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The Spectral Properties of p-Sombor (Laplacian) Matrix of Graphs

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Abstract The Sombor index, which was recently introduced into chemical graph theory, can predict physico-chemical properties of molecules. In this paper, we investigate the properties of (p-)Sombor index from an algebraic viewpoint. The p-Sombor matrix $\mathcal{S}_p(G)$ is the square matrix of order n whose (i,j)-entry is equal to $((d_i)^p+(d_j)^p)^{\frac{1}{p}}$ if $v_i\sim v_j$, and 0 otherwise, where d_i denotes the degree of vertex v_i in G. The matrix generalizes the famous Zagreb matrix ($p=1$), Sombor matrix ($p=2$) and inverse sum index matrix ($p=-1$). In this paper, we find a pair of p-Sombor noncospectral equienergetic graphs and determine some bounds for the p-Sombor (Laplacian) spectral radius. Then we describe the properties of connected graphs with k distinct p-Sombor Laplacian eigenvalues. At last, we determine the Sombor spectrum of some special graphs. As a by-product, we determine the spectral properties of Sombor matrix ($p=2$), Zagreb matrix ($p=1$) and inverse sum index matrix ($p=-1$).

Keywords p-Sombor matrix; p-Sombor Laplacian matrix; p-Sombor spectrum

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1. Introduction

Let $G=(V,E)$ be a simple graph with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and edge set $E(G)$. The degree $d(v_i)$ (or d_i) of vertex v_i denotes number of edges connecting with vertex v_i. We use $e=v_iv_j$ to denote edge connecting vertex v_i and vertex v_j. Let K_n, S_n, K_{n_1,n_2}, S_{n_1,n_2} denote complete graph, star graph, complete bipartite graph, and double star graph, respectively. $G\setminus\{e\}$ denotes deleting the edge e in graph G. We refer to [1] for all notations and terminologies utilized but not defined in this article.

As we know, the adjacent matrix $A(G)=[a_{ij}]_{n\times n}$ is defined as

$$a_{ij}=\begin{cases}1, & v_iv_j\in E(G),\\ 0, & \text{otherwise.}\end{cases}$$

Let $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$ be the adjacent eigenvalues of the graph G.

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The (adjacent) energy of a simple graph is introduced by Ivan Gutman [2]

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The Laplacian matrix is defined as follows: $L(G) = D(G) - A(G)$, where $D(G)$ is the degree of diagonal matrix. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the Laplacian eigenvalues of the graph G.

Recently, Gutman proposed the novel index, Sombor index [3], which is defined as

$$\mathrm{SO}(G) = \sum_{v_i v_j \in E(G)} \sqrt{d_i^2 + d_j^2}.$$

The Sombor index can help to exert modest discriminative potential and predict physico-chemical properties of molecules [4]. We can also see [5–8] for more details about Sombor index.

Soon after, Réti, Došlić and Ali introduced the p-Sombor index [9], which is defined as

$$\mathrm{SO}_p(G) = \sum_{v_i v_j \in E(G)} ((d_i)^p + (d_j)^p)^{\frac{1}{p}}.$$

The p-Sombor matrix [10] is defined as $\mathcal{S}_p = \mathcal{S}_p(G) = [s_{ij}^p]_{n\times n}$ $(p \neq 0)$, where

$$s_{ij}^p = \begin{cases} ((d_i)^p + (d_j)^p)^{\frac{1}{p}}, & v_i v_j \in E(G); \\ 0, & \text{otherwise}. \end{cases}$$

Let $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ be the eigenvalues of $\mathcal{S}_p$. The p-Sombor matrix generalizes the famous Zagreb matrix ($p = 1$), Sombor matrix ($p = 2$) and inverse sum index matrix ($p = -1$). The spectral properties of these matrices can be found in [11–14].

The p-Sombor energy [10] is defined as

$$S_pE(G) = \sum_{i=1}^{n} |\theta_i|.$$

The p-Sombor Laplacian matrix [10] is defined as

$$\mathcal{L}_p(G) = \mathcal{D}_p(G) - \mathcal{S}_p(G),$$

where

$$\mathcal{D}_p(G) = \mathrm{Diag}\Big(\sum_{j=1}^{n} s_{1j}^p, \sum_{j=1}^{n} s_{2j}^p, \ldots, \sum_{j=1}^{n} s_{nj}^p\Big).$$

Let $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$ be the eigenvalues of $\mathcal{L}_p(G)$. By giving a direction of every edge of G, we can obtain the directed graph $\mathcal{D}$. Let $S_R(G) = [s_{ie}]_{n\times m}$ be the association matrix of directed graph $\mathcal{D}$, where

$$s_{ie} = \begin{cases} ((d_i)^p + (d_j)^p)^{\frac{1}{2p}}, & e = \overrightarrow{v_j v_i}; \\ -((d_i)^p + (d_j)^p)^{\frac{1}{2p}}, & e = \overrightarrow{v_i v_j}; \\ 0, & \text{otherwise}. \end{cases}$$

Thus, we always have $\mathcal{L}_p(G) = S_R(G)[S_R(G)]^{\mathrm{T}}$.

