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Nonlinear Mixed Bi-Skew Jordan Triple Derivations on Prime $*$-Algebras

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Abstract Let $\mathcal{A}$ be a unital prime $*$-algebra with a nontrivial projection. In this paper, it is proved that a map $\Phi:\mathcal{A}\to\mathcal{A}$ satisfies

$$\Phi([A,B]_\diamond\circ C)=[\Phi(A),B]_\diamond\circ C+[A,\Phi(B)]_\diamond\circ C+[A,B]_\diamond\circ\Phi(C)$$

for all $A,B,C\in\mathcal{A}$ if and only if Φ is an additive $*$-derivation, where $A\circ B=A^*B+B^*A$ and $[A,B]_\diamond=A^*B-B^*A$.

Keywords mixed bi-skew Jordan triple derivations; $*$-derivations; prime $*$-algebras

MR(2020) Subject Classification 16W25; 16N60

1. Introduction

Let $\mathcal{A}$ be a $*$-algebra over the complex field $\mathbb{C}$. For $A,B\in\mathcal{A}$, we call the product $A\circ B=A^*B+B^*A$ the bi-skew Jordan product and $[A,B]_\diamond=A^*B-B^*A$ the bi-skew Lie product. These two new products have attracted many scholars to study [1–9]. Particular attention has been paid to understanding maps which preserve the bi-skew Jordan product and the bi-skew Lie product on C^*-algebras. Wang and Ji [1] proved that every bijective map preserving bi-skew Lie product between factor von Neumann algebras is a linear $*$-isomorphism or a conjugate linear $*$-isomorphism. Li et al. [9] proved that every bijective map preserving bi-skew Jordan product between von Neumann algebras with no central abelian projections is just the sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism. Taghavi and Gholampoor [5] studied surjective maps preserving bi-skew Jordan product between C^*-algebras.

Recall that an additive map $\Phi:\mathcal{A}\to\mathcal{A}$ is said to be an additive derivation if $\Phi(AB)=\Phi(A)B+A\Phi(B)$ for all $A,B\in\mathcal{A}$. Furthermore, Φ is said to be an additive $*$-derivation if it is an additive derivation and satisfies $\Phi(A^*)=\Phi(A)^*$ for all $A\in\mathcal{A}$. We say that $\Phi:\mathcal{A}\to\mathcal{A}$ is a nonlinear bi-skew Lie derivation or bi-skew Jordan derivation if

$$\Phi([A,B]_\diamond)=[\Phi(A),B]_\diamond+[A,\Phi(B)]_\diamond$$

or

$$\Phi(A\circ B)=\Phi(A)\circ B+A\circ\Phi(B)$$

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for all $A,B\in\mathcal{A}$. Recently, many authors have studied nonlinear bi-skew Lie derivations and bi-skew Jordan derivations. For example, Kong and Zhang [4] proved that any nonlinear bi-skew Lie derivation on a factor von Neumann algebra $\mathcal{A}$ with $\dim\mathcal{A}\geq 2$ is an additive $*$-derivation. Taghavi and Razeghi [8] investigated nonlinear bi-skew Lie derivations on prime $*$-algebras. Let Φ be a nonlinear bi-skew Lie derivation on a unital prime $*$-algebra with a nontrivial projection. They proved that if $\Phi(I)$ and $\Phi(iI)$ are self-adjoint, then Φ is an additive $*$-derivation. Darvish et al. [2] proved any nonlinear bi-skew Jordan derivation on prime $*$-algebras is an additive $*$-derivation. Khan [3] proved that any nonlinear bi-skew Lie triple derivation on a factor von Neumann algebra $\mathcal{A}$ with $\dim\mathcal{A}\geq 2$ is an additive $*$-derivation.

Recently, many authors have studied derivations corresponding to some mixed products. Zhou, Yang and Zhang [10] proved any map Φ from a unital $*$-algebra $\mathcal{A}$ containing a nontrivial projection to itself satisfying

$$\Phi([[A,B]_*,C])=[[\Phi(A),B]_*,C]+[[A,\Phi(B)]_*,C]+[[A,B]_*,\Phi(C)]$$

for all $A,B,C\in\mathcal{A}$, is an additive $*$-derivation, where $[A,B]=AB-BA$ is the usual Lie product of A and B and $[A,B]_*=AB-BA^*$ is the skew Lie product of A and B. Zhou and Zhang [11] proved that any map Φ on a factor von Neumann algebra $\mathcal{A}$ satisfying

$$\Phi([[A,B],C]_*)=[[\Phi(A),B],C]_*+[[A,\Phi(B)],C]_*+[[A,B],\Phi(C)]_*$$

for all $A,B,C\in\mathcal{A}$, is also an additive $*$-derivation. Zhao and Fang [7] gave a similar result on finite von Neumann algebras with no central summands of type I_1. Pang, Zhang and Ma [12] proved that if Φ is a second nonlinear mixed Jordan triple derivable mapping on a factor von Neumann algebra $\mathcal{A}$, that is,

$$\Phi(A\circ B\bullet C)=\Phi(A)\circ B\bullet C+A\circ\Phi(B)\bullet C+A\circ B\bullet\Phi(C)$$

for all $A,B,C\in\mathcal{A}$, then Φ is an additive $*$-derivation, where $A\circ B=AB+BA$ is the usual Jordan product of A and B and $A\bullet B=AB+BA^*$ is the Jordan $*$-product of A and B.

