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On the Distance Signless Laplacian Spectral Radius of Bicyclic Graphs

Nannan XU, Aimei YU*
School of Mathematics and Stastics, Beijing Jiaotong University, Beijing 100044*, P. R. China*

Abstract The distance signless Laplacian matrix of a connected graph G is defined as $\mathcal{Q}(G) = Tr(G) + D(G)$, where $Tr(G)$ is the diagonal matrix of the vertex transmissions in G and $D(G)$ is the distance matrix of G. The largest eigenvalue of the distance signless Laplacian matrix is called the distance signless Laplacian spectral radius of G. In this paper, we determine the unique graph with the maximum distance signless Laplacian spectral radius among all the bicyclic graphs with given order.

Keywords distance signless Laplacian matrix; spectral radius; bicyclic graph

MR(2020) Subject Classification 05C50

1. Introduction

We consider simple and undirected graphs. Undefined notations may be referred to [1]. Let G be a connected graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. If E_1 is a nonempty subset of $E(G)$, $G - E_1$ denotes the graph obtained from G by deleting all edges in E_1. If E_1 is a set of edges which are not in $E(G)$, $G + E_1$ denotes the graph obtained from G by adding all the edges in E_1. A spanning subgraph of a graph G is a subgraph whose vertex set is the entire vertex set of G. For $v \in V(G)$, $d_G(v)$ denotes the degree of v in G. We denote $\Delta(G) = \max\{d_G(v) : v \in V(G)\}$. We use C_n and P_n to represent a cycle and a path, respectively, each on n vertices.

For $v_i, v_j \in V(G)$, the distance between v_i and v_j in G, denoted by $d_G(v_i, v_j)$, is the length of the shortest path from v_i to v_j in G. The distance matrix of G is $D(G) = (d_{i,j})_{n\times n}$, where $d_{i,j} = d_G(v_i, v_j)$. For $v_i \in V(G)$, the transmission of v_i in G, denoted by $\mathrm{tr}_G(v_i)$, is the sum of distances from v_i to all other vertices of G. Let $\mathrm{tr}_{\max}(G)$ be the maximum vertex transmission of G. Let $Tr(G)$ be the $n \times n$ diagonal matrix of the vertex transmissons in G. Then the distance signless Laplacian matrix of G is $\mathcal{Q}(G) = Tr(G) + D(G)$. The eigenvalues of $\mathcal{Q}(G)$, denoted by $\rho_1(G), \rho_2(G), \ldots, \rho_n(G)$, are called the distance signless Laplacian eigenvalues of G. Since $\mathcal{Q}(G)$ is a real symmetric matrix, without loss of generality, suppose that $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_n(G)$. The largest distance signless Laplacian eigenvalue of G is called the distance signless Laplacian spectral radius of G, and is denoted by $\rho(G)$. Since $\mathcal{Q}(G)$ is an irreducible nonnegative matrix, by

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* Corresponding author
E-mail address: 18121587@bjtu.edu.cn (Nannan XU); yuaimeimath@163.com (Aimei YU)

Perron-Frobenius Theorem, $\rho(G)$ is simple and there is a unique unit positive eigenvector of $\mathcal{Q}(G)$ corresponding to $\rho(G)$, which is called the (distance signless Laplacian) principal eigenvector of G.

Aouchiche and Hansen [2] introduced the distance signless Laplacian matrix and studied the distance signless Laplacian eigenvlues of a connected graph. For a connected graph G, the value $c(G) = |E(G)| - |V(G)| + 1$ is called the cyclomatic number of G. When $c(G) = 0, 1, 2$, the graph is called a tree, a unicyclic graph and a bicyclic graph, respectively. Xing et al. [3] determined the graphs with the minimum distance signless Laplacian spectral radius among the trees, unicyclic graphs and bipartite graphs with fixed number of vertices, respectively, and also determined the graphs with the minimum distance signless Laplacian spectral radius among the connected graphs with fixed number of vertices and pendent vertices, and the connected graphs with fixed number of vertices and connectivity, respectively. Lin and Zhou [4] studied the effect of three types of graft transformations to decrease or increase the distance signless Laplacian spectral radius, and determined the unique graphs with the maximum distance signless Laplacian spectral radius among trees, and among trees with given maximum degree, respectively, and also determined the unique graphs with the minimum and the maximum distance signless Laplacian spectral radius among all non-starlike trees, among non-caterpillar trees, and among non-starlike non-caterpillar trees, respectively. Bapat et al. [5] proved that the distance signless Laplacian spectral radius was maximized at a dumbbell in the class of all trees with a given number of pendent vertices, and also determined the unique graphs with the maximum distance signless Laplacian spectral radius among unicyclic graphs. Xing and Zhou [6] determined the graphs with the minimum and the second-minimum distance signless Laplacian spectral radius among bicyclic graphs with given order. Lin and Lu [7] found a sharp lower bound as well as a sharp upper bound of the distance signless Laplacian spectral radius in terms of the clique number. Furthermore, both extremal graphs were uniquely determined. More results on the distance signless Laplacian spectral radius can be found in [8–12].

In this paper, we determine the unique graph with the maximum distance signless Laplacian spectral radius among all the bicyclic graphs with given order.

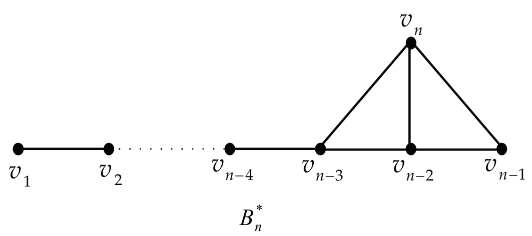

Figure 1 Graph B_n^*

Let $n \geq 4$ be an integer, and $P_{n-1} = v_1v_2\cdots v_{n-1}$ be a path with order $n-1$. We denote by B_n^* the bicyclic graph of order n obtained from P_{n-1} by adding one vertex v_n and three edges v_nv_{n-3}, v_nv_{n-2} and v_nv_{n-1}. Graph B_n^* is depicted in Figure 1. Let $\mathcal{B}_n$ be the set of bicyclic graphs of order n.

Theorem 1.1 *Let $n \geq 8$ and $G \in \mathcal{B}_n$. Then $\rho(G) \leq \rho(B_n^*)$ with equality if and only if $G \cong B_n^*$.*

The rest of this paper is organized as follows. In Section 2, we present some useful lemmas. In Section 3, we compare the distance signless Laplacian spectral radius of some special graphs with $\rho(B_n^*)$, and then use these results to prove Theorem 1.1.

