, O. PRAKASH, et al. *A study of constacyclic codes over the ring $\mathbb{Z}_4[u]/\langle u^2-3\rangle$*. Discrete Math. Algorithms Appl., 2018, **10**(4): 1850056, 10 pp.

[11] A. SHARMA, M. BHAINTWAL. *A class of skew-constacyclic codes over $\mathbb{Z}_4+u\mathbb{Z}_4$*. Int. J. Inf. Coding Theory, 2017, **4**(4): 289–303.

[12] B. YILDIZ, N. AYDIN. *On cyclic codes over $\mathbb{Z}_4+u\mathbb{Z}_4$ and $\mathbb{Z}_4$-images*. Int. J. Inf. Coding Theory, 2014, **2**(4): 226–237.

[13] W. W. PETERSON, E. J. WELDON. *Error Correcting Codes*. MIT Press, Cambridge, 1972.

[14] S. R. LÓPEZ-PERMOUTH, B. R. PARRA-AVILA, S. SZABO. *Dual generalizations of the concept of cyclicity of codes*. Adv. Math. Mommun, 2009, **3**: 227–234.

[15] M. MATSUOKA. *Polycyclic codes and sequential codes over finite commutative QF rings*. JP J. Algebra Number Theory Appl., 2011, **23**(1): 77–85.

[16] A. ALAHMADI, S. DOUGHERTY, A. LEROY, et al. *On the duality and the direction of polycyclic codes*. Adv. Math. Commun., 2016, **10**(4): 921–929.

[17] A. ALAHMADI, C. GÜNERI, H. SHOAIB, et al. *Long quasi-polycyclic t-CIS codes*. Adv. Math. Commun., 2018, **12**(1): 189–198.

[18] A. FOTUE-TABUE, E. MARTINEZ-MORO, J. T. BLACKFORD. *On polycyclic codes over a finite chain ring*. Adv. Math. Commun., 2020, **14**(3): 455–466.

[19] Li LIU, Yi XU. *Generalized quasi-polycyclic codes over finite fields*. J. Systems Sci. Math. Sci., 2019, **39**(3): 470–476.

[20] S. R. LÓPEZ-PERMOUTH, H. ÖZADAM, F. ÖZBUDAK, et al. *Polycyclic codes over Galois rings with applications to repeated-root constacyclic codes*. Finite Fields Appl., 2013, **19**: 16–38.

[21] Minjia SHI, Xiaoxiao LI, Z. SEPASDAR, et al. *Polycyclic codes as invariant subspaces*. Finite Fields Appl., 2020, **68**: 101760, 14 pp.

[22] T. ABUALRUB, I. SIAP. *Reversible cyclic codes over $\mathbb{Z}_4$*. Australas. J. Combin., 2007, **38**: 195–205.

[23] N. AYDIN, T. ASAMOV. *Table of the Z4 Database* [Online]. Available: http://www.asamov.com/Z4Codes/CODES/ShowCODESTablePage.aspx.