Motivated by the above mentioned works, in this paper, we will consider derivations corresponding to the new product of the mixture of the bi-skew Lie product and the bi-skew Jordan product. A map $\Phi:\mathcal{A}\to\mathcal{A}$ is said to be a nonlinear mixed bi-skew Jordan triple derivation if

$$\Phi([A,B]_\diamond\circ C)=[\Phi(A),B]_\diamond\circ C+[A,\Phi(B)]_\diamond\circ C+[A,B]_\diamond\circ\Phi(C)$$

for all $A,B,C\in\mathcal{A}$. Recall that an algebra $\mathcal{A}$ is prime if $A\mathcal{A}B=\{0\}$ for $A,B\in\mathcal{A}$ implies either $A=0$ or $B=0$. Let $\mathcal{A}$ be a unital prime $*$-algebra with a nontrivial projection. In this paper, we prove that Φ is a nonlinear mixed bi-skew Jordan triple derivation on $\mathcal{A}$ if and only if Φ is an additive $*$-derivation.

2. The main result and its proof

The main result in this paper reads as follows.

Theorem 2.1 *Let $\mathcal{A}$ be a unital prime $*$-algebra with a nontrivial projection P. Then a map*

$\Phi : \mathcal{A} \to \mathcal{A}$ satisfies

$$\Phi([A,B]_\diamond \circ C) = [\Phi(A), B]_\diamond \circ C + [A, \Phi(B)]_\diamond \circ C + [A,B]_\diamond \circ \Phi(C)$$

for all $A, B, C \in \mathcal{A}$ if and only if Φ is an additive ∗-derivation.

Let $P_1 = P$ and $P_2 = I - P$. Denote

$$\mathcal{A}_{ij} = P_i \mathcal{A} P_j, \quad i,j = 1,2.$$

Let

$$\mathcal{M} = \{A \in \mathcal{A} : A^* = A\},$$

$$\mathcal{N} = \{A \in \mathcal{A} : A^* = -A\},$$

$$\mathcal{M}_{12} = \{P_1 M P_2 + P_2 M P_1 : M \in \mathcal{M}\}$$

and

$$\mathcal{M}_{ii} = P_i \mathcal{M} P_i, \quad i = 1,2.$$

Thus, for any $M \in \mathcal{M}$, $M = M_{11} + M_{12} + M_{22}$, where $M_{11} \in \mathcal{M}_{11}$, $M_{12} \in \mathcal{M}_{12}$, $M_{22} \in \mathcal{M}_{22}$. Clearly, we only need to prove the necessity. We will complete the proof by several lemmas.

Lemma 2.2 *Let Φ be a nonlinear mixed bi-skew Jordan triple derivation on $\mathcal{A}$. Then $\Phi(0) = 0$.*

Proof Indeed, we have

$$\Phi(0) = \Phi([0,0]_\diamond \circ 0) = [\Phi(0), 0]_\diamond \circ 0 + [0, \Phi(0)]_\diamond \circ 0 + [0,0]_\diamond \circ \Phi(0) = 0. \quad \square$$

Lemma 2.3 *For any $M \in \mathcal{M}$, we have $\Phi(M) \in \mathcal{M}$.*

Proof For any $M \in \mathcal{M}$, $M = [M, \frac{i}{2}I]_\diamond \circ (\frac{i}{2}I)$. Since $[A,B]_\diamond \circ C \in \mathcal{M}$ for all $A, B, C \in \mathcal{A}$, we obtain

$$\begin{aligned}\Phi(M) &= \Phi([M, \tfrac{i}{2}I]_\diamond \circ (\tfrac{i}{2}I)) \\ &= [\Phi(M), \tfrac{i}{2}I]_\diamond \circ (\tfrac{i}{2}I) + [M, \Phi(\tfrac{i}{2}I)]_\diamond \circ (\tfrac{i}{2}I) + [M, \tfrac{i}{2}I]_\diamond \circ \Phi(\tfrac{i}{2}I) \in \mathcal{M}. \quad \square\end{aligned}$$