2. Some useful lemmas

In this section, we present some known results. Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. A column vector $x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^{\mathrm{T}}$ can be considered as a function defined on $V(G)$ which maps vertex v_i to x_{v_i} for $1 \leq i \leq n$. For convenience, we write $x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^{\mathrm{T}}$ as $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$. Then

$$x^{\mathrm{T}}\mathcal{Q}(G)x = \sum_{1 \leq i < j \leq n} d_G(v_i, v_j)(x_i + x_j)^2.$$

If $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is the eigenvector of $\rho(G)$, then for $1 \leq i \leq n$, we have the following eigenequation of G at $v_i \in V(G)$,

$$\rho(G)x_i = \mathrm{tr}_G(v_i)x_i + \sum_{v_j \in V(G) \setminus \{v_i\}} d_G(v_i, v_j)x_j.$$

For a unit column vector $x \in R^n$ with at least one nonnegative entry, by Rayleigh's principle, we have $\rho(G) \geq x^{\mathrm{T}}\mathcal{Q}(G)x$ with equality if and only if x is the principal eigenvector of $\mathcal{Q}(G)$ corresponding to $\rho(G)$.

In [4], Lin and Zhou proved the following five lemmas.

Lemma 2.1 ([4]) *Let G be a connected graph with η being an automorphism of G, and x a principal eigenvector of G. Then for $u, v \in V(G)$, $\eta(u) = v$ implies $x_u = x_v$.*

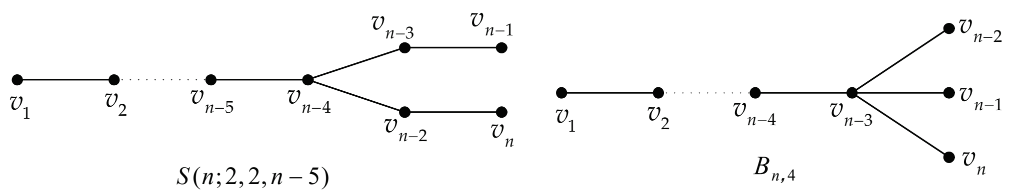

Figure 2 Graphs $S(n; 2, 2, n-5)$ and $B_{n,4}$

Let G be a graph and $v \in V(G)$. If $d_G(v) = 1$, v is a pendent vertex of G. If v is adjacent to a pendent vertex of G and $d_G(v) \geq 2$, v is called a support vertex. We denote by $\sup(G)$ the set of all support vertices in G. A caterpillar is a tree such that the removal of all pendent vertices yields a path. A tree that is not a caterpillar is said to be a non-caterpillar tree. Obviously, if T is a tree, and T has a vertex v such that $v \notin \sup(T)$ and $d_T(v) \geq 3$, then T is a non-caterpillar tree. Let $n, n_1, n_2, \ldots, n_r$ be $r+1$ positive integers with $\sum_{i=1}^{r} n_i + 1 = n$. We denote by $S(n; n_1, n_2, \ldots, n_r)$ the graph of order n obtained from vertex-disjoint paths $P_{n_1}, P_{n_2}, \ldots, P_{n_r}$ by adding an edge between a vertex u and a terminal vertex of P_{n_i} for each $i = 1, 2, \ldots, r$. For example, $S(n; 2, 2, n-5)$ is the graph shown in Figure 2.

Lemma 2.2 ([4]) *Let T be a non-caterpillar tree of order $n \geq 7$. Then $\rho(T) \leq \rho(S(n;2,2,n-5))$ with equality if and only if $T \cong S(n;2,2,n-5)$.*

For integer $2 \leq \Delta \leq n-1$, $B_{n,\Delta}$ denotes the tree of order n obtained from $P_{n-\Delta+1}$ by attaching $\Delta - 1$ pendent vertices to a terminal vertex of $P_{n-\Delta+1}$. For example, $B_{n,4}$ is the graph shown in Figure 2.

Lemma 2.3 ([4]) *Let T be a tree of order $n \geq 5$ with maximum degree Δ, where $2 \leq \Delta \leq n-1$. Then $\rho(T) \leq \rho(B_{n,\Delta})$ with equality if and only if $T \cong B_{n,\Delta}$.*

Let $r \geq 2$ be an integer, and $P_r = v_1v_2\cdots v_r$ be a path in G. If $d_G(v_1) \geq 3$, $d_G(v_r) = 1$, and $d_G(v_i) = 2$ for $2 \leq i \leq r-1$, P_r is called a pendent path of length $r-1$ in G. For a nontrivial connected graph G with $u \in V(G)$ and positive integers k and l, let $G_u(k,l)$ be the graph obtained from G by attaching two pendent paths of lengths k and l respectively at u, and $G_u(k,0)$ be the graph obtained from G by attaching a pendent path of length k at u.

Lemma 2.4 ([4]) *Let G be a nontrivial connected graph with $u \in V(G)$. For $k \geq l \geq 1$, $\rho(G_u(k,l)) < \rho(G_u(k+1,l-1))$.*

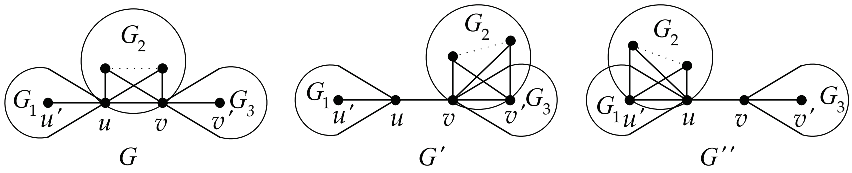

Figure 3 Graphs G, G' and G'' in Lemma 2.5

Lemma 2.5 ([4]) *Let G be a graph with three induced subgraphs G_1, G_2 and G_3 such that $V(G_1) \cap V(G_2) = \{u\}$, $V(G_2) \cap V(G_3) = \{v\}$, $\cup_{i=1}^{3} V(G_i) = V(G)$, $|V(G_1) \setminus \{u\}| \geq 1$, $|V(G_2) \setminus \{u,v\}| \geq 1$ and $|V(G_3) \setminus \{v\}| \geq 1$. Suppose that $uv \in E(G_2)$, $N_{G_2}(u) \setminus \{v\} = N_{G_2}(v) \setminus \{u\} = V_0$, $u' \in N_{G_1}(u)$ and $v' \in N_{G_3}(v)$. Let $G' = G - \{uw : w \in V_0\} + \{v'w : w \in V_0\}$ and $G'' = G - \{vw : w \in V_0\} + \{u'w : w \in V_0\}$. Graphs G, G' and G'' are depicted in Figure 3. Then $\rho(G) < \rho(G')$ or $\rho(G) < \rho(G'')$.*

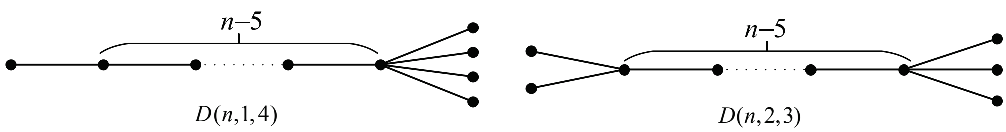

Figure 4 Graphs $D(n,1,4)$ and $D(n,2,3)$

Let a, b and n be three positive integers with $n \geq a+b+2$. Let P_{n-a-b} be a path of order $n-a-b$ with pendent vertices u and v. The dumbbell $D(n,a,b)$ is the graph of order n obtained from P_{n-a-b} by attaching a pendent vertices at u and attaching b pendent vertices at v. Let $\mathcal{T}_{n,k}$ be the class of all trees with n vertices and k pendent vertices. If $a+b=k$, $D(n,a,b) \in \mathcal{T}_{n,k}$. For example, $D(n,1,4)$ and $D(n,2,3)$ are the trees with 5 pendent vertices. Trees $D(n,1,4)$ and $D(n,2,3)$ are depicted in Figure 4.