Much work about the weighted adjacency matrix [15] had been considered, such as ABC matrix [16], sum-connectivity Laplacian matrix [17], extended adjacency matrix [18], p-Sombor matrix [10]. Motivated by above results, it is also interesting to obtain more spectral properties of p-Sombor (Laplacian) matrix.

2. Bounds of p-Sombor spectral radius

In [10], Liu et al. obtained the upper bounds of p-Sombor spectral radius ($p \geq 1$) among trees with n vertices. It should be noted that in [10, Theorem 4.7], they missed the condition $p \geq 1$.

Theorem 2.1 ([10]) *Let $G \in T_n$ and $\theta_1(G)$ be the p-Sombor spectral radius of G. If $p \geq 1$, then $\theta_1(G) \leq \theta_1(K_{1,n-1})$ with equality iff $G \cong K_{1,n-1}$.*

Let $C = (c_{ij})$, $D = (d_{ij})$ be real matrix. If $c_{ij} \leq d_{ij}$ for $1 \leq i, j \leq n$, we denote $C \preceq D$. In the following, we consider more bounds of p-Sombor spectral radius $\theta_1(G)$ with given parameters.

Theorem 2.2 *Let $G \in U_n$ with maximum degree Δ. Then $\theta_1(G) \leq 2^{\frac{1}{p}+1}(n-1)\sqrt{\Delta - 1}$ with equality iff $G \cong C_3$.*

Proof By the properties of p-Sombor matrix, $\mathcal{S}_p \preceq 2^{\frac{1}{p}}(n-1)A$, thus $\theta_1 \leq 2^{\frac{1}{p}}(n-1)\lambda_1$. Since $G \in U_n$, then $\lambda_1(G) \leq 2\sqrt{\Delta - 1}$, with equality iff $G \cong C_n$ (see [19]). Thus $\theta_1 \leq 2^{\frac{1}{p}+1}(n-1)\sqrt{\Delta - 1}$.

If $G \cong C_3$, then $\theta_1(C_3) = 2^{\frac{1}{p}+2} = 2^{\frac{1}{p}+1}(n-1)\sqrt{\Delta - 1}$. If $\theta_1(G) = 2^{\frac{1}{p}+1}(n-1)\sqrt{\Delta - 1}$, then $G \cong C_n$, then $d_u = 2$ for $u \in V(G)$. Then $2^{\frac{1}{p}+1}(n-1) = \theta_1(C_n) = 2^{\frac{1}{p}+1}\lambda_1(C_n) = 2^{\frac{1}{p}+2}$, thus $n = 3$, i.e., $G \cong C_3$. □

Theorem 2.3 *Let G be a connected graph with $|V(G)| = n$, $|E(G)| = m$. Then $\theta_1 \leq 2^{\frac{1}{p}}(n-1)\sqrt{2m - n + 1}$ with equality iff $G \cong K_n$.*

Proof By the properties of p-Sombor matrix, $\mathcal{S}_p(G) \preceq 2^{\frac{1}{p}}(n-1)A(G)$, thus $\theta_1 \leq 2^{\frac{1}{p}}(n-1)\lambda_1$. Since $\lambda_1 \leq \sqrt{2m - n + 1}$ with equality iff $G \cong K_{1,n-1}$ or $G \cong K_n$ (see [20]). Thus $\theta_1 \leq 2^{\frac{1}{p}}(n-1)\sqrt{2m + 1 - n}$ with equality iff $G \cong K_n$. □

Lemma 2.4 ([10]) *Let G be a graph with maximum degree (resp., minimum degree) Δ (resp., δ). Then $2^{\frac{1}{p}}\delta\lambda_1 \leq \theta_1 \leq 2^{\frac{1}{p}}\Delta\lambda_1$ with equality iff G is regular graph.*

Since $\delta \leq \lambda_1 \leq \Delta$, by Lemma 2.4, we immediately have

Corollary 2.5 *Let G be a graph with maximum degree (resp., minimum degree) Δ (resp., δ). Then $2^{\frac{1}{p}}\delta^2 \leq \theta_1 \leq 2^{\frac{1}{p}}\Delta^2$ with equality iff G is regular graph.*

Let U_n^* denote the collection of unicyclic even graphs. $U_4^{n-4} = C_4 \cdot S_{n-3}$ is the graphs obtained from cycle C_4 and star S_{n-3} by identifying one vertex of C_4 and the central of S_{n-3}.

Lemma 2.6 ([21]) *Let $G \in U_n^*$ ($n \geq 4$). Then $\lambda_1(G) \leq \lambda_1(U_4^{n-4})$, where $\lambda_1(U_4^{n-4})$ is the maximal root of polynomial $x^4 - nx^2 + 2n - 8$.*

By Lemma 2.6, we have $\theta_1(G) \leq 2^{\frac{1}{p}}(n-1)\lambda_1(G) \leq 2^{\frac{1}{p}}(n-1)\lambda_1(U_4^{n-4})$. Thus

Theorem 2.7 *Let $G \in U_n^*$ ($n \geq 4$). Then we have $\theta_1 \leq 2^{\frac{1}{p}}(n-1)\lambda_1(U_4^{n-4})$ with equality iff*

$G \cong C_4$. And $\lambda_1(U_4^{n-4})$ is the maximal root of polynomial $x^4 - nx^2 + 2n - 8$.