Lemma 2.4 *For any $A_{11} \in \mathcal{M}_{11}, M_{12} \in \mathcal{M}_{12}$ and $A_{22} \in \mathcal{M}_{22}$, we have*

$$\Phi(A_{11} + M_{12}) = \Phi(A_{11}) + \Phi(M_{12})$$

and

$$\Phi(M_{12} + A_{22}) = \Phi(M_{12}) + \Phi(A_{22}).$$

Proof Let $T = \Phi(A_{11} + M_{12}) - \Phi(A_{11}) - \Phi(M_{12})$. By Lemma 2.3, we have $T^* = T$. We only need to prove

$$T = T_{11} + T_{12} + T_{22} = 0.$$

Since $[iP_2, A_{11}]_\diamond = 0$, we obtain

$$\begin{aligned}&[\Phi(iP_2), A_{11} + M_{12}]_\diamond \circ (iI) + [iP_2, \Phi(A_{11} + M_{12})]_\diamond \circ (iI) + [iP_2, A_{11} + M_{12}]_\diamond \circ \Phi(iI) \\ &= \Phi([iP_2, A_{11} + M_{12}]_\diamond \circ (iI))\end{aligned}$$

$$= \Phi([iP_2, A_{11}]_\diamond \circ (iI)) + \Phi([iP_2, M_{12}]_\diamond \circ (iI))$$
$$= [\Phi(iP_2), A_{11} + M_{12}]_\diamond \circ (iI) + [iP_2, \Phi(A_{11}) + \Phi(M_{12})]_\diamond \circ (iI) +$$
$$[iP_2, A_{11} + M_{12}]_\diamond \circ \Phi(iI).$$

From this, we get $[iP_2, T]_\diamond \circ (iI) = 0$, and hence $T_{12} = T_{22} = 0$.

It follows from $[i(P_1 - P_2), M_{12}]_\diamond = 0$ that

$$[\Phi(i(P_1 - P_2)), A_{11} + M_{12}]_\diamond \circ (iI) + [i(P_1 - P_2), \Phi(A_{11} + M_{12})]_\diamond \circ (iI) +$$
$$[i(P_1 - P_2), A_{11} + M_{12}]_\diamond \circ \Phi(iI)$$
$$= \Phi([i(P_1 - P_2), A_{11} + M_{12}]_\diamond \circ (iI))$$
$$= \Phi([i(P_1 - P_2), A_{11}]_\diamond \circ (iI)) + \Phi([i(P_1 - P_2), M_{12}]_\diamond \circ (iI))$$
$$= [\Phi(i(P_1 - P_2)), A_{11} + M_{12}]_\diamond \circ (iI) + [i(P_1 - P_2), \Phi(A_{11}) + \Phi(M_{12})]_\diamond \circ (iI) +$$
$$[i(P_1 - P_2), A_{11} + M_{12}]_\diamond \circ \Phi(iI),$$

which implies that $[i(P_1 - P_2), T]_\diamond \circ (iI) = 0$. So $T_{11} = 0$, and then $T = 0$.

Similarly, we can show that $\Phi(M_{12} + A_{22}) = \Phi(M_{12}) + \Phi(A_{22})$. □

Lemma 2.5 *For any $A_{11} \in \mathcal{M}_{11}, M_{12} \in \mathcal{M}_{12}$ and $C_{22} \in \mathcal{M}_{22}$, we have*

$$\Phi(A_{11} + M_{12} + C_{22}) = \Phi(A_{11}) + \Phi(M_{12}) + \Phi(C_{22}).$$

Proof Let $T = \Phi(A_{11} + M_{12} + C_{22}) - \Phi(A_{11}) - \Phi(M_{12}) - \Phi(C_{22})$. By Lemma 2.3, we have $T^* = T$. Since $[iP_1, C_{22}]_\diamond = 0$, it follows from Lemma 2.4 that

$$[\Phi(iP_1), A_{11} + M_{12} + C_{22}]_\diamond \circ (iI) + [iP_1, \Phi(A_{11} + M_{12} + C_{22})]_\diamond \circ (iI) +$$
$$[iP_1, A_{11} + M_{12} + C_{22}]_\diamond \circ \Phi(iI)$$
$$= \Phi([iP_1, A_{11} + M_{12} + C_{22}]_\diamond \circ (iI))$$
$$= \Phi([iP_1, A_{11} + M_{12}]_\diamond \circ (iI)) + \Phi([iP_1, C_{22}]_\diamond \circ (iI))$$
$$= [\Phi(iP_1), A_{11} + M_{12} + C_{22}]_\diamond \circ (iI) + [iP_1, \Phi(A_{11}) + \Phi(M_{12}) + \Phi(C_{22})]_\diamond \circ (iI) +$$
$$[iP_1, A_{11} + M_{12} + C_{22}]_\diamond \circ \Phi(iI),$$

which yields that $[iP_1, T]_\diamond \circ (iI) = 0$. So $T_{11} = T_{12} = 0$. In the similar manner, we can get that $T_{22} = 0$. Hence $T = 0$. □