Lemma 2.6 ([5]) *If T is a tree with the maximum distance signless Laplacian spectral radius in $\mathcal{T}_{n,k}$, then $T \cong D(n,a,b)$ for some positive integer a and b, where $a+b=k$.*

Lemma 2.7 ([2]) *Let G be a connected graph on n vertices with m ($m \geq n$) edges. If $\tilde{G}$ is the connected graph obtained from G by deleting an edge, then $\rho(G) \leq \rho(\tilde{G})$.*

The Wiener index of a connected graph G, denoted by $W(G)$, is the sum of distances between all unordered pairs of vertices of G. A graph G is said to be transmission regular if $\mathrm{tr}_G(v)$ is a constant for each $v \in V(G)$.

Lemma 2.8 ([6]) *Let G be a connected graph on n vertices. Then*

$$\rho(G) \geq \frac{4W(G)}{n} = \frac{2\sum_{v\in V(G)} \mathrm{tr}_G(v)}{n}$$

with equality if and only if G is transmission regular.

Lemma 2.9 ([13]) *Let G be a connected graph with the maximum transmission* $\mathrm{tr}_{\max}(G)$. *Then* $\rho(G) \leq 2\mathrm{tr}_{\max}(G)$.

3. Distance signless Laplacian spectral radius of bicyclic graphs

In this section, we compare the distance signless Laplacian spectral radius of some special graphs with $\rho(B_n^*)$, and then use these results to determine the graph with the maximum distance signless Laplacian spectral radius among all the bicyclic graphs with given order.

n	8	9	10	11	12	13	14
$\rho(B_n^*)$	39.3676	51.2270	64.5786	79.4165	95.7361	113.5338	132.8065
$\rho(S(n;2,2,n-5))$	38.9173	50.2532	62.9956	77.3149	93.1762	110.4357	129.3655
$\rho(B_{n,4})$	38.1249	49.6770	62.7647	77.3678	93.4743	111.0759	130.1666
n	15	16	17	18	19		
$\rho(B_n^*)$	153.5517	175.7673	199.4514	224.6025	251.2191		
$\rho(S(n;2,2,n-5))$	149.6137	171.4942	194.8626	219.7151	246.0485		

Table 1 Some values of $\rho(B_n^*)$, $\rho(S(n;2,2,n-5))$ and $\rho(B_{n,4})$

Lemma 3.1 *Let $n \geq 8$. Then each of the following holds:*

(i) $\rho(S(n;2,2,n-5)) < \rho(B_n^*)$;

(ii) $\rho(B_{n,4}) < \rho(B_n^*)$.

Proof The vertices of B_n^*, $S(n;2,2,n-5)$ and $B_{n,4}$ are labeled as in Figures 1 and 2, respectively.

(i) For $8 \leq n \leq 19$, we can use computer to get $\rho(S(n;2,2,n-5))$ and $\rho(B_n^*)$ (see Table 1). From Table 1, it is easy to see that $\rho(S(n;2,2,n-5)) < \rho(B_n^*)$. In the following, we suppose that $n \geq 20$. Let $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be the principal eigenvector of $\mathcal{Q}(S(n; 2, 2, n-5))$ corresponding to $\rho(S(n; 2, 2, n-5))$. It is easy to see that

$$d_{B_n^*}(u,v) - d_{S(n;2,2,n-5)}(u,v) = \begin{cases} 1, & \text{if } u = v_{n-2} \text{ and } v \in \{v_1, v_2, \ldots, v_{n-4}\}, \\ -1, & \text{if } u = v_{n-2} \text{ and } v = v_{n-3}, \\ 1, & \text{if } u = v_{n-1} \text{ and } v \in \{v_1, v_2, \ldots, v_{n-3}\}, \\ -2, & \text{if} u = v_{n-1} \text{ and } v = v_{n-2}, \\ -2, & \text{if } u = v_n \text{ and } v = v_{n-3}, \\ -3, & \text{if } u = v_n \text{ and } v = v_{n-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} &\rho(B_n^*) - \rho(S(n;2,2,n-5)) \geq x^{\mathrm{T}}(\mathcal{Q}(B_n^*) - \mathcal{Q}(S(n;2,2,n-5)))x \\ &= \sum_{1 \leq i < j \leq n} (d_{B_n^*}(v_i, v_j) - d_{S(n;2,2,n-5)}(v_i, v_j))(x_i + x_j)^2 \\ &= \sum_{i=1}^{n-4} (x_{n-2} + x_i)^2 - (x_{n-2} + x_{n-3})^2 + \sum_{i=1}^{n-3} (x_{n-1} + x_i)^2 - 2(x_{n-1} + x_{n-2})^2 - \\ &\quad 2(x_n + x_{n-3})^2 - 3(x_n + x_{n-1})^2. \end{aligned} \tag{3.1}$$

By Lemma 2.1, $x_{n-3} = x_{n-2}$ and $x_{n-1} = x_n$. By Perron-Frobenius Theorem, $x_i > 0$ for $1 \leq i \leq n$. Combining these results with $n \geq 20$ and (3.1), we have

$$\begin{aligned} &\rho(B_n^*) - \rho(S(n;2,2,n-5)) \\ &\geq \sum_{i=1}^{n-4} (x_{n-3} + x_i)^2 + \sum_{i=1}^{n-4} (x_{n-1} + x_i)^2 - 7x_{n-3}^2 - 15x_{n-1}^2 - 6x_{n-3}x_{n-1} \\ &> 16x_{n-3}^2 + 16x_{n-1}^2 - 7x_{n-3}^2 - 15x_{n-1}^2 - 6x_{n-3}x_{n-1} \\ &= (3x_{n-3} - x_{n-1})^2 \geq 0. \end{aligned}$$

Thus $\rho(S(n;2,2,n-5)) < \rho(B_n^*)$.