Let $U(l,\Delta)$ be the graphs obtained by connecting $\Delta - 2$ pendent edges to every vertex of C_l.

Lemma 2.8 ([22]) *Let $G \in U(l,\Delta)$. Then $\lambda_1(G) = 1 + \sqrt{\Delta - 1}$.*

Since $\theta_1(G) \le 2^{\frac{1}{p}}\Delta\lambda_1(G)$, by Lemma 2.8, we have

Theorem 2.9 *Let $G \in U(l,\Delta)$. Then $\theta_1(G) \le 2^{\frac{1}{p}}\Delta(1 + \sqrt{\Delta - 1})$ with equality iff $G \cong C_l$.*

3. The p-Sombor equienergetic graphs

The joint graph $G_1 \vee G_2$ is derived from vertices of G_1, G_2 and connecting each of the vertices of G_1 and the vertices of G_2.

Theorem 3.1 *Let G_1 be an r_1-regular graph with $|V(G_1)| = n_1$, G_2 be an r_2-regular graph with $|V(G_2)| = n_2$. Then the p-Sombor eigenvalues of $G_1 \vee G_2$ are*

$$\begin{gathered}
2^{\frac{1}{p}}(r_1 + n_2)\lambda_i(G_1), \quad i = 2,3,\ldots,n_1;\\
2^{\frac{1}{p}}(r_2 + n_1)\lambda_i(G_2), \quad i = 2,3,\ldots,n_2;\\
2^{\frac{1}{p}-1}((n_1 + r_2)r_2 + (n_2 + r_1)r_1) \pm \\
\sqrt{2^{\frac{2}{p}-2}((n_1 + r_2)r_2 - (n_2 + r_1)r_1)^2 + n_1n_2((n_1 + r_2)^p + (n_2 + r_1)^p)^{\frac{2}{p}}}.
\end{gathered}$$

Proof By the definition of p-Sombor matrix, we have

$$\mathcal{S}_p(G_1 \vee G_2) = \begin{pmatrix} 2^{\frac{1}{p}}(r_1 + n_2)A(G_1) & ((r_1 + n_2)^p + (r_2 + n_1)^p)^{\frac{1}{p}} J_{n_1 \times n_2} \\ ((r_1 + n_2)^p + (r_2 + n_1)^p)^{\frac{1}{p}} J_{n_2 \times n_1} & 2^{\frac{1}{p}}(r_2 + n_1)A(G_2) \end{pmatrix}.$$

Thus, by the properties of quotient matrix, the p-Sombor eigenvalues of $G_1 \vee G_2$ are

$$\begin{gathered}
2^{\frac{1}{p}}(r_1 + n_2)\lambda_i(G_1), \quad i = 2,3,\ldots,n_1;\\
2^{\frac{1}{p}}(r_2 + n_1)\lambda_i(G_2), \quad i = 2,3,\ldots,n_2;\\
2^{\frac{1}{p}-1}((n_1 + r_2)r_2 + (n_2 + r_1)r_1) \pm \\
\sqrt{2^{\frac{2}{p}-2}((n_1 + r_2)r_2 - (n_2 + r_1)r_1)^2 + n_1n_2((n_1 + r_2)^p + (n_2 + r_1)^p)^{\frac{2}{p}}}. \quad \square
\end{gathered}$$

By the definition of p-Sombor energy and Theorem 3.1, then

Theorem 3.2 *Let G_1 be an r_1-regular graph with $|V(G_1)| = n_1$, G_2 be an r_2-regular graph with $|V(G_2)| = n_2$. Then the p-Sombor energy of $G_1 \vee G_2$ is*

$$\begin{gathered}
2^{\frac{1}{p}}(r_1 + n_2)\mathcal{E}(G_1) + 2^{\frac{1}{p}}(r_2 + n_1)\mathcal{E}(G_2) - 2^{\frac{1}{p}}(r_1 + n_2)r_1 - 2^{\frac{1}{p}}(r_2 + n_1)r_2 + \\
\sqrt{2^{\frac{2}{p}}((n_1 + r_2)r_2 - (n_2 + r_1)r_1)^2 + 4n_1n_2((n_1 + r_2)^p + (n_2 + r_1)^p)^{\frac{2}{p}}}.
\end{gathered}$$

We call G and G' (non-isomorphic) equienergetic if $|V(G)| = n = |V(G')|$, G' and G have no identical spectra and they have the same energy. Similarly, we call G and G' (non-isomorphic) p-Sombor equienergetic if $|V(G)| = n = |V(G')|$, G' and G have no identical p-Sombor spectra and they have the same p-Sombor energy. We find some p-Sombor equienergetic graphs.