Lemma 2.6 *For any $M_{12}, B_{12} \in \mathcal{M}_{12}$, we have*

$$\Phi(M_{12} + B_{12}) = \Phi(M_{12}) + \Phi(B_{12}).$$

Proof Let $M_{12}, B_{12} \in \mathcal{M}_{12}$. Then $M_{12} = iU_{12} - iU_{12}^*$, $B_{12} = iV_{12} - iV_{12}^*$, where $U_{12}, V_{12} \in \mathcal{A}_{12}$. Since

$$[P_1 + U_{12} + U_{12}^*, P_2 + V_{12} + V_{12}^*]_\diamond \circ (-\frac{i}{2}I) = M_{12} + B_{12} + iM_{12}B_{12} - iB_{12}M_{12},$$

where

$$M_{12} + B_{12} \in \mathcal{M}_{12}$$

and

$$iM_{12}B_{12} - iB_{12}M_{12} = P_1(iU_{12}V_{12}^* - iV_{12}U_{12}^*)P_1 + P_2(iU_{12}^*V_{12} - iV_{12}^*U_{12})P_2 \in \mathcal{M}_{11} + \mathcal{M}_{22},$$

by Lemma 2.5, we have

$$\begin{aligned}
&\Phi(M_{12} + B_{12}) + \Phi(iM_{12}B_{12} - iB_{12}M_{12})\\
&= \Phi(M_{12} + B_{12} + iM_{12}B_{12} - iB_{12}M_{12})\\
&= \Phi([P_1 + U_{12} + U_{12}^*, P_2 + V_{12} + V_{12}^*]_\diamond \circ (-\frac{i}{2}I))\\
&= [\Phi(P_1) + \Phi(U_{12} + U_{12}^*), P_2 + V_{12} + V_{12}^*]_\diamond \circ (-\frac{i}{2}I)+\\
&\quad [P_1 + U_{12} + U_{12}^*, \Phi(P_2) + \Phi(V_{12} + V_{12}^*)]_\diamond \circ (-\frac{i}{2}I)+\\
&\quad [P_1 + U_{12} + U_{12}^*, P_2 + V_{12} + V_{12}^*]_\diamond \circ \Phi(-\frac{i}{2}I)\\
&= \Phi([P_1, P_2]_\diamond \circ (-\frac{i}{2}I)) + \Phi([P_1, V_{12} + V_{12}^*]_\diamond \circ (-\frac{i}{2}I))+\\
&\quad \Phi([U_{12} + U_{12}^*, P_2]_\diamond \circ (-\frac{i}{2}I)) + \Phi([U_{12} + U_{12}^*, V_{12} + V_{12}^*]_\diamond \circ (-\frac{i}{2}I))\\
&= \Phi(B_{12}) + \Phi(M_{12}) + \Phi(iM_{12}B_{12} - iB_{12}M_{12}),
\end{aligned}$$

which implies that $\Phi(M_{12} + B_{12}) = \Phi(M_{12}) + \Phi(B_{12})$. □

Lemma 2.7 *For any $A_{ii}, B_{ii} \in \mathcal{M}_{ii}$, $i = 1, 2$, we have*

$$\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).$$

Proof Let $T = \Phi(A_{11} + B_{11}) - \Phi(A_{11}) - \Phi(B_{11})$. By Lemma 2.3, we have $T^* = T$. Since $[iP_2, A_{11}]_\diamond = [iP_2, B_{11}]_\diamond = 0$, we obtain

$$\begin{aligned}
&[\Phi(iP_2), A_{11} + B_{11}]_\diamond \circ (iI) + [iP_2, \Phi(A_{11} + B_{11})]_\diamond \circ (iI)+\\
&\quad [iP_2, A_{11} + B_{11}]_\diamond \circ \Phi(iI)\\
&= \Phi([iP_2, A_{11} + B_{11}]_\diamond \circ (iI))\\
&= \Phi([iP_2, A_{11}]_\diamond \circ (iI)) + \Phi([iP_2, B_{11}]_\diamond \circ (iI))\\
&= [\Phi(iP_2), A_{11} + B_{11}]_\diamond \circ (iI) + [iP_2, \Phi(A_{11}) + \Phi(B_{11})]_\diamond \circ (iI)+\\
&\quad [iP_2, A_{11} + B_{11}]_\diamond \circ \Phi(iI).
\end{aligned}$$

So $[iP_2, T]_\diamond \circ (iI) = 0$, and hence $T_{12} = T_{22} = 0$. Now we have $T = T_{11}$.