(ii) For $8 \leq n \leq 14$, we can use computer to get $\rho(B_{n,4})$ and $\rho(B_n^*)$ (see Table 1). From Table 1, it is easy to see that $\rho(B_{n,4}) < \rho(B_n^*)$. In the following, we suppose that $n \geq 15$. Let $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be the principal eigenvector of $\mathcal{Q}(B_{n,4})$ corresponding to $\rho(B_{n,4})$. It is easy to see that

$$d_{B_n^*}(u,v) - d_{B_{n,4}}(u,v) = \begin{cases} 1, & \text{if } u = v_{n-1} \text{ and } v \in \{v_1, v_2, \ldots, v_{n-3}\}, \\ -1, & \text{if } u = v_{n-1} \text{ and } v = v_{n-2}, \\ -1, & \text{if } u = v_n \text{ and } v = v_{n-2}, \\ -1, & \text{if } u = v_n \text{ and } v = v_{n-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\rho(B_n^*) - \rho(B_{n,4}) \geq x^{\mathrm{T}}(\mathcal{Q}(B_n^*) - \mathcal{Q}(B_{n,4}))x$$

$$= \sum_{1\le i<j\le n} (d_{B_n^*}(v_i,v_j) - d_{B_{n,4}}(v_i,v_j))(x_i+x_j)^2$$

$$= \sum_{i=1}^{n-3}(x_{n-1}+x_i)^2 - (x_{n-1}+x_{n-2})^2 - (x_n+x_{n-2})^2 - (x_n+x_{n-1})^2. \tag{3.2}$$

By Lemma 2.1, we have $x_{n-2} = x_{n-1} = x_n$. By Perron-Frobenius Theorem, $x_i > 0$ for $1 \le i \le n$. Combining these results with $n \ge 15$ and (3.2),

$$\rho(B_n^*) - \rho(B_{n,4}) \ge \sum_{i=1}^{n-3}(x_{n-1}+x_i)^2 - 12x_{n-1}^2 > 12x_{n-1}^2 - 12x_{n-1}^2 = 0.$$

Thus $\rho(B_{n,4}) < \rho(B_n^*)$. □

For a connected graph G, we denote by $p(G)$ the number of the pendent vertices in G.

Lemma 3.2 *Let T be a tree of order $n \ge 8$.*

(i) If T is a non-caterpillar tree, then $\rho(T) < \rho(B_n^*)$.

(ii) If T is a tree with maximum degree $\Delta(T) \ge 4$, then $\rho(T) < \rho(B_n^*)$.

(iii) If $p(T) \ge 5$, then $\rho(T) < \rho(B_n^*)$.

(iv) Let G be a bicyclic graph of order $n \ge 8$ and let T be a spanning tree of G. If T is a non-caterpillar tree, or $\Delta(T) \ge 4$, or $p(T) \ge 5$, then $\rho(G) < \rho(B_n^*)$.

Proof (i) If T is a non-caterpillar tree with order $n \ge 8$, by Lemma 2.2, we have $\rho(T) \le \rho(S(n;2,2,n-5))$ with equality if and only if $T \cong S(n;2,2,n-5)$. By Lemma 3.1 (i), $\rho(S(n;2,2,n-5)) < \rho(B_n^*)$. Hence we have $\rho(T) < \rho(B_n^*)$.

(ii) Let $\Delta(T) \ge 4$. If $\Delta(T) = 4$, by Lemma 2.3, $\rho(T) \le \rho(B_{n,4})$. If $\Delta(T) \ge 5$, by Lemmas 2.3 and 2.4, we have $\rho(T) \le \rho(B_{n,\Delta}) < \rho(B_{n,4})$. Hence for any case, $\rho(T) \le \rho(B_{n,4})$. By Lemma 3.1 (ii), $\rho(B_{n,4}) < \rho(B_n^*)$. Thus $\rho(T) < \rho(B_n^*)$.

(iii) Let $p(T) \ge 5$. By Lemma 2.6, we have $\rho(T) \le \rho(D(n,a,b))$ for some positive integers a and b, where $a + b = p(T) \ge 5$. Without loss of generality, suppose that $a \le b$. Then $b \ge 3$, and $D(n,a,b)$ is a tree with maximum degree $\Delta \ge 4$. By Lemma 3.2 (ii), $\rho(D(n,a,b)) < \rho(B_n^*)$. Thus we have $\rho(T) < \rho(B_n^*)$.

(iv) Let G be a bicyclic graph of order $n \ge 8$ and let T be a spanning tree of G. By Lemma 2.7, $\rho(G) \le \rho(T)$. If T is a non-caterpillar tree, or $\Delta(T) \ge 4$, or $p(T) \ge 5$, by (i)–(iii) of Lemma 3.2, we have $\rho(T) < \rho(B_n^*)$. Thus $\rho(G) < \rho(B_n^*)$. □

Let n and i be two integers with $2 \le i \le n-2$. Let $P_{n-1} = v_1v_2\cdots v_{n-1}$ be a path with order $n-1$. We denote by B_i the bicyclic graph with n vertices obtained from P_{n-1} by adding one vertex v_n and three edges v_nv_{i-1}, v_nv_i and v_nv_{i+1}. Let $\mathcal{B}_n^1 = \{B_i : 2 \le i \le n-2\}$.

Lemma 3.3 *For any $B_i \in \mathcal{B}_n^1$, we have $\rho(B_i) \le \rho(B_n^*)$ with equality if and only if $i = 2$ or $i = n-2$.*

Proof Since $B_2 \cong B_n^*$ and $B_{n-2} \cong B_n^*$, it is sufficient to prove that for $3 \le i \le n-3$, B_i is not the graph with the maximum distance signless Laplacian spectral radius in $\mathcal{B}_n^1$. In the following, we suppose that $3 \le i \le n-3$. Let $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be the principal eigenvector of $\mathcal{Q}(B_i)$

corresponding to $\rho(B_i)$. By Perron-Frobenius Theorem, $x_i > 0$ for $1 \le i \le n$. It is easy to see that

$$d_{B_{i-1}}(u,v) - d_{B_i}(u,v) = \begin{cases} -1, & \text{if } u = v_n \text{ and } v \in \{v_1, v_2, \ldots, v_{i-2}\}, \\ 1, & \text{if } u = v_n \text{ and } v \in \{v_{i+1}, v_{i+2}, \ldots, v_{n-1}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} \rho(B_{i-1}) - \rho(B_i) &\ge x^{\mathrm{T}}(\mathcal{Q}(B_{i-1}) - \mathcal{Q}(B_i))x \\ &= \sum_{1\le i<j\le n} (d_{B_{i-1}}(v_i,v_j) - d_{B_i}(v_i,v_j))(x_i + x_j)^2 \\ &= -\sum_{j=1}^{i-2}(x_n + x_j)^2 + (x_n + x_{i+1})^2 + \sum_{j=i+2}^{n-1}(x_n + x_j)^2. \end{aligned} \tag{3.3}$$

If

$$\sum_{j=1}^{i-2}(x_n + x_j)^2 \le \sum_{j=i+2}^{n-1}(x_n + x_j)^2,$$

by (3.3), we have $\rho(B_{i-1}) - \rho(B_i) \ge (x_n + x_{i+1})^2 > 0$, i.e., $\rho(B_{i-1}) > \rho(B_i)$.

If

$$\sum_{j=1}^{i-2}(x_n + x_j)^2 > \sum_{j=i+2}^{n-1}(x_n + x_j)^2,$$

from

$$d_{B_{i+1}}(u,v) - d_{B_i}(u,v) = \begin{cases} 1, & \text{if } u = v_n \text{ and } v \in \{v_1, v_2, \ldots, v_{i-1}\}, \\ -1, & \text{if } u = v_n \text{ and } v \in \{v_{i+2}, v_{i+3}, \ldots, v_{n-1}\}, \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} \rho(B_{i+1}) - \rho(B_i) &\ge x^{\mathrm{T}}(\mathcal{Q}(B_{i+1}) - \mathcal{Q}(B_i))x \\ &= \sum_{1\le i<j\le n} (d_{B_{i+1}}(v_i,v_j) - d_{B_i}(v_i,v_j))(x_i + x_j)^2 \\ &= \sum_{j=1}^{i-2}(x_n + x_j)^2 + (x_n + x_{i-1})^2 - \sum_{j=i+2}^{n-1}(x_n + x_j)^2 \\ &> (x_n + x_{i-1})^2 > 0, \end{aligned}$$

i.e., $\rho(B_{i+1}) > \rho(B_i)$.

