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On the (m, r, s)-Halves of a Riordan Array and Applications

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Abstract Given a Riordan array, its vertical half and horizontal half are studied separately before. In the present paper, we introduce the (m, r, s)-halves of a Riordan array. This allows us to discuss the vertical half and horizontal half in a uniform context. As applications, we find several new identities involving Fibonacci, Pell and Jacobsthal sequences by applying the (m, r, s)-halves of Pascal and Delannoy matrices.

Keywords Riordan array; central coefficients; Pascal matrix; Delannoy matrix; Fibonacci numbers; Pell numbers; Jacobsthal numbers

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1. Introduction

Finding some new identities [1–4] is a very important problem in combinatorics. In this paper, we use Riordan arrays and (m, r, s)-halves of a Riordan array to find some identities. We begin by reviewing some facts about Riordan arrays. An infinite lower triangular matrix $G = (g_{n,k})_{n,k\in\mathbb{N}}$ is called a Riordan array if its column k has generating function $d(t)h(t)^k$, where $d(t) = \sum_{n=0}^{\infty} d_n t^n$ and $h(t) = \sum_{n=1}^{\infty} h_n t^n$ are formal power series with $d_0 \neq 0$ and $h_1 \neq 0$. The Riordan array corresponding to the pair $d(t)$ and $h(t)$ is denoted by $(d(t), h(t))$, and its generic entry is $g_{n,k} = [t^n]d(t)h(t)^k$, where $[t^n]$ denotes the coefficient operator. The set of all Riordan arrays forms a group under ordinary row-by-column product with the multiplication identity $(1, t)$, called the Riordan group. The multiplication rule of Riordan arrays is given by

$$(d(t), h(t))(g(t), f(t)) = (d(t)g(h(t)), f(h(t))). \tag{1.1}$$

If $(b_n)_{b\in\mathbb{N}}$ is any sequence having $b(t) = \sum_{n=0}^{\infty} b_n t^n$ as its generating function, then for every Riordan array $(d(t), h(t)) = (g_{n,k})_{n,k\in\mathbb{N}}$

$$\sum_{k=0}^{n} g_{n,k} b_k = [t^n]d(t)b(h(t)). \tag{1.2}$$

This is called the fundamental theorem of Riordan arrays [5–8] and it can be rewritten as

$$(d(t), h(t))b(t) = d(t)b(h(t)). \tag{1.3}$$

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For example, the Pascal matrix $P = \left(\binom{n}{k}\right)_{n,k\geq 0}$ is the element $(\frac{1}{1-t}, \frac{t}{1-t})$ of the Riordan group and Delannoy matrix can be expressed as $(\frac{1}{1-t}, \frac{t+t^2}{1-t})$ (see [9,10]), which are registered as sequence A007318 and A008288 in OEIS [11], respectively. In the sequel, sequences are frequently referred to by their OEIS number.

Most studies on the Riordan matrices were related to combinatorics [6–8,12–15] or algebraic structures [5,16–19]. The vertical halves of Riordan arrays and the horizontal halves of Riordan arrays were introduced in Yang et al. [18,20,21] and Barry [6,9,16], respectively.

Definition 1.1 *Let $G = (p(t), tq(t)) = (g_{n,k})_{n,k\geq 0}$ be a Riordan array.*

(i) The central coefficients of the Riordan array G are the elements $g_{2n,n}$;

(ii) The vertical half of G is defined as the infinite lower triangular matrix $(v_{n,k})_{n,k\geq 0}$ with general (n,k)-th term $v_{n,k} = g_{2n-k,n}$;

(iii) The horizontal half of G is defined as the infinite lower triangular matrix $(h_{n,k})_{n,k\geq 0}$ with general (n,k)-th term $h_{n,k} = g_{2n,n+k}$.

The following (m,r)-vertical halves of Riordan arrays and the (m,r)-horizontal halves of Riordan arrays were introduced in Yang et al. [12,18,22].

Definition 1.2 *Let $G = (p(t), tq(t)) = (g_{n,k})_{n,k\geq 0}$ be a Riordan array and let $m > r \geq 0$ be integers.*

(i) The (m,r)-central coefficients of $G = (g_{n,k})_{n,k\in\mathbb{N}}$ are the entries $g_{(m+1)n+r,mn+r}$;

(ii) The (m,r)-vertical half of G is defined as the matrix $G^{[m,r]}$ with general (n,k)-th term $g_{(m+1)n+r-k,mn+r}$;

(iii) The (m,r)-horizontal half of G is defined as the matrix $G^{(m,r)}$ with general (n,k)-th term $g_{(m+1)n+r,mn+k+r}$.

Obviously, the $(1,0)$-vertical half is the vertical half and the $(1,0)$-horizontal half is the horizontal half. In [18,22], the following results are obtained.

