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The Uniform Asymptotics for the Tail of Poisson Shot Noise Process with Dependent and Heavy-Tailed Shocks

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Abstract This paper considers the uniform asymptotic tail behavior of a Poisson shot noise process with some dependent and heavy-tailed shocks. When the shocks are bivariate upper tail asymptotic independent nonnegative random variables with long-tailed and dominatedly varying tailed distributions, and the shot noise function has both positive lower and upper bounds, a uniform asymptotic formula for the tail probability of the process has been established. Furthermore, when the shocks have continuous and consistently varying tailed distributions, the positive lower-bound condition on the shot noise function can be removed. For the case that the shot noise function is not necessarily upper-bounded, a uniform asymptotic result is also obtained when the shocks follow a pairwise negatively quadrant dependence structure.

Keywords Poisson shot noise process; dependent shock; heavy-tailed distribution; uniform asymptotics

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1. Introduction

This paper will consider the following stochastic process

$$S(t)=\sum_{k=1}^{\infty}X_kh(t,\tau_k)\mathbf{1}_{\{\tau_k\le t\}},\quad t\ge 0, \tag{1.1}$$

where $\{X_k,k\ge 1\}$ is a sequence of nonnegative and identically distributed random variables, $\{\tau_k,k\ge 1\}$ is another sequence of nonnegative random variables, independent of $\{X_k,k\ge 1\}$, $h(t,s)$ is a nonnegative Borel measurable function and $\mathbf{1}_A$ is the indicator function of a set A. Assume that $N(t)=\sup\{n\ge 1:\tau_n\le t\}$, $t\ge 0$, is a Poisson process with intensity $\lambda(t)>0$,

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$t \geq 0$ and cumulative intensity $m(t) = \int_0^t \lambda(s)\mathrm{d}s$, $t > 0$. Then $S(t)$ is called a Poisson shot noise process and $h(t,s)$ is called the shot noise function, which is used to indicate the effect of each shock X_k, $k \geq 1$ on the system up to time t. If $\{N(t), \geq 0\}$ is a general counting process then $S(t)$ is generally called a shot noise process. For some other formulations of the shot noise process with different degree of generality one can see [1–6] and references there in.

Shot noise processes are not a new phenomenon in probabilistic modeling. For example, in the insurance risk theory, a shot noise process is used to model the aggregate claim of an insurance company. Weng et al. [5] gave some examples to explain that in this context, X_k is used to denote the k-th claim size and the shot noise function $h(t,s)$ is imposed to capture the interest factor, the factor of delay, or both factors simultaneously. The applications of shot noise process to insurance and financial risk theory can be found in [1, 2, 7–9] and so on. Most of the above researches require that the shot noise processes have independent shocks. This assumption does not correspond to the actual circumstances of the insurance and financial business. Weng et al. [5] considered the dependent shocks. They obtained the tail behavior of the Poisson shot noise process (1.1), where the shocks X_k, $k \geq 1$, are bivariate upper tail independent.

In this paper we will still investigate the Poisson shot noise process (1.1) with dependent shocks X_k, $k \geq 1$. When the shocks X_k, $k \geq 1$ have heavy-tailed distributions, we will give the uniform asymptotics of the tail of the Poisson shot noise process (1.1). The rest of the paper is organized as follows. Section 2 includes preliminaries and main results. Section 3 gives the proofs of main results.

2. Preliminaries and main results

Hereafter, all limit relationship is $x \to \infty$, unless stated otherwise. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$; write $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \leq 1$; write $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \geq 1$; write $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$; write $a(x) = O(b(x))$ if $\limsup a(x)/b(x) < \infty$. Furthermore, for two bivariate functions $a(x,t)$ and $b(x,t)$, we write $a(x,t) \sim b(x,t)$ uniformly for all t from some nonempty set Δ as $x \to \infty$, if

$$\lim_{x\to\infty} \sup_{t\in\Delta} |\frac{a(x,t)}{b(x,t)} - 1| = 0;$$

write $a(x,t) \lesssim b(x,t)$ uniformly for all $t \in \Delta$ as $x \to \infty$, if

$$\limsup_{x\to\infty} \sup_{t\in\Delta} \frac{a(x,t)}{b(x,t)} \leq 1$$

and write $a(x,t) \gtrsim b(x,t)$ uniformly for all $t \in \Delta$ as $x \to \infty$, if

$$\liminf_{x\to\infty} \inf_{t\in\Delta} \frac{a(x,t)}{b(x,t)} \geq 1.$$

For real numbers x and y, let $x \wedge y = \min\{x, y\}$.