For any $D_{12} \in \mathcal{A}_{12}$, let $M = D_{12} + D_{12}^*$. Then

$$[A_{11}, iM]_\diamond \circ (iI),\ [B_{11}, iM]_\diamond \circ (iI) \in \mathcal{M}_{12}.$$

It follows from Lemma 2.6 that

$$\begin{aligned}
&[\Phi(A_{11} + B_{11}), iM]_\diamond \circ (iI) + [A_{11} + B_{11}, \Phi(iM)]_\diamond \circ (iI)+\\
&\quad [A_{11} + B_{11}, iM]_\diamond \circ \Phi(iI)\\
&= \Phi([A_{11} + B_{11}, iM]_\diamond \circ (iI))
\end{aligned}$$

$$= \Phi([A_{11}, iM]_\diamond \circ (iI)) + \Phi([B_{11}, iM]_\diamond \circ (iI))$$
$$= [\Phi(A_{11}) + \Phi(B_{11}), iM]_\diamond \circ (iI) + [A_{11} + B_{11}, \Phi(iM)]_\diamond \circ (iI) +$$
$$[A_{11} + B_{11}, iM]_\diamond \circ \Phi(iI),$$

which implies that $[T, iM]_\diamond \circ (iI) = 0$, that is, $T_{11}D_{12} + D_{12}^* T_{11} = 0$. Multiplying the above equation by P_2 from the right, we have $T_{11}D_{12} = 0$ for all $D_{12} \in \mathcal{A}_{12}$. It follows from the primeness of $\mathcal{A}$ that $T_{11} = 0$, and so $T = 0$.

Similarly, we can prove that $\Phi(A_{22} + B_{22}) = \Phi(A_{22}) + \Phi(B_{22})$. □

Remark 2.8 From Lemmas 2.5–2.7, we can show that Φ is additive on $\mathcal{M}$.

Lemma 2.9 *Let Φ be a nonlinear mixed bi-skew Jordan triple derivation on $\mathcal{A}$. Then $\Phi(iI) = 0$.*

Proof For any $M \in \mathcal{M}$, it follows from Lemma 2.3 and Remark 2.8 that

$$\begin{aligned} 4\Phi(M) &= \Phi(4M) = \Phi([M, iI]_\diamond \circ (iI)) \\ &= [\Phi(M), iI]_\diamond \circ (iI) + [M, \Phi(iI)]_\diamond \circ (iI) + [M, iI]_\diamond \circ \Phi(iI) \\ &= 4\Phi(M) + 2i(\Phi(iI)^* M - M\Phi(iI)) + 2i(\Phi(iI)^* M - M\Phi(iI)) \\ &= 4\Phi(M) + 4i(\Phi(iI)^* M - M\Phi(iI)). \end{aligned}$$

So $\Phi(iI)^* M - M\Phi(iI) = 0$ for all $M \in \mathcal{M}$. Let $M = I$. Then $\Phi(iI) = \Phi(iI)^* \in \mathcal{M}$. Now we have $\Phi(iI)M = M\Phi(iI)$ for all $M \in \mathcal{M}$. Since for any $B \in \mathcal{A}$, $B = M_1 + iM_2$ with $M_1 = \frac{B+B^*}{2} \in \mathcal{M}$ and $M_2 = \frac{B-B^*}{2i} \in \mathcal{M}$, it follows that $\Phi(iI)B = B\Phi(iI)$ for all $B \in \mathcal{A}$. Hence

$$\Phi(iI) \in \mathcal{Z}(\mathcal{A}) \cap \mathcal{M}. \tag{2.1}$$

For any $M \in \mathcal{M}$, from Lemma 2.3, we see that

$$0 = \Phi([M, iI]_\diamond \circ I) = [M, iI]_\diamond \circ \Phi(I) = 2i(\Phi(I)^* M - M\Phi(I)).$$

In the same manner, we obtain

$$\Phi(I) \in \mathcal{Z}(\mathcal{A}) \cap \mathcal{M}. \tag{2.2}$$

Let $\Phi(iP_1) = W_1 + iW_2$, where $W_1, W_2 \in \mathcal{N}$. It follows from Eq. (2.1) that

$$\begin{aligned} 0 &= \Phi([iI, iP_1]_\diamond \circ (\frac{i}{2} I)) \\ &= [\Phi(iI), iP_1]_\diamond \circ (\frac{i}{2} I) + [iI, \Phi(iP_1)]_\diamond \circ (\frac{i}{2} I) \\ &= 2\Phi(iI)P_1 - 2iW_2, \end{aligned}$$

which impies that $iW_2 = \Phi(iI)P_1$, and so

$$\Phi(iP_1) = W_1 + \Phi(iI)P_1. \tag{2.3}$$

In view of Eqs. (2.2) and (2.3), we find that

$$\begin{aligned} 4\Phi(P_1) &= \Phi([I, iP_1]_\diamond \circ (iI)) = [\Phi(I), iP_1]_\diamond \circ (iI) + [I, \Phi(iP_1)]_\diamond \circ (iI) \\ &= 4\Phi(I)P_1 - 4iW_1, \end{aligned} \tag{2.4}$$

which yields that

$$\Phi(P_1) = \Phi(I)P_1 - iW_1. \tag{2.5}$$

On the other hand, by Eqs. (2.3) and (2.5), we obtain

$$\begin{aligned} 4\Phi(P_1) &= \Phi([P_1, iP_1]_\diamond \circ (iI)) \\ &= [\Phi(P_1), iP_1]_\diamond \circ (iI) + [P_1, \Phi(iP_1)]_\diamond \circ (iI) \\ &= 4\Phi(I)P_1 - 4i(P_1W_1 + W_1P_1). \end{aligned} \tag{2.6}$$