Hence when $3 \le i \le n-3$, B_i is not the graph with the maximum distance signless Laplacian spectral radius in $\mathcal{B}_n^1$, which implies $\rho(B_i) \le \rho(B_n^*)$ with equality if and only if $i = 2$ or $i = n-2$. □

Let n, i and j be three integers with $2 \le i < j \le n-2$, and let $P_{n-2} = v_2 v_3 \cdots v_{n-1}$ be a path with order $n-2$. We denote by $B_{i,j}$ the bicyclic graph with n vertices obtained from P_{n-2} by adding two vertices v_1, v_n and four edges v_1v_i, v_1v_{i+1}, v_nv_j and v_nv_{j+1}. For example, $B_{2,n-2}$ is the graph depicted in Figure 5. Let $\mathcal{B}_n^2 = \{B_{i,j} : 2 \le i < j \le n-2\}$.

Lemma 3.4 *For any $B_{i,j} \in \mathcal{B}_n^2$, we have $\rho(B_{i,j}) < \rho(B_n^*)$.*

Proof Suppose that G^* maximizes the distance signless Laplacian spectral radius in $\mathcal{B}_n^2$. Let $\mathcal{S} = \{B_{2,3}, B_{2,n-2}\} \cup \{B_{i,i+1} : 3 \le i \le n-3\}$. We claim

$$G^* \in \mathcal{S}. \tag{3.4}$$

Let $G^* \cong B_{i,j}$. If $i = 2$ and $4 \le j \le n-3$, by Lemma 2.5, we have $\rho(G^*) < \rho(B_{i,j-1})$ or $\rho(G^*) < \rho(B_{i,j+1})$, which is a contradiction. So when $i = 2$, we have $j = 3$ or $j = n-2$, i.e., $G^* \in \{B_{2,3}, B_{2,n-2}\}$. If $3 \le i \le n-3$ and $i+2 \le j \le n-2$, by Lemma 2.5, $\rho(G^*) < \rho(B_{i-1,j})$ or $\rho(G^*) < \rho(B_{i+1,j})$, which is a contradiction. So when $3 \le i \le n-3$, we must have $j = i+1$, i.e., $G^* \in \{B_{i,i+1} : 3 \le i \le n-3\}$. From above, we have $G^* \in \mathcal{S}$.

Now we prove that for any graph $G \in \mathcal{S}$, we have $\rho(G) < \rho(B_n^*)$. If $G \cong B_{2,3}$, let $G_0 = G - \{v_1v_2, v_nv_4\}$; if $G \cong B_{i,i+1}$ where $3 \le i \le n-3$, let $G_0 = G - \{v_1v_i, v_nv_{i+2}\}$. Then G_0 is a spanning tree of G with maximum degree $\Delta(G_0) \ge 4$. By Lemma 3.2 (iv), we have $\rho(G) < \rho(B_n^*)$. Suppose that $G \cong B_{2,n-2}$. The vertices of B_n^* and $B_{2,n-2}$ are labeled as in Figures 1 and 5, respectively. Let $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be the principal eigenvector of $\mathcal{Q}(B_{2,n-2})$ corresponding to $\rho(B_{2,n-2})$. It is easy to see that

$$d_{B_n^*}(u,v) - d_{B_{2,n-2}}(u,v) = \begin{cases} 1, & \text{if } u = v_1 \text{ and } v \in \{v_3, v_4, \ldots, v_{n-1}\}, \\ -1, & \text{if } u = v_n \text{ and } v \in \{v_2, v_3, \ldots, v_{n-3}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} \rho(B_n^*) - \rho(B_{2,n-2}) &\ge x^{\mathrm{T}}(\mathcal{Q}(B_n^*) - \mathcal{Q}(B_{2,n-2}))x \\ &= \sum_{1 \le i < j \le n} (d_{B_n^*}(v_i, v_j) - d_{B_{2,n-2}}(v_i, v_j))(x_i + x_j)^2 \\ &= \sum_{i=3}^{n-1} (x_1 + x_i)^2 - \sum_{i=2}^{n-3} (x_n + x_i)^2 \\ &= \sum_{i=3}^{n-3} (x_1 + x_i)^2 + (x_1 + x_{n-2})^2 + (x_1 + x_{n-1})^2 - (x_n + x_2)^2 - \sum_{i=3}^{n-3} (x_n + x_i)^2. \end{aligned} \tag{3.5}$$

By Lemma 2.1, we have $x_1 = x_2 = x_{n-1} = x_n$. By Perron-Frobenius Theorem, $x_i > 0$ for $1 \le i \le n$. Combining these results with (3.5), we have

$$\rho(B_n^*) - \rho(B_{2,n-2}) \ge (x_1 + x_{n-2})^2 > 0.$$

Thus $\rho(B_n^*) > \rho(B_{2,n-2})$. Hence for any graph $G \in \mathcal{S}$, we have $\rho(G) < \rho(B_n^*)$. From (3.4), we have $\rho(B_{i,j}) < \rho(B_n^*)$ for any $B_{i,j} \in \mathcal{B}_n^2$. □

Let C_n be a cycle with n vertices. Let C_n^1 be the unicyclic graph of order n obtained from $C_{n-1} = v_1v_2 \cdots v_{n-1}v_1$ by adding one vertex v_n and one edge v_nv_1 to C_{n-1}. Suppose that n, i, j are three integers with $1 \le i < j \le n-2$. We denote by $C_{i,j}$ the unicyclic graph with n vertices obtained from $C_{n-2} = v_1v_2 \cdots v_{n-2}v_1$ by adding two vertices v_{n-1}, v_n and two edges $v_{n-1}v_i, v_nv_j$ to C_{n-2}. Let $\mathcal{C}_n^2 = \{C_{i,j} : 1 \le i < j \le n-2\}$ and $\mathcal{U}_n^* = \{C_n, C_n^1\} \cup \mathcal{C}_n^2$.