Theorem 3.3 *The graphs $G_1 \vee K_t$ and $G_2 \vee K_t$ are p-Sombor equienergetic graphs, where G_1 and G_2 are depicted in Figure 1.*

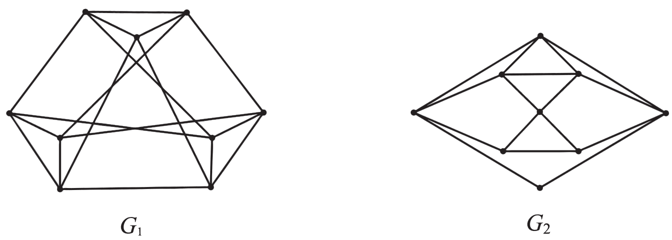

Figure 1 Two equienergetic 4-regular graphs

Proof The graphs G_1, G_2 are 4-regular equienergetic graphs which was introduced in [23]. $\mathcal{E}(G_2) = 16 = \mathcal{E}(G_1)$, $S_pE(G_2) = 2^{\frac{1}{p}+6} = S_pE(G_1)$. By Theorem 3.2,

$$\begin{aligned} S_pE(G_1 \vee K_t) =& S_pE(G_2 \vee K_t) = 3 \times 2^{\frac{1}{p}+2}(t+4) + 2^{\frac{1}{p}}(t+8)(t-1) + \\ & \sqrt{2^{\frac{2}{p}}(t^2+3t-24)^2 + 36t((t+4)^p + (t+8)^p)^{\frac{2}{p}}}. \end{aligned}$$

Thus, the graphs $G_1 \vee K_t$ and $G_2 \vee K_t$ are p-Sombor noncospectral equienergetic graphs. □

Corollary 3.4 *(i) The p-Sombor spectrum of K_{n_1,n_2} is $\{0^{[n_1+n_2-2]}, \pm\sqrt{n_1n_2}(n_1^p + n_2^p)^{\frac{1}{p}}\}$.*

(ii) The p-Sombor spectrum of S_n is $\{0^{[n-2]}, \pm\sqrt{n-1}((n-1)^p + 1)^{\frac{1}{p}}\}$.

(iii) The p-Sombor spectrum of $CS_{w,n-w} = K_w \vee \overline{K_{n-w}}$ (i.e., complete split graph) is

$$\begin{aligned} &\{0^{[n-1-w]}, -2^{\frac{1}{p}}(n-1)^{[w-1]}, 2^{\frac{1}{p}-1}(w-1)(n-1) \pm \\ &\sqrt{2^{\frac{2}{p}-2}(w-1)^2(n-1)^2 + w(n-w)(w^p + (n-1)^p)^{\frac{2}{p}}}\}. \end{aligned}$$

By Corollary 3.4, we can obtain the p-Sombor energy of these special graphs.

Corollary 3.5 *(i) The p-Sombor energy of K_{n_1,n_2} is $2\sqrt{n_1n_2}(n_1^p + n_2^p)^{\frac{1}{p}}$.*

(ii) The p-Sombor energy of S_n is $2\sqrt{n-1}(1 + (n-1)^p)^{\frac{1}{p}}$.

(iii) The p-Sombor energy of $CS_{w,n-w}$ is

$$2^{\frac{1}{p}}(w-1)(n-1) + 2\sqrt{2^{\frac{2}{p}-2}(w-1)^2(n-1)^2 + w(n-w)(w^p + (n-1)^p)^{\frac{2}{p}}}.$$

Note that when $p = 2$, we can obtain the Sombor spectrum, Sombor energy of S_n and K_{n_1,n_2} which is the results of [24, Theorem 2.8 and 2.6].

Note that $K_n \setminus \{e\} \cong CS_{n-2,2}$ for any $e = uv \in K_n$. By Case (iii) of Corollary 3.5, we can calculate the Sombor energy of $K_n \setminus \{e\}$. We have $S_2E(K_n) = 2\sqrt{2}(n-1)^2$ and

$$S_2E(K_n \setminus \{e\}) = \sqrt{2}(n-3)(n-1) + \sqrt{2}\sqrt{(n-3)^2(n-1)^2 + 4(n-2)((n-2)^2 + (n-1)^2)}.$$

Thus $S_2E(K_n) > S_2E(K_n \setminus \{e\})$ $(n \geq 3)$. We partially answer the problem proposed by Ghanbari in [24] about what is the relationship between $S_2E(G)$ and $S_2E(G \setminus \{e\})$.

4. Spectral properties of p-Sombor Laplacian matrices

In the following, we determine some spectral properties of p-Sombor Laplacian matrices.

Proposition 4.1 *The p-Sombor Laplacian matrix $\mathcal{L}_p(G)$ is a positive semi-definite matrix.*

Proof Let $\mathcal{L}_p = \mathcal{D}_p - \mathcal{S}_p$, where $\mathcal{D}_p(G) = \mathrm{Diag}(\sum_{j=1}^n s_{1j}^p, \sum_{j=1}^n s_{2j}^p, \ldots, \sum_{j=1}^n s_{nj}^p)$.

$$[\mathcal{S}_p(G)]_{ij} \triangleq s_{ij}^p = \begin{cases} ((d_i)^p + (d_j)^p)^{\frac{1}{p}}, & v_iv_j \in E(G); \\ 0, & \text{otherwise.} \end{cases}$$