Comparing Eqs. (2.4) and (2.6), we have $P_1W_1 + W_1P_1 = W_1$, and so

$$P_1W_1P_1 = P_2W_1P_2 = 0. \tag{2.7}$$

From Eqs. (2.3) and (2.7), we get that

$$\Phi(iP_1) = W_1 + \Phi(iI)P_1 = \Phi(iI)P_1 + P_1W_1P_2 + P_2W_1P_1. \tag{2.8}$$

For any $A_{12} \in \mathcal{A}_{12}$, putting $M = A_{12} + A_{12}^*$, then $M \in \mathcal{M}$. It follows from Lemma 2.3 and Remark 2.8 that

$$\begin{aligned} -2\Phi(M) &= \Phi([iP_1, M]_\diamond \circ (iI)) \\ &= [\Phi(iP_1), M]_\diamond \circ (iI) + [iP_1, \Phi(M)]_\diamond \circ (iI) \\ &= -2(i\Phi(iP_1)^*M - iM\Phi(iP_1) + \Phi(M)P_1 + P_1\Phi(M)). \end{aligned} \tag{2.9}$$

Multiplying Eq. (2.9) by P_1 from the left and by P_2 from the right, then by Eq. (2.8), we have $\Phi(iI)A_{12} = 0$. It follows from the primeness of $\mathcal{A}$ that $\Phi(iI)P_1 = 0$. On the other hand, by Eq. (2.1), we also get $\Phi(iI)A_{12}^* = 0$. By the primeness of $\mathcal{A}$, $\Phi(iI)P_2 = 0$. Now we obtain $\Phi(iI) = \Phi(iI)P_1 + \Phi(iI)P_2 = 0$. □

Lemma 2.10 *(1) For any $N \in \mathcal{N}$, we have $\Phi(N)^* = -\Phi(N)$ and $\Phi(iN) = i\Phi(N) + i\Phi(I)N$;*

(2) Φ is additive on $\mathcal{N}$;

(3) For any $H, K \in \mathcal{N}$, we have $\Phi(H + iK) = \Phi(H) + i\Phi(K) + i\Phi(I)K$.

Proof (1) For any $N \in \mathcal{N}$, it follows from Lemma 2.9 that

$$0 = \Phi([iI, N]_\diamond \circ (iI)) = [iI, \Phi(N)]_\diamond \circ (iI) = -2(\Phi(N)^* + \Phi(N)).$$

So $\Phi(N)^* = -\Phi(N)$ for all $N \in \mathcal{N}$.

For any $N \in \mathcal{N}$, by Remark 2.8, Lemma 2.9 and Eq. (2.2), we get

$$4\Phi(iN) = \Phi([N, I]_\diamond \circ (iI)) = [\Phi(N), I]_\diamond \circ (iI) + [N, \Phi(I)]_\diamond \circ (iI) = 4i(\Phi(N) + \Phi(I)N).$$

That is,

$$\Phi(iN) = i\Phi(N) + i\Phi(I)N \tag{2.10}$$

for all $N \in \mathcal{N}$.

(2) For any $H, K \in \mathcal{N}$, we can get from Remark 2.8 and Eq. (2.10) that

$$i\Phi(H + K) + i\Phi(I)(H + K) = \Phi(i(H + K))$$

$$= \Phi(iH) + \Phi(iK) = i(\Phi(H) + \Phi(K)) + i\Phi(I)(H + K).$$

Hence $\Phi(H + K) = \Phi(H) + \Phi(K)$ for all $H, K \in \mathcal{N}$.

(3) For any $H, K \in \mathcal{N}$, by Remark 2.8, Lemma 2.9 and Eq. (2.10), we have

$$\begin{aligned} 4(i\Phi(K) + i\Phi(I)K) &= \Phi(4iK) = \Phi([H + iK, iI]_\diamond \circ (iI)) \\ &= [\Phi(H + iK), iI]_\diamond \circ (iI) = 2(\Phi(H + iK) + \Phi(H + iK)^*) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} 4(i\Phi(H) + i\Phi(I)H) &= \Phi(4iH) = \Phi([H + iK, I]_\diamond \circ (iI)) \\ &= [\Phi(H + iK), I]_\diamond \circ (iI) + [H + iK, \Phi(I)]_\diamond \circ (iI) \\ &= 2i(\Phi(H + iK) - \Phi(H + iK)^*) + 4i\Phi(I)H. \end{aligned} \tag{2.12}$$

In view of Eqs. (2.11) and (2.12), we obtain

$$\Phi(H + iK) = \Phi(H) + i\Phi(K) + i\Phi(I)K. \quad \square$$

Lemma 2.11 *(1) For any $A \in \mathcal{A}$, we have $\Phi(A)^* = \Phi(A)$;*

(2) Φ is additive on $\mathcal{A}$.