Lemma 1.3 *Let $G = (p(t), tq(t)) = (g_{n,k})_{n,k\geq 0}$ be a Riordan array and let $f(t)$ be the generating function defined by the functional equation $f(t) = tq(f(t))^m$. Then we have*

(i) The (m,r)-vertical half of G is given by

$$G^{[m,r]} = (\frac{tf'(t)p(f(t))q(f(t))^r}{f(t)}, f(t)). \tag{1.4}$$

(ii) The (m,r)-horizontal half of G is given by

$$G^{(m,r)} = (\frac{p(f(t))q(f(t))^r}{1 - mtq(f(t))^{m-1}q'(f(t))}, tq(f(t))^{m+1}). \tag{1.5}$$

In [23,24], He introduced the vertical half Riordan array operator (VHRAO) Ψ and the horizontal half Riordan array operator (HHRAO) $\widehat{\Psi}$ as follows:

$$\Psi : (p(t), tq(t)) \to (\frac{tf'(t)p(f(t))}{f(t)}, f(t)), \tag{1.6}$$

$$\widehat{\Psi} : (p(t), tq(t)) \to (\frac{tf'(t)p(f(t))}{f(t)}, tq(f(t))^2), \tag{1.7}$$

where $f(t)$ is the compositional inverse of $\frac{t}{q(t)}$, i.e., $f(t)$ is determined by the functional equation $f(t) = tq(f(t))$.

In this paper, we will introduce the (m, r, s)-halves $G^{(m,r,s)}$ of a Riordan array $G = (g_{n,k})_{n,k\geq 0}$, and the definition will be presented in the next section. We will give characterizations for the iteration of vertical and horizontal half Riordan array transformation operators by using the (m, r, s)-half Riordan array. In Section 3, we study (m, r, s)-half Riordan arrays of Delannoy matrix and show that (m, r, s)-half of Delannoy matrix $G = (\frac{1}{1-t}, \frac{t+t^2}{1-t})$ can be represented in terms of the generating function $R(t)$ of $(m + 1)$-Schröder numbers, which satisfies the equation $R(t) = 1 + tR(t)^m + tR(t)^{m+1}$. In Section 4, we show that (m, r, s)-half of Pascal matrix $G = (\frac{1}{1-t}, \frac{t}{1-t})$ can be represented in terms of the generating function $\mathcal{B}_{m+1}(t)$ of $(m+1)$-Catalan numbers, which satisfies the equation $\mathcal{B}_{m+1}(t) = 1 + t\mathcal{B}_{m+1}(t)^{m+1}$. Several new identities involving Fibonacci, Jacobsthal and Pell sequences are obtained by applying the vertical halves of Pascal and Delannoy matrices, respectively.

2. The (m, r, s)-halves of a Riordan array

In this section, we will introduce and study the (m, r, s)-halves of a Riordan array.

Definition 2.1 *Let $G = (p(t), tq(t)) = (g_{n,k})_{n,k\geq 0}$ be a Riordan array and let $m > r \geq 0$ be integers and s a positive fractional number such that ms is integral number. The (m, r, s)-half of G is defined as the matrix $G^{(m,r,s)}$ with general (n, k)-th term $g_{(m+1)n+(ms-m-1)k+r,mn+(ms-m)k+r}$.*

Example 2.2 Choosing $m = 1$ and $r = 0$, we have $G^{(1,0,s)} = (g_{2n+(s-2)k,n+(s-1)k})$. In particular,

(i) $G^{(1,0,1)} = (g_{2n-k,n})$ is the vertical half of G;

(ii) $G^{(1,0,2)} = (g_{2n,n+k})$ is the horizontal half of G;

(iii) $G^{(1,0,3)} = (g_{2n+k,n+2k})$;

(iv) $G^{(1,0,4)} = (g_{2n+2k,n+3k})$.

Example 2.3 Choosing $s = 1$ or $s = \frac{m+1}{m}$, we have

(i) $G^{(m,r,1)}$ is the (m, r)-vertical half of G;

(ii) $G^{(m,r,\frac{m+1}{m})}$ is the (m, r)-horizontal half of G.

Theorem 2.4 *Let $G = (p(t), tq(t)) = (g_{n,k})_{n,k\geq 0}$ be a Riordan array and let $f(t)$ be the generating function defined by the functional equation $f(t) = tq(f(t))^m$. Then the (m, r, s)-half Riodran array of G is given by*

$$G^{(m,r,s)} = (\frac{p(f(t))q(f(t))^r}{1 - mtq(f(t))^{m-1}q'(f(t))}, tq(f(t))^{ms}) \tag{2.1}$$

$$= (\frac{tf'(t)p(f(t))q(f(t))^r}{f(t)}, t(\frac{f(t)}{t})^s). \tag{2.2}$$

Proof Considering the relation $f(t) = tq(f(t))^m$ and using the Lagrange inversion formula [25], we have