Let $\varphi = (\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_n))^{\mathrm{T}}$ be the unit column vector with respect to vertices $V(G) = \{v_1, v_2, \ldots, v_n\}$. There exists a real number λ such that $\lambda\varphi(x) = (\mathcal{L}_p(G)\varphi)(x)$. Since

$$\begin{aligned} &\varphi^{\mathrm{T}}\mathcal{L}_p(G)\varphi = \varphi^{\mathrm{T}}\mathcal{D}_p(G)\varphi - \varphi^{\mathrm{T}}\mathcal{S}_p(G)\varphi, \\ &\varphi^{\mathrm{T}}\mathcal{S}_p(G)\varphi = \sum_{v_i\sim v_j} s_{ij}^p\varphi(v_i)\varphi(v_j), \\ &\varphi^{\mathrm{T}}\mathcal{D}_p(G)\varphi = \sum_{i=1}^n \Big(\sum_{k=1}^n s_{ik}^p)\varphi^2(v_i\Big) = \sum_{v_i\sim v_j} s_{ij}^p(\varphi^2(v_i) + \varphi^2(v_j)). \end{aligned}$$

Thus

$$\varphi^{\mathrm{T}}\mathcal{L}_p(G)\varphi = \sum_{v_i\sim v_j} s_{ij}^p(\varphi(v_i) - \varphi(v_j))^2 \geq 0. \quad \square$$

By Proposition 4.1, we have that the multiplicity of zero among p-Sombor Laplacian eigenvalues is equal to the number of connected components.

Similar to the incidence matrix $R(G)$, for matrix $S_R(G)$, we also have

Lemma 4.2 *Let G be a connected graph with $|V(G)| = n$. Then* $\mathrm{rank}(S_R(G)) = n - 1$.

Proof Let $x = (x_1, x_2, \ldots, x_n)$ be a vector, such that $x^{\mathrm{T}}S_R(G) = \mathbf{0}$. Then for any $v_i \sim v_j$, and $v_i \to v_j$ in the digraph D, we have $-x_i(d_i^p + d_j^p)^{\frac{1}{2p}} + x_j(d_j^p + d_i^p)^{\frac{1}{2p}} = 0$. Thus $x_i = x_j$ for $v_i \sim v_j$. Since G is a connected graph, we have $x_1 = x_2 = \cdots = x_n$. Then $x = k(1, 1, \ldots, 1)^{\mathrm{T}}$. Thus $\mathrm{rank}(S_R(G)) \geq n - 1$. By the definition of $S_R(G)$, we know the rows of $S_R(G)$ are linearly dependent, such that $\mathrm{rank}(S_R(G)) \leq n - 1$. Thus we have $\mathrm{rank}(S_R(G)) = n - 1$. $\square$

By Lemma 4.2, $\mathrm{rank}(\mathcal{L}_p(G)) = \mathrm{rank}(S_R(G)[S_R(G)]^{\mathrm{T}}) = \mathrm{rank}(S_R(G)) = n - k$, where k denotes the number of connected components. Thus, multiplicity of 0 as an eigenvalue of $\mathcal{L}_p$ is equal to the number of components in G.

In the following, we describe the properties of connected graphs with k distinct p-Sombor Laplacian eigenvalues. The proof is similar to that of [16, Lemma 2.2], the detail is omitted.

Theorem 4.3 *Let G be a connected graph with $|V(G)| = n$. Then $\mathcal{L}_p(G)$ has k $(2 \leq k \leq n)$ distinct eigenvalues iff there exist $k - 1$ distinct non-zero numbers $t_1, t_2, \ldots, t_{k-1}$ such that*

$$\prod_{i=1}^{k-1}(\mathcal{L}_p(G) - t_iI) = (-1)^{k-1}\frac{\prod_{i=1}^{k-1} t_i}{n}J,$$

where J denotes all-one matrix, I is a unit matrix.

An (s,t)-semiregular bipartite graph is a bipartite graph (X,Y) with $|X|=s$, $|X|=t$ and the degree of each vertex in the same partite set is the same.

Theorem 4.4 *(i) If G is a k-regular graph, then $\eta_i(G)=2^{\frac{1}{p}}k\mu_i(G)$, $i=1,2,\ldots,n$. Specifically, if G is K_n, $\eta_n(K_n)=0$,*

$$\eta_1(K_n)=\eta_2(K_n)=\cdots=\eta_{n-1}(K_n)=2^{\frac{1}{p}}(n-1)n.$$

If $G\cong C_n$, then

$$\eta_i(C_n)=2^{\frac{1}{p}+2}(1-\cos\frac{2\pi i}{n}),\quad i=0,1,\ldots,n-1.$$

(ii) If G is an (s,t)-semiregular bipartite graph, then $\eta_i(G)=(s^p+t^p)^{\frac{1}{p}}\mu_i(G)$, $i=1,2,\ldots,n$. Specifically, if G is $K_{a,b}$ $(a+b=n, a\geq b)$,

$$\eta_1(K_{a,b})=n(a^p+b^p)^{\frac{1}{p}},\ \eta_2(K_{a,b})=\eta_3(K_{a,b})=\cdots=\eta_b(K_{a,b})=a(a^p+b^p)^{\frac{1}{p}},$$

$$\eta_{b+1}(K_{a,b})=\eta_{b+2}(K_{a,b})=\cdots=\eta_{n-1}(K_{a,b})=b(a^p+b^p)^{\frac{1}{p}}$$

and $\eta_n(K_{a,b})=0$.