Proof (1) For any $A \in \mathcal{A}$, $A = A_1 + iA_2$, where $A_1, A_2 \in \mathcal{N}$. Then we can get from Eq. (2.2) and Lemma 2.10 that

$$\begin{aligned} \Phi(A)^* &= \Phi(A_1 + iA_2)^* = (\Phi(A_1) + i\Phi(A_2) + i\Phi(I)A_2)^* \\ &= -\Phi(A_1) + i\Phi(A_2) + i\Phi(I)A_2 = \Phi(-A_1 + iA_2) = \Phi(A^*) \end{aligned}$$

for all $A \in \mathcal{A}$.

(2) For any $A, B \in \mathcal{A}$, $A = A_1 + iA_2$, $B = B_1 + iB_2$, where $A_i, B_i \in \mathcal{N}$ $(i = 1, 2)$. It follows from Lemma 2.10 that

$$\begin{aligned} \Phi(A + B) &= \Phi((A_1 + B_1) + i(A_2 + B_2)) \\ &= \Phi(A_1 + B_1) + i\Phi(A_2 + B_2) + i\Phi(I)(A_2 + B_2) \\ &= (\Phi(A_1) + i\Phi(A_2) + i\Phi(I)A_2) + (\Phi(B_1) + i\Phi(B_2) + i\Phi(I)B_2) \\ &= \Phi(A) + \Phi(B). \end{aligned}$$

Hence Φ is additive on $\mathcal{A}$. $\square$

Lemma 2.12 *(1) $\Phi(I) = 0$;*

(2) For any $A \in \mathcal{A}$, we have $\Phi(iA) = i\Phi(A)$.

Proof (1) In view of Eqs. (2.5) and (2.7), we have

$$\Phi(P_1) = \Phi(I)P_1 - iP_1W_1P_2 - iP_2W_1P_1. \tag{2.13}$$

For any $A_{12} \in \mathcal{A}_{12}$, it follows from Lemmas 2.9–2.11 that

$$2(i(\Phi(A_{12})^* - \Phi(A_{12})) + i\Phi(I)(A_{12}^* - A_{12}))$$

$$\begin{aligned}
&= 2\Phi(i(A_{12}^* - A_{12})) = \Phi([P_1, A_{12} + A_{12}^*]_\diamond \circ (iI)) \\
&= [\Phi(P_1), A_{12} + A_{12}^*]_\diamond \circ (iI) + [P_1, \Phi(A_{12} + A_{12}^*)]_\diamond \circ (iI) \\
&= 2i[(A_{12} + A_{12}^*)(\Phi(I)P_1 - iP_1W_1P_2 - iP_2W_1P_1) - \\
&\quad (\Phi(I)P_1 - iP_1W_1P_2 - iP_2W_1P_1)(A_{12} + A_{12}^*) + \\
&\quad (\Phi(A_{12})^* + \Phi(A_{12}))P_1 - P_1(\Phi(A_{12}) + \Phi(A_{12})^*)].
\end{aligned}$$

Multiplying by P_1 from the left and by P_2 from the right, we obtain

$$P_1\Phi(A_{12})^*P_2 = 0. \tag{2.14}$$

On the other hand, we also have

$$\begin{aligned}
&2(\Phi(A_{12}) + \Phi(A_{12})^*) = \Phi([P_1, i(A_{12} - A_{12}^*)]_\diamond \circ (iI)) \\
&= [\Phi(P_1), i(A_{12} - A_{12}^*)]_\diamond \circ (iI) + [P_1, \Phi(i(A_{12} - A_{12}^*))]_\diamond \circ (iI) \\
&= 2[(A_{12}^* - A_{12})(\Phi(I)P_1 - iP_1W_1P_2 - iP_2W_1P_1) + \\
&\quad (\Phi(I)P_1 - iP_1W_1P_2 - iP_2W_1P_1)(A_{12} - A_{12}^*) + \\
&\quad (\Phi(A_{12})^* - \Phi(A_{12}) + (A_{12}^* - A_{12})\Phi(I))P_1 + \\
&\quad P_1(\Phi(A_{12}) - \Phi(A_{12})^* + \Phi(I)(A_{12} - A_{12}^*))].
\end{aligned}$$

Multiplying by P_1 from the left and by P_2 from the right, we obtain $\Phi(I)A_{12} = 0$ by the Eq. (2.14). It follows from the primeness of $\mathcal{A}$ that $\Phi(I)P_1 = 0$. On the other hand, by Eq. (2.2), we also get $\Phi(I)A_{12}^* = 0$. So $\Phi(I)P_2 = 0$. Now we obtain $\Phi(I) = \Phi(I)P_1 + \Phi(I)P_2 = 0$.