$$\begin{aligned}
&[t^n]\frac{p(f(t))q(f(t))^r}{1-mtq(f(t))^{m-1}q'(f(t))}(tq(f(t))^{ms})^k\\
&=[t^n]\frac{p(f(t))q(f(t))^{m+r}q(f(t))^{(ms)k}}{q(f(t))^m-mf(t)q(f(t))^{m-1}q'(f(t))}(\frac{f(t)}{q(f(t))^m})^k\\
&=[t^n]\frac{p(t)q(t)^{m+m(s-1)k+r}t^k}{q(t)^m-mtq(t)^{m-1}q'(t)}q(t)^{mn-m}(q(t)^m-mtq(t)^{m-1}q'(t))\\
&=[t^{n-k}]p(t)q(t)^{mn+m(s-1)k+r}\\
&=[t^{(m+1)n+(ms-m-1)k+r}]p(t)(tq(t))^{mn+m(s-1)k+r}\\
&=g_{(m+1)n+(ms-m-1)k+r,mn+(ms-m)k+r}.
\end{aligned}$$

Hence the proof follows. □

Theorem 2.5 *Let $G=(p(t),tq(t))=(g_{n,k})_{n,k\geq 0}$ be a Riordan array. Then we have*

$$(G^{(m,r,s)})^{-1}=(1,\overline{tq(t)^{ms-m}})(\frac{q(t)-mtq'(t)}{p(t)q(t)^{r+1}},\frac{t}{q(t)^m}),\tag{2.3}$$

where $\overline{tq(t)^{ms-m}}$ is the composition inverse of $tq(t)^{ms-m}$.

Proof Let $f(t)$ be the generating function defined by the functional equation $f(t)=tq(f(t))^m$. By the above theorem, we can obtain the following decomposition.

$$\begin{aligned}
G^{(m,r,s)}&=(\frac{p(f(t))q(f(t))^r}{1-mtq(f(t))^{m-1}q'(f(t))},tq(f(t))^{ms})\\
&=(\frac{p(f(t))q(f(t))^r}{1-mtq(f(t))^{m-1}q'(f(t))},f(t))(1,\bar{f}\cdot q(t)^{ms})\\
&=(\frac{p(f(t))q(f(t))^r}{1-mtq(f(t))^{m-1}q'(f(t))},f(t))(1,\frac{t}{q(t)^m}q(t)^{ms})\\
&=(\frac{p(f(t))q(f(t))^r}{1-mtq(f(t))^{m-1}q'(f(t))},f(t))(1,tq(t)^{ms-m})\\
&=G^{(m,r,1)}(1,tq(t)^{ms-m}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(G^{(m,r,s)})^{-1}&=(1,tq(t)^{ms-m})^{-1}(G^{(m,r,1)})^{-1}\\
&=(1,\overline{tq(t)^{ms-m}})(\frac{q(t)-mtq'(t)}{p(t)q(t)^{r+1}},\frac{t}{q(t)^m}),
\end{aligned}$$

where we used the fact [18]

$$(G^{(m,r,1)})^{-1}=(\frac{q(t)-mtq'(t)}{p(t)q(t)^{r+1}},\frac{t}{q(t)^m}).$$

This completes the proof. □

Theorem 2.6 *Let the VHRA operator Ψ be defined by (1.6) and let $\Psi^m=\Psi\Psi^{m-1}$ for $m\geq 2$, with $\Psi^1=\Psi$. Then, for any Riordan array $G=(g_{n,k})_{n,k\geq 0}$*

$$\Psi^m G=G^{(m,0,\frac{1}{m})}.\tag{2.4}$$

Proof We will give an inductive proof for (2.4). From Theorem 2.4 we obtain (2.4) for $m=1$. Assume (2.4) holds for m, that is

$$\Psi^m G = G^{(m,0,\frac{1}{m})}.$$

If we denote by $h_{n,k}$ the (n,k)-th entry of $\Psi^m G$, then $h_{n,k} = g_{(m+1)n-mk,mn+(1-m)k}$. Let $\Psi^{m+1}G = (l_{n,k})_{n,k\in\mathbb{N}}$. Then $l_{n,k} = h_{2n-k,n} = g_{(m+2)n-(m+1)k,(m+1)n-mk}$. This implies that $\Psi^{m+1}G = G^{(m+1,0,\frac{1}{m+1})}$. Hence, (2.4) is also true for $m+1$, completing the proof of (2.4). □

Theorem 2.7 *Let the HHRA operator $\widehat{\Psi}$ be defined by (1.7) and let $\widehat{\Psi}^m = \widehat{\Psi}\widehat{\Psi}^{m-1}$ for $m \geq 2$, with initial $\widehat{\Psi}^1 = \widehat{\Psi}$. Then, for any Riordan array $G = (g_{n,k})_{n,k\geq 0}$, we have*

$$\widehat{\Psi}^m G = G^{(2^m-1,0,1+\frac{1}{2^m-1})}. \tag{2.5}$$

Proof The proof is similar to that of Theorem 2.6. □

3. Halves of Pascal matrix

For any integer $m \geq 0$, the m-Catalan numbers or Fuss-Catalan numbers [13, 26–28] are defined by the formula