Proof (i) Since G is a k-regular graph, then $\mathcal{L}_p(G)=2^{\frac{1}{p}}kL(G)$, thus $\eta_i=2^{\frac{1}{p}}k\mu_i$. If $G\cong K_n$, then $\mu_1(K_n)=\mu_2(K_n)=\cdots=\mu_{n-1}(K_n)=n$, $\mu_n(K_n)=0$. If $G\cong C_n$, then $\mu_i(C_n)=2(1-\cos\frac{2\pi i}{n})$, $i=0,1,\ldots,n-1$.

(ii) If G is an (s,t)-semiregular bipartite graph, then $\mathcal{L}_p(G)=(s^p+t^p)^{\frac{1}{p}}L(G)$, thus $\eta_i(G)=(s^p+t^p)^{\frac{1}{p}}\mu_i(G)$, $i=1,2,\ldots,n$. If $G\cong K_{a,b}$ $(a+b=n, a\geq b)$, then $n=\mu_1(K_{a,b})$, $\mu_2(K_{a,b})=\mu_3(K_{a,b})=\cdots=\mu_b(K_{a,b})=a$, $\mu_{b+1}(K_{a,b})=\mu_{b+2}(K_{a,b})=\cdots=\mu_{n-1}(K_{a,b})=b$ and $\mu_n(K_{a,b})=0$. $\square$

Theorem 4.5 *Let G be a graph with $|V(G)|=n$. Then G has exactly one p-Sombor Laplacian eigenvalue iff $G\cong nK_1$.*

Proof Since $\mathcal{L}_p(G)$ is a positive semi-definite matrix (by Proposition 4.1), then $\eta_n(G)=0$. If G has one p-Sombor Laplacian eigenvalue, $\eta_1=\eta_2=\cdots=\eta_n=0$, then $\mathcal{L}_p(G)=\mathbf{0}$, thus $G\cong nK_1$. If $G\cong nK_1$, then $\mathcal{L}_p(G)=\mathbf{0}$, thus $\eta_1=\eta_2=\cdots=\eta_n=0$. $\square$

Theorem 4.6 *Let G be a connected graph with $|V(G)|=n(\geq 2)$. Then G has two distinct p-Sombor Laplacian eigenvalues iff $G\cong K_n$.*

Proof By Theorem 4.3, we know G has two distinct p-Sombor Laplacian eigenvalues iff there exists non-zero number t_1 such that $\mathcal{L}_p(G)-t_1I=-\frac{t_1}{n}J$. Then off-diagonal entries of $\mathcal{L}_p(G)$ are not 0, thus $G\cong K_n$. If $G\cong K_n$, by Theorem 4.4, K_n has two p-Sombor Laplacian eigenvalues 0 and $2^{\frac{1}{p}}(n-1)n$. $\square$

Lemma 4.7 ([10]) *Let G be a connected graph with $|V(G)|=n$. Then $\mathrm{SO}_p\leq 2^{\frac{1}{p}-1}n(n-1)^2$ with equality iff $G\cong K_n$.*

By Lemma 4.7, we can obtain the sharp bounds of the second minimal p-Sombor Laplacian eigenvalue η_{n-1}.

Theorem 4.8 *Let G be a connected graph with $|V(G)| = n$. Then $\eta_{n-1} \leq 2^{\frac{1}{p}}(n-1)n$ with equality iff $G \cong K_n$.*

Proof Since $\eta_n + \sum_{i=1}^{n-1} \eta_i = 2\mathrm{SO}_p(G)$, then $\eta_{n-1} \leq \frac{2}{n-1}\mathrm{SO}_p(G)$. By Lemma 4.7,

$$\mathrm{SO}_p(G) \leq \binom{n}{2} 2^{\frac{1}{p}}(n-1) = 2^{\frac{1}{p}-1}n(n-1)^2.$$

Thus $\eta_{n-1}(G) \leq 2^{\frac{1}{p}}n(n-1)$ with equality iff $G \cong K_n$. $\square$

5. The Sombor spectrum of some special graphs

In the following, we introduce a way of matrix decomposition introduced in [25]. Let M be an $n \times n$ symmetric matrix given in (5.1) where block $E \in R^{t\times t}$, block $\gamma \in R^{t\times s}$, block $F \in R^{s\times s}$, block $Q \in R^{s\times s}$ and $n = zs + t$, z denotes number of copies of F. Denote by $\sigma(X)$ spectrum of matrix X, and $\sigma^k(X)$ the multiset with k copies of $\sigma(X)$.

$$M = \begin{pmatrix} E & \gamma & \gamma & \cdots & \gamma \\ \gamma^T & F & Q & \cdots & Q \\ \gamma^T & Q & F & \cdots & Q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma^T & Q & Q & \cdots & F \end{pmatrix} \tag{5.1}$$

Lemma 5.1 ([25]) *Let M be the matrix in (5.1). Then*

(i) The spectrum $\sigma(F-Q) \subseteq \sigma(M)$ with multiplicity $z-1$, where z denotes the number of copies of the block in matrix M.

(ii) The spectrum $\sigma(M) \setminus \sigma^{z-1}(F-Q) = \sigma(M')$ is the set of remaining $s+t$ eigenvalues of M, and

$$M' = \begin{pmatrix} E & \sqrt{z}\gamma \\ \sqrt{z}\gamma^T & F + (z-1)Q \end{pmatrix}.$$

The conclusion of Lemma 5.1 means that $\sigma(M) = \sigma^{z-1}(F-Q) \bigcup \sigma(M')$.