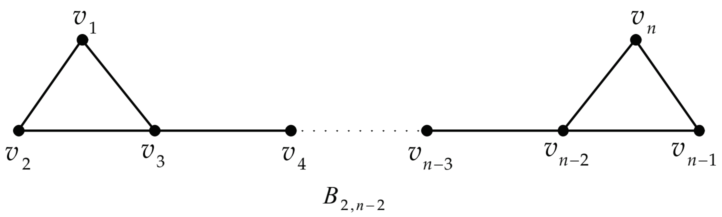

Figure 5 Graph $B_{2,n-2}$

Lemma 3.5 *Let G be a bicyclic graph of order $n \geq 8$ and let G' be a spanning unicyclic subgraph of G. If $G' \in \mathcal{U}_n^*$, we have $\rho(G) < \rho(B_n^*)$.*

Proof It is known that the Wiener index of path P_n is $W(P_n) = \frac{n(n-1)(n+1)}{6}$ (see [14]). So $W(B_n^*) = W(P_{n-1}) + \mathrm{tr}_{B_n^*}(v_n) = \frac{n(n-2)(n-1)}{6} + 2 + \frac{(n-3)(n-2)}{2}$. By Lemma 2.8, we have

$$\rho(B_n^*) \geq \frac{4W(B_n^*)}{n} = \frac{2}{3}n^2 + \frac{20}{n} - \frac{26}{3}. \tag{3.6}$$

Let $n \geq 8$, $G \in \mathcal{B}_n$ and G' be a spanning unicyclic subgraph of G such that $G' \in \mathcal{U}_n^*$. By Lemma 2.7, $\rho(G) \leq \rho(G')$. Hence it suffices to prove that $\rho(G') < \rho(B_n^*)$. In the following, we distinguish three cases:

Case 1. $G' \cong C_n$.

Since C_n is a transmission regular graph, by Lemma 2.8, we have $\rho(C_n) = 2\mathrm{tr}_{C_n}(v)$ for any $v \in V(C_n)$, i.e.,

$$\rho(C_n) = \begin{cases} \frac{n^2}{2}, & \text{if } n \text{ is even,} \\ \frac{n^2-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Then by (3.6), we have $\rho(B_n^*) \geq \frac{2}{3}n^2 + \frac{20}{n} - \frac{26}{3} > \frac{n^2}{2} \geq \rho(C_n)$ when $n \geq 6$.

Case 2. $G' \cong C_n^1$.

Since

$$\mathrm{tr}_{\max}(C_n^1) = \mathrm{tr}_{C_n^1}(v_n) = \begin{cases} \frac{n^2+2n-4}{4}, & \text{if } n \text{ is even,} \\ \frac{n^2+2n-3}{4}, & \text{if } n \text{ is odd,} \end{cases}$$

by Lemma 2.9, we have

$$\rho(C_n^1) \leq 2\mathrm{tr}_{\max}(C_n^1) \leq \frac{n^2+2n-3}{2}. \tag{3.7}$$

When $n \geq 10$, by (3.6), $\rho(B_n^*) \geq \frac{2}{3}n^2 + \frac{20}{n} - \frac{26}{3} > \frac{n^2+2n-3}{2} \geq \rho(C_n^1)$. When $n = 8, 9$, from Table 1 and (3.7), we have $\rho(B_8^*) \approx 39.3676 > 38.5 \geq \rho(C_8^1)$ and $\rho(B_9^*) \approx 51.2270 > 48 \geq \rho(C_9^1)$. Thus $\rho(B_n^*) > \rho(C_n^1)$ for $n \geq 8$.

Case 3. $G' \in \mathcal{C}_n^2$.

It is routine to verify that

$$\mathrm{tr}_{\max}(G') = \mathrm{tr}_{G'}(v_n) \leq \begin{cases} \mathrm{tr}_{C_{1,\frac{n}{2}}}(v_n) = \frac{n^2+2n}{4}, & \text{if } n \text{ is even,} \\ \mathrm{tr}_{C_{1,\frac{n-1}{2}}}(v_n) = \frac{n^2+2n-3}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

Then by Lemma 2.9,

$$\rho(G') \leq 2\mathrm{tr}_{\max}(G') \leq \frac{n^2+2n}{2}. \tag{3.8}$$

When $n \geq 11$, by (3.6) and (3.8),

$$\rho(B_n^*) \geq \frac{2}{3}n^2 + \frac{20}{n} - \frac{26}{3} > \frac{n^2+2n}{2} \geq \rho(G').$$

When $n = 8$, we have $\mathcal{C}_8^2 = \{C_{1,2}, C_{1,3}, C_{1,4}\}$. By direct calculation, $\rho(C_{1,2}) \leq 2\mathrm{tr}_{\max}(C_{1,2}) = \mathrm{tr}_{C_{1,2}}(v_8) = 36$, $\rho(C_{1,3}) \leq 2\mathrm{tr}_{\max}(C_{1,3}) = 2\mathrm{tr}_{C_{1,3}}(v_8) = 38$ and $\rho(C_{1,4}) \approx 32.7178$. Then from Table 1, $\rho(B_8^*) \approx 39.3676 > \rho(G')$ for $G' \in \mathcal{C}_8^2$. From Table 1 and (3.8), when $G' \in \mathcal{C}_9^2$, we have $\rho(B_9^*) \approx 51.2270 > 49.5 \geq \rho(G')$, and when $G' \in \mathcal{C}_{10}^2$, we have $\rho(B_{10}^*) \approx 64.5786 > 60 \geq \rho(G')$. Hence $\rho(B_n^*) > \rho(G')$ for $G' \in \mathcal{C}_n^2$ and $n \geq 8$. This completes the proof of Lemma 3.5. $\Box$

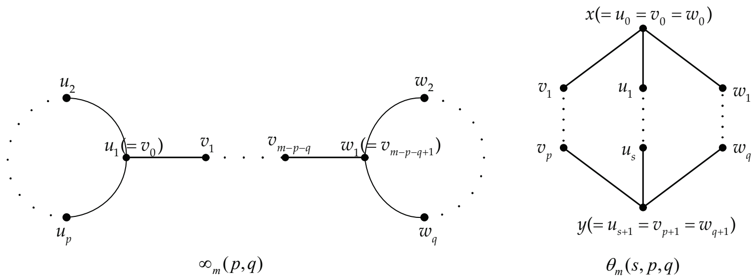

Figure 6 Graphs $\infty_m(p,q)$ and $\theta_m(s,p,q)$

Let m, p and q be integers with $3 \leq p \leq q \leq m-2$. Let $C_p = u_1u_2\cdots u_pu_1$ and $C_q = w_1w_2\cdots w_qw_1$ be two cycles, and $P = v_0v_1\cdots v_{m-p-q}v_{m-p-q+1}$ be a path of length $m-p-q+1$. We denote by $\infty_m(p,q)$ the bicyclic graph with m vertices obtained from C_p, C_q and P by identifying u_1 with v_0 and identifying w_1 with $v_{m-p-q+1}$, where $m-p-q+1 \geq 0$ and $m-p-q+1 = 0$ means identifying u_1 and w_1. Graph $\infty_m(p,q)$ is depicted in Figure 6.

Let m, s, p and q be integers with $0 \leq s \leq p \leq q \leq m-3$ and $s+p+q = m-2$. Let $P' = u_0u_1\cdots u_su_{s+1}$, $P'' = v_0v_1\cdots v_pv_{p+1}$ and $P''' = w_0w_1\cdots w_qw_{q+1}$ be three paths of length $s+1$, $p+1$ and $q+1$, respectively. We denote by $\theta_m(s,p,q)$ the bicyclic graph with m vertices obtained from P', P'' and P''' by identifying u_0, v_0 and w_0 to a new vertex x and identifying u_{s+1}, v_{p+1} and w_{q+1} to a new vertex y. Graph $\theta_m(s,p,q)$ is depicted in Figure 6.