$$C_n^{(m)} = \frac{1}{mn+1}\binom{mn+1}{n}, \quad n = 0,1,2,\ldots. \tag{3.1}$$

The generating function $\mathcal{B}_m(t) = \sum_{n=0}^{\infty} \frac{1}{mn+1}\binom{mn+1}{n} t^n$ satisfies the functional equation

$$\mathcal{B}_m(t) = 1 + t\mathcal{B}_m(t)^m. \tag{3.2}$$

It can be checked in [15, 27] that the following identities are valid

$$\mathcal{B}_m(t)^s = \sum_{n=0}^{\infty} \frac{s}{mn+s}\binom{mn+s}{n} t^n, \tag{3.3}$$

$$\frac{\mathcal{B}_m(t)^{s+1}}{1-(m-1)t\mathcal{B}_m(t)^m} = \sum_{n=0}^{\infty}\binom{mn+s}{n} t^n, \tag{3.4}$$

$$\mathcal{B}_{m-s}(t\mathcal{B}_m(t)^s) = \mathcal{B}_m(t). \tag{3.5}$$

Theorem 3.1 *The (m,r,s)-half of Pascal matrix $G = (\frac{1}{1-t}, \frac{t}{1-t})$ is*

$$G^{(m,r,s)} = (\frac{\mathcal{B}_{m+1}(t)^{r+1}}{1-mt\mathcal{B}_{m+1}(t)^{m+1}}, t\mathcal{B}_{m+1}(t)^{ms}).$$

Proof For the Riordan array $G = (\frac{1}{1-t}, \frac{t}{1-t})$, $p(t) = q(t) = \frac{1}{1-t}$. If $f(t)$ is determined by $f(t) = tq(f(t))^m$, then

$$f(t) = \frac{t}{(1-f(t))^m}, \frac{f(t)}{1-f(t)} = \frac{t}{(1-f(t))^{m+1}}, \frac{1}{1-f(t)} = 1 + \frac{t}{(1-f(t))^{m+1}}.$$

By (3.2), we have

$$\frac{1}{1-f(t)} = \mathcal{B}_{m+1}(t), f(t) = t\mathcal{B}_{m+1}(t)^m.$$

Let $G^{(m,r,s)} = (d(t), h(t))$. Then, from Theorem 2.4, we get

$$d(t) = f'(t)p(f(t))(\frac{f(t)}{t})^{\frac{r-m}{m}} = \frac{\mathcal{B}_{m+1}(t)^{r+1}}{1 - mt\mathcal{B}_{m+1}(t)^{m+1}}$$

and $h(t) = t(\frac{f(t)}{t})^s = t\mathcal{B}_{m+1}(t)^{ms}$. From which the conclusion follows. □

Theorem 3.2 *Let $G = (\frac{1}{1-t}, \frac{t}{1-t})$. Then*

$$(G^{(m,r,s)})^{-1} = (1, t(1-t)^{m-ms})^{-1}((1-(m+1)t)(1-t)^r, t(1-t)^m).$$

Proof From [18], we know that $(G^{(m,r,1)})^{-1} = ((1-(m+1)t)(1-t)^r, t(1-t)^m)$. Hence, using Theorem 2.5, we have

$$\begin{aligned}(G^{(m,r,s)})^{-1} &= (1, \frac{t}{(1-t)^{ms-m}})^{-1}(G^{(m,r,1)})^{-1} \\ &= (1, t(1-t)^{m-ms})^{-1}((1-(m+1)t)(1-t)^r, t(1-t)^m).\end{aligned}$$

This completes the proof. □

Corollary 3.3 *The $(m, 0, \frac{k}{m})$-half of Pascal matrix $G = (\frac{1}{1-t}, \frac{t}{1-t})$ is*

$$G^{(m,0,\frac{k}{m})} = (\frac{\mathcal{B}_{m+1}(t)}{1 - mt\mathcal{B}_{m+1}(t)^{m+1}}, t\mathcal{B}_{m+1}(t)^k)$$

and its inverse is given by

$$(G^{(m,0,\frac{k}{m})})^{-1} = ((m+1)\mathcal{B}_{m-k+1}(t)^{-1} - m, t\mathcal{B}_{m-k+1}(t)^{-k}).$$

Corollary 3.4 *Denote $C(t) = \mathcal{B}_2(t) = \frac{1-\sqrt{1-4t}}{2t}$ and $(t) = \frac{\mathcal{B}_2(t)}{1-t\mathcal{B}_2(t)^2} = \frac{1}{\sqrt{1-4t}}$. Then, we have*

$$\begin{aligned}G^{(1,0,1)} &= (B(t), tC(t)), \\ G^{(1,0,2)} &= (B(t), tC(t)^2), \\ G^{(1,0,3)} &= (B(t), tC(t)^3), \\ G^{(1,1,1)} &= (B(t)C(t), tC(t)), \\ G^{(1,1,2)} &= (B(t)C(t), tC(t)^2), \\ G^{(1,1,3)} &= (B(t)C(t), tC(t)^3).\end{aligned}$$