In the following, we consider the Sombor spectrum of the graphs star plus an edge S_n^+, double star S_{n_1,n_2} $(n_1 + n_2 + 2 = n)$ and $K_{n_1,n_2} \setminus \{e\}$.

Theorem 5.2 *The Sombor spectrum of S_n^+ consists of $0^{[n-4]}$ and the zeros of polynomial $x^4 - (n^3 - 3n^2 + 4n + 12)x^2 - 4\sqrt{2}(n^2 - 2n + 5)x + 8(n-3)(n^2 - 2n + 2)$.*

Proof The Sombor matrix of S_n^+ is

$$\begin{bmatrix} 0 & 2\sqrt{2} & \sqrt{(n-1)^2+4} & 0 & 0 & \cdots & 0 \\ 2\sqrt{2} & 0 & \sqrt{(n-1)^2+4} & 0 & 0 & \cdots & 0 \\ \sqrt{(n-1)^2+4} & \sqrt{(n-1)^2+4} & 0 & \sqrt{(n-1)^2+1} & \sqrt{(n-1)^2+1} & \cdots & \sqrt{(n-1)^2+1} \\ 0 & 0 & \sqrt{(n-1)^2+1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{(n-1)^2+1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \sqrt{(n-1)^2+1} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Let

$$E = \begin{bmatrix} 0 & 2\sqrt{2} & \sqrt{(n-1)^2+4} \\ 2\sqrt{2} & 0 & \sqrt{(n-1)^2+4} \\ \sqrt{(n-1)^2+4} & \sqrt{(n-1)^2+4} & 0 \end{bmatrix},$$

$$\gamma = \begin{bmatrix} 0 \\ 0 \\ \sqrt{(n-1)^2+1} \end{bmatrix},$$

$F = \mathbf{0}$ and $Q = \mathbf{0}$. By Lemma 5.1, 0 is the Sombor eigenvalues of S_n^+ with multiplicity $n-4$. For the remaining Sombor eigenvalues, we need to consider the following matrix

$$\begin{bmatrix} 0 & 2\sqrt{2} & \sqrt{(n-1)^2+4} & 0 \\ 2\sqrt{2} & 0 & \sqrt{(n-1)^2+4} & 0 \\ \sqrt{(n-1)^2+4} & \sqrt{(n-1)^2+4} & 0 & \sqrt{n-3}\sqrt{(n-1)^2+1} \\ 0 & 0 & \sqrt{n-3}\sqrt{(n-1)^2+1} & 0 \end{bmatrix}. \tag{5.2}$$

It is easy to calculate the characteristic polynomial of (5.2). The remaining four Sombor eigenvalues are zeros of $x^4-(n^3-3n^2+4n+12)x^2-4\sqrt{2}(n^2-2n+5)x+8(n-3)(n^2-2n+2)$. $\square$

Theorem 5.3 *The Sombor spectrum of S_{n_1,n_2} consists of $0^{[n_1+n_2-2]}$ and the zeros of polynomial $x^4-((n_1+1)^3+(n_2+1)^3+n_1+n_2)x^2+n_1n_2((n_1+1)^2+1)((n_2+1)^2+1)$.*

Proof The Sombor matrix of S_{n_1,n_2} is

$$\begin{bmatrix} 0 & u & \sqrt{(n_1+1)^2+1} & \cdots & \sqrt{(n_1+1)^2+1} & 0 & \cdots & 0 \\ u & 0 & 0 & \cdots & 0 & \sqrt{(n_2+1)^2+1} & \cdots & \sqrt{(n_2+1)^2+1} \\ \sqrt{(n_1+1)^2+1} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{(n_1+1)^2+1} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{(n_2+1)^2+1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \sqrt{(n_2+1)^2+1} & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $u=\sqrt{(n_1+1)^2+(n_2+1)^2}$. Let

$$E = \begin{bmatrix} 0 & \sqrt{(n_1+1)^2+(n_2+1)^2} & \sqrt{(n_1+1)^2+1} & \cdots & \sqrt{(n_1+1)^2+1} \\ \sqrt{(n_1+1)^2+(n_2+1)^2} & 0 & 0 & \cdots & 0 \\ \sqrt{(n_1+1)^2+1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{(n_1+1)^2+1} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$\gamma=\begin{bmatrix}0\\ \sqrt{(n_2+1)^2+1}\\ 0\\ \vdots\\ 0\end{bmatrix},$$

$F=\mathbf{0}$ and $Q=\mathbf{0}$. By Lemma 5.1, 0 is the Sombor eigenvalues of S_{n_1,n_2} with multiplicity n_2-1. For the remaining Sombor eigenvalues, we need to consider the following matrix

$$\begin{bmatrix}0 & \sqrt{(n_1+1)^2+(n_2+1)^2} & \sqrt{(n_1+1)^2+1} & \cdots & \sqrt{(n_1+1)^2+1} & 0\\ \sqrt{(n_1+1)^2+(n_2+1)^2} & 0 & 0 & \cdots & 0 & \sqrt{n_2}\sqrt{(n_2+1)^2+1}\\ \sqrt{(n_1+1)^2+1} & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \sqrt{(n_1+1)^2+1} & 0 & 0 & \cdots & 0 & 0\\ 0 & \sqrt{n_2}\sqrt{(n_2+1)^2+1} & 0 & \cdots & 0 & 0\end{bmatrix}. \tag{5.3}$$