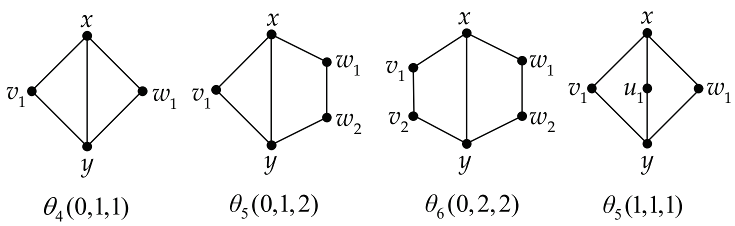

Figure 7 Graphs $\theta_4(0,1,1), \theta_5(0,1,2), \theta_6(0,2,2)$ and $\theta_5(1,1,1)$

For a bicyclic graph G of order n, the base of G, denoted by $\hat{G}$, is the unique minimal bicyclic subgraph of G. It is easy to see that there are no pendent vertices in $\hat{G}$ and G can be obtained from $\hat{G}$ by attaching trees to some vertices of $\hat{G}$. We denote by $T(v)$ the tree attaching to $v \in V(\hat{G})$. It is easy to see that $\sum_{v\in V(\hat{G})}(|V(T(v))|-1) + |V(\hat{G})| = n$.

Lemma 3.6 *Let G be a bicyclic graph of order $n \geq 8$ and $\hat{G}$ be the base of G. Denote $\mathcal{M} = \{\theta_4(0,1,1), \theta_5(0,1,2), \theta_6(0,2,2), \theta_5(1,1,1)\}$. If $G \ncong B_n^*$ and $\hat{G} \in \mathcal{M}$, then $\rho(G) < \rho(B_n^*)$.*

Proof Let G and $\hat{G}$ satisfy the assumptions. Let $U = \{v \in V(\hat{G}) : |V(T(v))| \geq 2\}$. Assume that the vertices of graphs in $\mathcal{M}$ are labeled as in Figure 7. If $y \in U$, let $G_0 = G - \{xv_1, xw_1\}$, then G_0 is a spanning tree of G with maximum degree $\Delta(G_0) \geq 4$. By Lemma 3.2 (iv), we have $\rho(G) < \rho(B_n^*)$. Hence in the following, we assume that $x \notin U$ and $y \notin U$. Let G' be the bicyclic graph obtained from G by changing every tree $T(v)$ into a pendent path of length $|V(T(v))| - 1$ at $v \in V(\hat{G})$. By Lemma 2.4, $\rho(G) \leq \rho(G')$. In the following, it suffices to prove $\rho(G') < \rho(B_n^*)$. Since $n \geq 8$, $|U| \geq 1$. If $\hat{G} \cong \theta_4(0,1,1)$, by symmetry, we only need to consider $U = \{v_1\}$ or $U = \{v_1, w_1\}$. If $U = \{v_1\}$, by $G \ncong B_n^*$ and Lemma 2.4, $\rho(G) < \rho(B_n^*)$. If $U = \{v_1, w_1\}$, by Lemma 3.3, $\rho(G') < \rho(B_n^*)$.

Now we assume that $\hat{G} \in \{\theta_5(0,1,2), \theta_6(0,2,2), \theta_5(1,1,1)\}$. By Lemma 3.2 (iv), in the following, it suffices to prove that G' has a spanning tree G_0 such that G_0 is a non-caterpillar tree.

Case 1. $\hat{G} \cong \theta_5(0,1,2)$.

If there is a vertex $v \in V(\hat{G})$ such that $|V(T(v))| \geq 3$, we assume that $v = v_1$ and let $G_0 = G' - \{xy, w_1w_2\}$. Since $d_{G_0}(v_1) = 3$ and $v_1 \notin \sup(G_0)$, G_0 is a spanning non-caterpillar tree of G'. For other cases, the argument is similar. If $|V(T(v))| = 2$ for every $v \in U$, by $n \geq 8$, we have $U = \{v_1, w_1, w_2\}$. Let $G_0 = G' - \{xv_1, w_1w_2\}$. Since $d_{G_0}(y) = 3$ and $y \notin \sup(G_0)$, G_0 is a spanning non-caterpillar tree of G'.

Case 2. $\hat{G} \cong \theta_6(0,2,2)$.

By symmetry, we assume $v_1 \in U$, and let $G_0 = G' - \{v_1v_2, yw_2\}$. Since $d_{G_0}(x) = 3$ and $x \notin \sup(G_0)$, G_0 is a spanning non-caterpillar tree of G'.

Case 3. $\hat{G} \cong \theta_5(1,1,1)$.

By symmetry, we assume that $\{v_1, w_1\} \subseteq U$ or $U = \{v_1\}$. If $\{v_1, w_1\} \subseteq U$, let $G_0 = G' - \{xv_1, xw_1\}$. Since $d_{G_0}(y) = 3$ and $y \notin \sup(G_0)$, G_0 is a spanning non-caterpillar tree of G'. If $U = \{v_1\}$, by $n \geq 8$, we have $|V(T(v_1))| \geq 4$, and then we let $G_0 = G' - \{xu_1, yw_1\}$. Since $d_{G_0}(v_1) = 3$ and $v_1 \notin \sup(G_0)$, G_0 is a spanning non-caterpillar tree of G'. This completes the proof. □

Proof of Theorem 1.1 Suppose that G and $\hat{G}$ satisfy the assumptions. Let $U = \{v \in V(\hat{G}) : |V(T(v))| \geq 2\}$. Denote by G' the graph obtained from G by changing each tree $T(v)$ into a pendent path of length $|V(T(v))| - 1$ at $v \in \hat{G}$. By Lemma 2.4, $\rho(G) \leq \rho(G')$. To prove $\rho(G) < \rho(B_n^*)$, it suffices to prove that $\rho(G') < \rho(B_n^*)$. Note that by Lemmas 3.2 (iv), 3.4 and 3.5, we have $\rho(G') < \rho(B_n^*)$, if G' satisfies one of the following conditions:

(C1) G' has a spanning tree G_0 such that G_0 is a non-caterpillar tree or $p(G_0) \geq 5$ or $\Delta(G_0) \geq 4$;

(C2) $G' \in \mathcal{B}_n^2$;

(C3) G' has a spanning unicyclic subgraph $G_0 \in \mathcal{U}_n^*$.

So to prove $\rho(G') < \rho(B_n^*)$, it suffices to prove that G' satisfies one of the above three conditions.

We distinguish two cases:

Case 1. $\hat{G} \cong \infty_m(p,q)$ $(p \le q)$.