In [12], by applying $G^{(1,0,2)} = (B(t), tC(t)^2)$ and $G^{(1,1,2)} = (B(t)C(t), tC(t)^2)$, Brietzke provides a new proof of some identities obtained by Andrews in [29], namely

$$F_n = \sum_{i=-\infty}^{\infty} (-1)^i \binom{n-1}{\lfloor \frac{1}{2}(n-1-5i) \rfloor}, \tag{3.6}$$

$$F_n = \sum_{i=-\infty}^{\infty} (-1)^i \binom{n}{\lfloor \frac{1}{2}(n-1-5i) \rfloor}, \tag{3.7}$$

where F_n are Fibonacci numbers. The Fibonacci numbers $(F_n)_{n\in\mathbb{N}}$ (A000045) (see [30]) are defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

In this section, we use the vertical half of the Pascal matrix to propose and prove some identities involving the Fibonacci numbers, Jacobsthal numbers and binomial coefficients.

Theorem 3.5 *For $n \geq 0$, we have*

$$F_{3n+1} = \sum_{j=0}^{n} \binom{2n-j}{n} (F_{2j} + F_{j-1}), \tag{3.8}$$

$$F_{3n+2} = \sum_{j=0}^{n} \binom{2n-j}{n} (F_{2j+1} + F_{j}), \tag{3.9}$$

$$F_{3n+3} = \sum_{j=0}^{n} \binom{2n-j}{n} (F_{2j+2} + F_{j+1}). \tag{3.10}$$

Proof Consider the vertical half of the Pascal matrix, it is the Riordan array $G^{(1,0,1)} = (B(t), tC(t))$, with (n,k)-th entry being $g_{2n-k,n} = \binom{2n-k}{n}$. The inverse is given by $(G^{(1,0,1)})^{-1} = (1-2t, t(1-t))$.

Since

$$(1-2t, t(1-t))\frac{1-t}{1-4t-t^2} = \frac{(1-2t)(1-t+t^2)}{(1-3t+t^2)(1-t-t^2)},$$

$$(1-2t, t(1-t))\frac{1+t}{1-4t-t^2} = \frac{(1-2t)(1+t-t^2)}{(1-3t+t^2)(1-t-t^2)},$$

$$(1-2t, t(1-t))\frac{2}{1-4t-t^2} = \frac{2(1-2t)}{(1-3t+t^2)(1-t-t^2)},$$

we can get that

$$(B(t), tC(t))\frac{(1-2t)(1-t+t^2)}{(1-3t+t^2)(1-t-t^2)} = \frac{1-t}{1-4t-t^2}, \tag{3.11}$$

$$(B(t), tC(t))\frac{(1-2t)(1+t-t^2)}{(1-3t+t^2)(1-t-t^2)} = \frac{1+t}{1-4t-t^2}, \tag{3.12}$$

$$(B(t), tC(t))\frac{2(1-2t)}{(1-3t+t^2)(1-t-t^2)} = \frac{2}{1-4t-t^2}. \tag{3.13}$$

From the following partial decomposition

$$\frac{(1-2t)(1-t+t^2)}{(1-3t+t^2)(1-t-t^2)} = \frac{t}{1-3t+t^2} + \frac{1}{1-t-t^2} - \frac{t}{1-t-t^2},$$

we have

$$[t^n]\frac{(1-2t)(1-t+t^2)}{(1-3t+t^2)(1-t-t^2)} = F_{2n} + F_{n+1} - F_n = F_{2n} + F_{n-1}.$$

Thus $\frac{(1-2t)(1-t+t^2)}{(1-3t+t^2)(1-t-t^2)}$ is the generation function of sequence $(F_{2n} + F_{n-1})_{n\in\mathbb{N}}$. In the same way we obtain that $\frac{(1-2t)(1+t-t^2)}{(1-3t+t^2)(1-t-t^2)}$ is the generation function of the sequence $(F_n + F_{2n+1})_{n\in\mathbb{N}}$ (A087124), and $\frac{2(1-2t)}{(1-3t+t^2)(1-t-t^2)}$ is the generation function of the sequence $(F_n + F_{2n})_{n\in\mathbb{N}}$ (A051450). Hence, from (1.2) and Eqs. (3.11)–(3.13), and using Corollary 3.3, we obtain our results (3.8)–(3.10), respectively. □

The Jacobsthal numbers J_n are defined recursively as follows [11]

$$J_{n+1} = J_n + 2J_{n-1}, \quad n \geq 1; \; J_0 = 0, \; J_1 = 1.$$

The generating function of Jacobsthal sequence is $J(t)=\sum_{n=0}^{\infty}J_nt^n=\frac{t}{1-t-2t^2}$. Using the vertical half of the Pascal matrix, we derive the following identities involving the Jacobsthal numbers.