We do a primary row transformation and column transformation for the matrix of (5.3), and obtain the following matrix (5.4)

$$\begin{bmatrix}0 & \sqrt{(n_1+1)^2+(n_2+1)^2} & 0 & \sqrt{(n_1+1)^2+1} & \cdots & \sqrt{(n_1+1)^2+1}\\ \sqrt{(n_1+1)^2+(n_2+1)^2} & 0 & \sqrt{n_2}\sqrt{(n_2+1)^2+1} & 0 & \cdots & 0\\ 0 & \sqrt{n_2}\sqrt{(n_2+1)^2+1} & 0 & 0 & \cdots & 0\\ \sqrt{(n_1+1)^2+1} & 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \sqrt{(n_1+1)^2+1} & 0 & 0 & 0 & \cdots & 0\end{bmatrix}. \tag{5.4}$$

By Lemma 5.1, 0 is the Sombor eigenvalues of S_{n_1,n_2} with multiplicity n_1-1. For the remaining Sombor eigenvalues, we need to consider the following matrix

$$\begin{bmatrix}0 & \sqrt{(n_1+1)^2+(n_2+1)^2} & 0 & \sqrt{n_1}\sqrt{(n_1+1)^2+1}\\ \sqrt{(n_1+1)^2+(n_2+1)^2} & 0 & \sqrt{n_2}\sqrt{(n_2+1)^2+1} & 0\\ 0 & \sqrt{n_2}\sqrt{(n_2+1)^2+1} & 0 & 0\\ \sqrt{n_1}\sqrt{(n_1+1)^2+1} & 0 & 0 & 0\end{bmatrix}. \tag{5.5}$$

It is easy to calculate the characteristic polynomial of (5.5). The remaining four Sombor eigenvalues are zeros of $x^4-((n_1+1)^3+(n_2+1)^3+n_1+n_2)x^2+n_1n_2((n_2+1)^2+1)((n_1+1)^2+1)$. $\Box$

Similar to the proof of Theorems 5.2 and 5.3, we also have the following result.

Theorem 5.4 *The Sombor spectrum of $K_{n_1,n_2}\setminus\{e\}$ consists of $0^{[n_1+n_2-4]}$ and the zeros of polynomial*

$x^4-\{(n_1-1)(n_1^2+n_2^2)(n_2-1)+(n_1-1)(n_2^2+(n_1-1)^2)+(n_1^2+(n_2-1)^2)(n_2-1)\}x^2+(n_1-1)(n_1^2+(n_2-1)^2)(n_2-1)((n_1-1)^2+n_2^2)$.

By Case (i) of Corollary 3.5, we have

$$S_2E(K_{n_1,n_2})=2\sqrt{n_1n_2}\sqrt{n_1^2+n_2^2}.$$

Suppose that $\pm x_1$, $\pm x_2$ are zeros of $x^4-\{(n_1-1)(n_1^2+n_2^2)(n_2-1)+(n_1-1)(n_2^2+(n_1-1)^2)+(n_1^2+(n_2-1)^2)(n_2-1)\}x^2+(n_1-1)(n_1^2+(n_2-1)^2)(n_2-1)((n_1-1)^2+n_2^2)$ of Theorem 5.4. Then

$$x_1^2+x_2^2=(n_1-1)(n_1^2+n_2^2)(n_2-1)+(n_1-1)(n_2^2+(n_1-1)^2)+(n_1^2+(n_2-1)^2)(n_2-1).$$

$$x_1^2\cdot x_2^2=(n_1-1)(n_1^2+(n_2-1)^2)(n_2-1)((n_1-1)^2+n_2^2).$$

Thus

$$\begin{aligned}(S_2E(K_{n_1,n_2}\setminus\{e\}))^2&=4(|x_1|+|x_2|)^2\\&=4\{(n_1-1)(n_1^2+n_2^2)(n_2-1)+((n_1-1)^2+n_2^2)(n_1-1)+(n_1^2+(n_2-1)^2)(n_2-1)+\\&\quad 2\sqrt{(n_1-1)(n_1^2+(n_2-1)^2)(n_2-1)((n_1-1)^2+n_2^2)}\}.\end{aligned}$$

It is difficult to compare the $S_2E(K_{n_1,n_2})$ and $S_2E(K_{n_1,n_2}\setminus\{e\})$ for any n_1 and n_2. But for special n_1 and n_2, we can compare them. For example, we let $G\cong K_{1,3}$. Then

$$S_2E(K_{1,3})=2\sqrt{3}\sqrt{10}=2\sqrt{30},$$

$$S_2E(K_{1,3}\setminus\{e\})=2\sqrt{10},$$

thus

$$S_2E(K_{1,3})>S_2E(K_{1,3}\setminus\{e\}).$$

Problem 5.5 *Whether it is true that $S_2E(K_{n_1,n_2})>S_2E(K_{n_1,n_2}\setminus\{e\})$ for any K_{n_1,n_2}.*

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