The vertices of $\infty_m(p,q)$ are labeled as in Figure 6. If $q \ge 5$, let $G_0 = G' - \{u_1u_2, w_3w_4\}$. Since $w_1 \notin \sup(G_0)$ and $d_{G_0}(w_1) \ge 3$, G_0 is a spanning non-caterpillar tree of G', i.e., G' satisfies (C1). If there is a vertex $v_i \in U$ for $0 \le i \le m-p-q+1$, let $G_0 = G' - \{u_2u_3, w_2w_3\}$. Then G_0 is a spanning tree of G' with $p(G_0) \ge 5$, i.e., G' satisfies (C1). If $m-p-q+1 = 0$, let $G_0 = G' - \{u_2u_3, w_2w_3\}$. Then G_0 is a spanning tree of G' with $\Delta(G_0) \ge 4$, i.e., G' satisfies (C1). Thus in the following we assume that $3 \le p \le q \le 4$, $v_i \notin U$ for $0 \le i \le m-p-q+1$ and $m-p-q+1 \ge 1$.

Subcase 1.1. $(p,q) = (3,3)$, i.e., $\hat{G} \cong \infty(3,3)$.

If $\{u_2, u_3\} \subseteq U$, let $G_0 = G' - \{u_2u_3, w_1w_2\}$. Since $u_1 \notin \sup(G_0)$ and $d_{G_0}(u_1) = 3$, then G_0 is a spanning non-caterpillar tree of G', i.e., G' satisfies (C1). For $\{w_2, w_3\} \subseteq U$, the argument is similar. Thus by symmetry we can assume that $U = \{u_2\}$ or $U = \{u_2, w_2\}$. Then $G' \in \mathcal{B}_n^2$, i.e., G' satisfies (C2).

Subcase 1.2. $(p,q) = (3,4)$ or $(p,q) = (4,4)$.

If $w_2 \in U$, let $G_0 = G' - \{w_2w_3, u_1u_2\}$. Since $w_1 \notin \sup(G_0)$ and $d_{G_0}(w_1) = 3$, then G_0 is a spanning non-caterpillar tree of G', i.e., G' satisfies (C1). For $w_4 \in U$, the argument is similar. Assume that $\{w_2, w_4\} \cap U = \emptyset$. Since $m-p-q+1 \ge 1$, $|V(G') \setminus (\{w_2, w_4\} \cup V(T(w_3)))| \ge 4$. Let x be the principal eigenvector of $\mathcal{Q}(G')$ corresponding to $\rho(G')$. By Lemma 2.1, $x_{w_2} = x_{w_4}$. By Perron-Frobenius Theorem, $x_v > 0$ for $v \in V(G')$. Let $G_0 = G' - \{w_1w_2\} + \{w_2w_4\}$. Since

$$d_{G_0}(u,v) - d_{G'}(u,v) = \begin{cases} 1, & \text{if } u = w_2 \text{ and } v \in \{V(G') \setminus (\{w_2, w_4\} \cup V(T(w_3)))\}, \\ -1, & \text{if } u = w_2 \text{ and } v = w_4, \\ 0, & \text{otherwise}, \end{cases}$$

we have

$$\begin{aligned} \rho(G_0) - \rho(G') &\ge x^{\mathrm{T}}(\mathcal{Q}(G_0) - \mathcal{Q}(G'))x \\ &= \sum_{v \in V(G') \setminus (\{w_2, w_4\} \cup V(T(w_3)))} (x_{w_2} + x_v)^2 - (x_{w_2} + x_{w_4})^2 \\ &> 4x_{w_2}^2 - 4x_{w_2}^2 = 0, \end{aligned}$$

i.e., $\rho(G') < \rho(G_0)$.

If $p = 3$, then $\hat{G_0} \cong \infty_m(3,3)$. By the argument similar to Subcase 1.1, we have $\rho(G_0) < \rho(B_n^*)$. Hence $\rho(G') < \rho(B_n^*)$ when $\hat{G} \cong \infty_m(3,4)$.

If $p = 4$, then $\hat{G_0} \cong \infty_m(4,3)$. By the argument similar to the above, we can get $\rho(G_0) < \rho(B_n^*)$. Thus $\rho(G') < \rho(B_n^*)$ when $\hat{G} \cong \infty_m(4,4)$.

Case 2. $\hat{G} \cong \theta_m(s,p,q)$.

Assume that the vertices of $\theta_m(s,p,q)$ are labeled as in Figure 6. If $\hat{G} \in \{\theta_4(0,1,1), \theta_5(0,1,2), \theta_6(0,2,2), \theta_5(1,1,1)\}$ and $G \ncong B_n^*$, from Lemma 3.6, we can get $\rho(G) < \rho(B_n^*)$.

If $s = 0, p \ge 2$, and $q \ge 3$, let $G_0 = G' - \{xv_1, w_1w_2\}$; if $s \ge 1, p \ge 2$, and $q \ge 2$, let $G_0 = G' - \{xv_1, xw_1\}$. For any case, G_0 is a spanning non-caterpillar tree of G', i.e., G' satisfies

(C1).

Now we assume that $s = 0$, $p = 1$, $q \geq 3$ or $s = 1$, $p = 1$, $q \geq 2$. If $y \in U$, let $G_0 = G' - \{xv_1, xw_1\}$. Then G_0 is a spanning tree of G' with $\Delta(G_0) \geq 4$, i.e., G' satisfies (C1). For $x \in U$, the argument is similar. Hence in the following, we assume that $\{x, y\} \cap U = \emptyset$.

If $v_1 \in U$, when $s = 0$, $p = 1$, $q \geq 3$, let $G_0 = G' - \{xv_1, w_1w_2\}$; when $s = 1$, $p = 1$, $q \geq 2$, let $G_0 = G' - \{xv_1, xw_1\}$. Then G_0 is a spanning non-caterpillar tree of G', i.e., G' satisfies (C1). When $s = 1$, $p = 1$, $q \geq 2$, the argument is similar for $u_1 \in U$. So in the following we suppose that $\{x, y, u_1, v_1\} \cap U = \emptyset$.

If there is vertex w_i $(1 \leq i \leq q)$ such that $|V(T(w_i))| \geq 3$, i.e., G' has a pendent path of length no less than 2 at vertex w_i $(1 \leq i \leq q)$, let $G_1 = G' - \{xu_1\}$. Then the length of the unique cycle in G_1 is at least 5. So we can find one edge uv on the cycle of G_1 such that $d_{G_1}(w_i, u) \geq 2$ and $d_{G_1}(w_i, v) \geq 2$. Let $G_0 = G_1 - \{uv\}$. Then G_0 is a spanning non-caterpillar tree of G'. Now we assume tht $|V(T(w_i))| = 2$ for each $w_i \in U$.

If $s = 0$, $p = 1$, $q \geq 3$, when $p(G) \geq 3$, let $G_0 = G - \{xv_1, xw_1\}$, and when $p(G) \leq 2$, let $G_2 = G - \{xy\}$. If $s = 1$, $p = 1$, $q \geq 2$, when $p(G) \geq 2$, let $G_0 = G - \{xu_1, xv_1\}$, and when $p(G) \leq 1$, let $G_2 = G - \{xu_1\}$. Then G_0 is a spanning tree of G' with $p(G_0) \geq 5$ (G' satisfies (C1)), and $G_2 \in \mathcal{U}_n^*$ (G' satisfies (C3)). This completes the proof of Theorem 1.1. $\square$

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