Theorem 3.6 *For $n\geq 0$, we have*

$$J_{2n+2}=\sum_{j=0}^{n}\sum_{i=0}^{\lfloor\frac{j}{3}\rfloor}\binom{2n-j}{n}\binom{j+2}{3i+2}; \tag{3.14}$$

$$J_{2n+1}=\sum_{j=0}^{n}\sum_{i=0}^{\lfloor\frac{j+1}{3}\rfloor}\binom{2n-j}{n}\binom{j+1}{3i}. \tag{3.15}$$

Proof It is known [11,31] that $\sum_{n=0}^{\infty}J_{2n+2}t^n=\frac{1}{1-5t+4t^2}$ and $\sum_{n=0}^{\infty}J_{2n+1}t^n=\frac{1-2t}{1-5t+4t^2}$. Let $\frac{1}{(1-2t)(1-t+t^2)}=\sum_{n=0}^{\infty}g_nt^n$ and $\frac{1-2t+2t^2}{(1-2t)(1-t+t^2)}=\sum_{n=0}^{\infty}\bar{g}_nt^n$. Then

$$g_n=\sum_{i=0}^{\lfloor\frac{n}{3}\rfloor}\binom{n+2}{3i+2}\quad\text{and}\quad\bar{g}_n=\sum_{i=0}^{\lfloor\frac{n+1}{3}\rfloor}\binom{n+1}{3i}.$$

By a straightforward computation we get

$$(1-2t,t(1-t))\frac{1}{1-5t+4t^2}=\frac{1}{(1-2t)(1-t+t^2)},$$
$$(1-2t,t(1-t))\frac{1-2t}{1-5t+4t^2}=\frac{1-2t+2t^2}{(1-2t)(1-t+t^2)},$$

which is equivalent to

$$(B(t),tC(t))\frac{1}{(1-2t)(1-t+t^2)}=\frac{1}{1-5t+4t^2},$$
$$(B(t),tC(t))\frac{1-2t+2t^2}{(1-2t)(1-t+t^2)}=\frac{1-2t}{1-5t+4t^2},$$

from which (3.14) and (3.15) follow. □

Theorem 3.7 *For $n\geq 0$, we have*

$$\sum_{j=0}^{n}\binom{2n-j}{n}=\binom{2n+1}{n}, \tag{3.16}$$

$$\sum_{j=0}^{n}\binom{2n-j}{n}2^j=4^n. \tag{3.17}$$

Proof By using the identities $C(t)=\frac{1}{1-tC(t)}$ and $B(t)=\frac{1}{1-2tC(t)}$, we have

$$(B(t),tC(t))\frac{1}{1-t}=B(t)C(t),\ (B(t),tC(t))\frac{1}{(1-2t)}=\frac{1}{1-4t}.$$

So the results follow by the fundamental theorem of Riordan arrays. □

Theorem 3.8 *For $n\geq 0$, we have*

$$\sum_{j=0}^{n}(-1)^j\binom{2n+j}{n+2j}=1+\sum_{j=1}^{n-1}\binom{2j}{j-1}. \tag{3.18}$$

Proof By using the identities $C(t) = 1 + tC(t)^2$, we have

$$\begin{aligned}(B(t), tC(t)^3)\frac{1}{1+t} &= \frac{B(t)}{1+tC(t)^3} = \frac{B(t)}{1+C(t)(C(t)-1)}\\ &= \frac{B(t)}{1+C(t)^2-C(t)} = \frac{B(t)}{C(t)^2-tC(t)^2}\\ &= \frac{B(t)C(t)^{-2}}{1-t}.\end{aligned}$$

From (3.4), $[t^i]B(t)C(t)^{-2} = \binom{2i-2}{i}$. Thus,

$$[t^n]\frac{B(t)C(t)^{-2}}{1-t} = \sum_{i=0}^{n}\binom{2i-2}{i} = 1 + \sum_{j=1}^{n-1}\binom{2j}{j-1}.$$

By Corollary 3.4, we know that the general entry of $(B(t), tC(t)^3)$ is $\binom{2n+k}{n+2k}$. Then the result follows by the fundamental theorem of Riordan arrays. □

Note that the sequence $(1 + \sum_{j=1}^{n-1}\binom{2j}{j-1})_{n\geq 0}$ is registered as A279561 in OEIS [11], which counts the number of inversion sequences avoiding the patterns 021 and 120 (see [32, 33]).