(2) For any $N \in \mathcal{N}$, by Lemma 2.10 (1) and $\Phi(I) = 0$, we have

$$\Phi(iN) = i\Phi(N). \tag{2.15}$$

For any $A \in \mathcal{A}$, $A = A_1 + iA_2$, where $A_1, A_2 \in \mathcal{N}$. From Lemma 2.11 (2) and Eq. (2.15), we have

$$\Phi(iA) = \Phi(i(A_1 + iA_2)) = \Phi(iA_1 - A_2) = i(\Phi(A_1) + i\Phi(A_2)) = i\Phi(A)$$

for all $A \in \mathcal{A}$. □

Lemma 2.13 *Φ is a derivation on $\mathcal{A}$.*

Proof For any $A, B \in \mathcal{A}$, by Lemmas 2.11 (2) and 2.12 (2), we have

$$\begin{aligned}
&2(\Phi(A^*B + B^*A)) = \Phi([A, iB]_\diamond \circ (iI)) \\
&= [\Phi(A), iB]_\diamond \circ (iI) + [A, i\Phi(B)]_\diamond \circ (iI) \\
&= 2(\Phi(A)^*B + B^*\Phi(A) + A^*\Phi(B) + \Phi(B)^*A),
\end{aligned}$$

which impies that

$$\Phi(A^*B + B^*A) = \Phi(A)^*B + B^*\Phi(A) + A^*\Phi(B) + \Phi(B)^*A. \tag{2.16}$$

On the other hand, we also have

$$-2i(\Phi(A^*B - B^*A)) = \Phi([iA, iB]_\diamond \circ (iI))$$

$$= [i\Phi(A), iB]_\diamond \circ (iI) + [iA, i\Phi(B)]_\diamond \circ (iI)$$
$$= -2i(\Phi(A)^*B - B^*\Phi(A) + A^*\Phi(B) - \Phi(B)^*A),$$

which yields that

$$\Phi(A^*B - B^*A) = \Phi(A)^*B - B^*\Phi(A) + A^*\Phi(B) - \Phi(B)^*A. \tag{2.17}$$

By summing Eqs. (2.16) and (2.17), we obtain

$$\Phi(A^*B) = \Phi(A)^*B + A^*\Phi(B).$$

Then we can get from Lemma 2.11 (1) that

$$\Phi(AB) = \Phi(A)B + A\Phi(B). \quad \square$$

Now, from Lemmas 2.11 and 2.13, we obtain that Φ is an additive $*$-derivation on $\mathcal{A}$. This completes the proof of Theorem 2.1.

3. Corollaries

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$, and $\mathcal{A} \subseteq B(\mathcal{H})$ be a von Neumann algebra. $\mathcal{A}$ is a factor if its center is $\mathbb{C}I$. It is well known that a factor von Neumann algebra is prime. Now we can get the following corollary.

Corollary 3.1 *Let $\mathcal{A}$ be a factor von Neumann algebra with dim$(\mathcal{A}) \geq 2$. Then $\Phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear mixed bi-skew Jordan triple derivation if and only if Φ is an additive $*$-derivation.*

We denote the subalgebra of all bounded finite rank operators by $\mathcal{F}(\mathcal{H}) \subseteq B(\mathcal{H})$. We call a subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ a standard operator algebra if it contains $\mathcal{F}(\mathcal{H})$. Now we have the following corollary.

Corollary 3.2 *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$ containing the identity operator I. Suppose that $\mathcal{A}$ is closed under the adjoint operation. Then $\Phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear mixed bi-skew Jordan triple derivation if and only if Φ is a linear $*$-derivation. Moreover, there exists an operator $T \in B(\mathcal{H})$ satisfying $T + T^* = 0$ such that $\Phi(A) = AT - TA$ for all $A \in \mathcal{A}$, i.e., Φ is inner.*

Proof Since $\mathcal{A}$ is prime, we know that Φ is an additive $*$-derivation. It follows from [13] that Φ is a linear inner derivation, i.e., there exists an operator $S \in B(\mathcal{H})$ such that $\Phi(A) = AS - SA$. Using the fact $\Phi(A^*) = \Phi(A)^*$, we have

$$A^*S - SA^* = \Phi(A^*) = \Phi(A)^* = -A^*S^* + S^*A^*$$

for all $A \in \mathcal{A}$. This leads to $A^*(S + S^*) = (S + S^*)A^*$. Hence, $S + S^* = \lambda I$ for some $\lambda \in \mathbb{R}$. Let us set $T = S - \frac{1}{2}\lambda I$. One can check that $T + T^* = 0$ such that $\Phi(A) = AT - TA$.

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