Theorem 3.9 *For $n \geq 0$, we have*

$$\sum_{j=0}^{n}(-1)^j\binom{2n+j+1}{n+2j+1} = 1 + \frac{1}{2}\sum_{j=1}^{n}\binom{2j}{j}. \tag{3.19}$$

Proof By using the identities $C(t) = 1 + tC(t)^2$, we have

$$\begin{aligned}(B(t)C(t), tC(t)^3)\frac{1}{1+t} &= \frac{B(t)C(t)}{1+tC(t)^3} = \frac{B(t)C(t)}{1+C(t)(C(t)-1)}\\ &= \frac{B(t)C(t)}{1+C(t)^2-C(t)} = \frac{B(t)C(t)}{C(t)^2-tC(t)^2}\\ &= \frac{B(t)C(t)^{-1}}{1-t}.\end{aligned}$$

From (3.4), $[t^i]B(t)C(t)^{-1} = \binom{2i-1}{i}$. Then we can obtain that

$$[t^n]\frac{B(t)C(t)^{-1}}{1-t} = \sum_{i=0}^{n}\binom{2i-1}{i} = 1 + \sum_{i=1}^{n}\binom{2i-1}{i} = 1 + \frac{1}{2}\sum_{i=1}^{n}\binom{2i}{i}.$$

We also have that the general entry of $(B(t)C(t), tC(t)^3)$ is $\binom{2n+k+1}{n+2k+1}$ by Corollary 3.4. Thus the result follows by the fundamental theorem of Riordan arrays. □

Note that the sequence $(1 + \sum_{i=1}^{n}\binom{2i-1}{i})_{n\geq 0}$ is registered as A024718 in OEIS [11], which counts the total number of leaves in all rooted ordered trees with at most n edges [34]. It also counts the number of UH-free Schröeder paths of semilength n with horizontal steps only at level less than two [35].

4. Halves of Delannoy matrix

Let $p(t) = \frac{1}{1-t}$ and $q(t) = \frac{1+t}{1-t}$. Then $G = (p(t), tq(t)) = (\frac{1}{1-t}, \frac{t+t^2}{1-t})$ is the Delannoy matrix [10, 21, 36, 37]. If $f(t) = tq(f(t))^m$, then $f(t) = t(\frac{1+f(t)}{1-f(t)})^m$. We let $R(t) = \frac{1+f(t)}{1-f(t)}$. Then

$f(t)=tR(t)^m$ and $R(t)$ satisfies the equation $R(t)=1+tR(t)^m+tR(t)^{m+1}$. From [38], $R(t)$ is the generating function of $(m+1)$-Schröder numbers. Using this generating function, we have the following characterization for the (m,r,s)-half of $G=(\frac{1}{1-t},\frac{t+t^2}{1-t})$.

Theorem 4.1 *The (m,r,s)-half of the Delannoy matrix $G=(\frac{1}{1-t},\frac{t+t^2}{1-t})$ is given by*

$$G^{(m,r,s)}=(\frac{(1-tR(t)^m)R(t)^r}{(1-tR(t)^m)^2-2mtR(t)^{m-1}},tR(t)^{ms})$$

and its inverse can be factorized as

$$(G^{(m,r,s)})^{-1}=(1,t(\frac{1-t}{1+t})^{m-ms})^{-1}(\frac{(1-t)^r(1-2mt-t^2)}{(1+t)^{r+1}},\frac{t(1-t)^m}{(1+t)^m}).$$

Proof Let $p(t)=\frac{1}{1-t}$ and $q(t)=\frac{1+t}{1-t}$. If $f(t)=tq(f(t))^m$, then $f(t)=t(\frac{1+f(t)}{1-f(t)})^m$. Let $R(t)=q(f(t))=\frac{1+f(t)}{1-f(t)}$. Then $f(t)=tR(t)^m$ and $R(t)$ satisfies the equation $R(t)=1+tR(t)^m+tR(t)^{m+1}$. Therefore, from Theorems 2.4 and 2.5, we get that

$$\begin{aligned}G^{(m,r,s)}&=(\frac{p(f(t))q(f(t))^r}{1-mtq(f(t))^{m-1}q'(f(t))},tq(f(t))^{ms})\\&=(\frac{(1-tR(t)^m)R(t)^r}{(1-tR(t)^m)^2-2mtR(t)^{m-1}},tR(t)^{ms}),\\(G^{(m,r,s)})^{-1}&=(1,\overline{tq(t)^{ms-m}})(G^{(m,r,1)})^{-1}\\&=(1,\overline{(\frac{t(1+t)}{1-t})^{ms-m}})(\frac{(1-t)^r(1-2mt-t^2)}{(1+t)^{r+1}},\frac{t(1-t)^m}{(1+t)^m})\\&=(1,t(\frac{1-t}{1+t})^{m-ms})^{-1}(\frac{(1-t)^r(1-2mt-t^2)}{(1+t)^{r+1}},\frac{t(1-t)^m}{(1+t)^m}),\end{aligned}$$

this completes the proof. □

Theorem 4.2 *The $(m,0,\frac{1}{m})$-half of the Delannoy matrix $G=(\frac{1}{1-t},\frac{t+t^2}{1-t})$ is given by*

$$G^{(m,0,\frac{1}{m})}=(\frac{1-tR(t)^m}{(1-tR(t)^m)^2-2mtR(t)^{m-1}},tR(t))$$

and its inverse can be factorized as

$$(G^{(m,0,\frac{1}{m})})^{-1}=(1,t(\frac{1-t}{1+t})^{m-1})^{-1}(\frac{(1-2mt-t^2)}{(1+t)},\frac{t(1-t)^m}{(1+t)^m}).$$

The Delannoy matrix $D=(\frac{1}{1-t},\frac{t+t^2}{1-t})$ has the (n,k)-th entry $d_{n,k}=\sum_{j=0}^{n-k}\binom{k}{j}\binom{n-j}{k}$. It is well-known that $\sum_{k=0}^{n}d_{n,k}=P_{n+1}$, where the Pell numbers [39] P_n are defined by

$$\sum_{n=0}^{\infty}P_nt^n=\frac{t}{1-2t-t^2}.$$

The $(1,0,1)$-half of Delannoy matrix is given by

$$G^{(1,0,1)}=(\frac{1}{\sqrt{1-6t+t^2}},\frac{1-t-\sqrt{1-6t+t^2}}{2}),$$

its generic element is $d_{2n-k,n}=\sum_{j=0}^{n-k}\binom{n}{j}\binom{2n-k-j}{n}$ and

$$(G^{(1,0,1)})^{-1}=(\frac{1-2t-t^2}{1+t},\frac{t(1-t)}{1+t}).$$

Theorem 4.3 *For $n \geq 0$, we have*

$$P_{2n+2} = \sum_{k=0}^{n} d_{2n-k,n}(2P_{k+1} - 2P_k), \tag{4.1}$$

$$P_{2n+1} = \sum_{k=0}^{n} d_{2n-k,n}(P_{k+1} + P_{k-1}), \tag{4.2}$$

where $P_{-1} = P_0 = 0$.

Proof Since

$$\left(\frac{1-2t-t^2}{1+t}, \frac{t(1-t)}{1+t}\right)\frac{2t}{1-6t+t^2} = \frac{2t-2t^2}{1-2t-t^2},$$

$$\left(\frac{1-2t-t^2}{1+t}, \frac{t(1-t)}{1+t}\right)\frac{1-t}{1-6t+t^2} = \frac{1+t^2}{1-2t-t^2},$$

we have

$$\left(\frac{1}{\sqrt{1-6t+t^2}}, \frac{1-t-\sqrt{1-6t+t^2}}{2}\right)\frac{2t-2t^2}{1-2t-t^2} = \frac{2t}{1-6t+t^2},$$

$$\left(\frac{1}{\sqrt{1-6t+t^2}}, \frac{1-t-\sqrt{1-6t+t^2}}{2}\right)\frac{1+t^2}{1-2t-t^2} = \frac{1-t}{1-6t+t^2}.$$

Hence, from the generating functions

$$\frac{2t-2t^2}{1-2t-t^2} = \sum_{n=0}^{\infty}(2P_{n+1} - 2P_n)t^n$$

and

$$\frac{1+t^2}{1-2t-t^2} = \sum_{n=0}^{\infty}(P_{n+1} + P_{n-1})t^n,$$

as well as (1.2), we arrive at the desired results. □

Theorem 4.4 *For $n \geq 0$, we have*

$$Q_{2n} = \sum_{k=0}^{n} d_{2n-k,n}(Q_k - Q_{k-1} - 0^k), \tag{4.3}$$

$$Q_{2n+1} = \sum_{k=0}^{n} d_{2n-k,n}(Q_k + Q_{k-1} - 0^k), \tag{4.4}$$

where the Pell-Lucas numbers Q_n are defined by $\sum_{n=0}^{\infty} Q_n t^n = \frac{1-t}{1-2t-t^2}$, with $Q_{-1} = 0$.

Proof Making use of (1.3), we obtain

$$\left(\frac{1-2t-t^2}{1+t}, \frac{t(1-t)}{1+t}\right)\frac{2-3t}{1-6t+t^2} = \frac{1-2t+3t^2}{1-2t-t^2},$$

$$\left(\frac{1-2t-t^2}{1+t}, \frac{t(1-t)}{1+t}\right)\frac{1+t}{1-6t+t^2} = \frac{1+2t-t^2}{1-2t-t^2},$$

which are equivalent to

$$\left(\frac{1}{\sqrt{1-6t+t^2}}, \frac{1-t-\sqrt{1-6t+t^2}}{2}\right)\frac{1-2t+3t^2}{1-2t-t^2} = \frac{2-3t}{1-6t+t^2},$$

$$(\frac{1}{\sqrt{1-6t+t^2}},\frac{1-t-\sqrt{1-6t+t^2}}{2})\frac{1+2t-t^2}{1-2t-t^2}=\frac{1+t}{1-6t+t^2}.$$

It can be verified that the generating functions

$$\frac{1-2t+3t^2}{1-2t-t^2}=\sum_{n=0}^{\infty}(Q_k-Q_{k-1}-0^k)t^n$$

and

$$\frac{1+2t-t^2}{1-2t-t^2}=\sum_{n=0}^{\infty}(Q_k+Q_{k-1}-0^k)t^n.$$

Hence, the results follow from (1.2). □

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