This paper will investigate the heavy-tailed shocks. For a proper distribution V on $(-\infty, \infty)$, let $\overline{V} = 1 - V$ be its tail. A random variable ξ or its corresponding distribution V satisfying $\overline{V}(x) > 0$ for all $x \in (-\infty, \infty)$ is called heavy-tailed if for all $\beta > 0$, $\mathbb{E}e^{\beta\xi} = \infty$, otherwise, we say that the random variable ξ (or V) is light-tailed. One of the heavy-tailed subclasses is the class $\mathcal{D}$ of distributions with dominatedly varying tails. Say that a distribution V on $(-\infty,\infty)$ belongs to the class $\mathcal{D}$, if for any $0<y<1$,

$$\overline{V}(xy)=O(\overline{V}(x)).$$

Another important subclass of the heavy-tailed distribution class is the class $\mathcal{L}$ of distributions with long tails. Say that a distribution V on $(-\infty,\infty)$ belongs to the class $\mathcal{L}$, if for any $y>0$,

$$\overline{V}(x+y)\sim\overline{V}(x).$$

A smaller class is the class $\mathcal{C}$ of distributions with consistently varying tails. Say that a distribution V on $(-\infty,\infty)$ belongs to the class $\mathcal{C}$, if

$$\lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{V}(xy)}{\overline{V}(x)}=1,$$

or, equivalently,

$$\lim_{y\downarrow 1}\liminf_{x\to\infty}\frac{\overline{V}(xy)}{\overline{V}(x)}=1.$$

A subclass of the class $\mathcal{C}$ is the class of distributions with regularly varying tails. Say that a distribution V on $(-\infty,\infty)$ belongs to the class $\mathcal{R}_{-\alpha}$ for some $0\le\alpha<\infty$, if

$$\overline{V}(xy)\sim y^{-\alpha}\overline{V}(x)$$

holds for all $y>0$. Let $\mathcal{R}$ denote the union of all $\mathcal{R}_{-\alpha}$ over the range $0\le\alpha<\infty$. It is well known that $\mathcal{R}\subset\mathcal{C}\subset\mathcal{L}\cap\mathcal{D}$ (see [10–13]).

For a distribution V on $(-\infty,\infty)$, denote its upper Matuszewska index by

$$J_V^+=-\lim_{y\to\infty}\frac{\log\overline{V}_*(y)}{\log y}\quad\text{with}\quad\overline{V}_*(y):=\liminf_{x\to\infty}\frac{\overline{V}(xy)}{\overline{V}(x)},\quad y>1.$$

From [14, Chapter 2.1], we know that $V\in\mathcal{D}$ if and only if $J_V^+<\infty$.

When the shocks, X_k, $k\ge 1$ are bivariate upper tail independent, Weng et al. [5] obtained the tail behavior of the shot noise process (1.1) under the following assumptions.

Assumption 2.1 $\{X_k,k\ge 1\}$ *are nonnegative and identically distributed as a generic random variable* X *with a common distribution* F *and satisfy the following bivariate upper tail independent condition:*

$$\lim_{x\to\infty}\frac{\mathbb{P}(X_i>x,X_j>x)}{\mathbb{P}(X_i>x)}=0\ \ \textit{for all}\ \ i\ne j\ge 1.$$

Before giving the next assumption, we first give the definition of exchangeable random variables [15, Section 7.2]. Say that n random variables $\xi_1,\dots,\xi_n$ is exchangeable if $\xi_{k_1},\dots,\xi_{k_n}$ has the same joint distribution for all permutation $(k_1,\dots,k_n)$ of $(1,\dots,n)$. The infinite sequence of random variables $\{\xi_k,k\ge 1\}$ is said to be exchangeable if every finite subsequence $\xi_1,\dots,\xi_n$ is exchangeable.

Assumption 2.2 $\{X_k,k\ge 1\}$ *are exchangeable.*

Assumption 2.3 *For a real number $t > 0$, let $Z(t)$ denote a random variable with density function $\lambda(s)/m(t)$, $0 < s < t$ and $\{Z_k(t), k \geq 1\}$ are independent copies of $Z(t)$ and are independent of all other random variables or processes involved in this paper.*

Assumption 2.4 *For a fixed real number $T > 0$, the shot noise function $h(t,s) : [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfies these conditions:*

(i) There exist two constants a and b with $0 < a \leq b < \infty$ such that $a \leq h(t,s) \leq b$ for any $0 < s \leq t \leq T$;

(ii) $h(t,s) = 0$ for $s > t$.

Assumption 2.4′ *For a fixed real number $T > 0$, the shot noise function $h(t,s) : [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfies these conditions:*

(i) There exists a constant $b > 0$ such that $0 < h(t,s) \leq b$ for any $0 < s \leq t \leq T$;

(ii) $h(t,s) = 0$ for $s > t$.

This paper still investigates the dependent shocks X_k, $k \geq 1$ for the shot noise process (1.1). When the shocks, X_k, $k \geq 1$ have a stronger dependence structure, i.e., the bivariate upper tail asymptotic independent in the following Assumption 2.1′ than the bivariate upper tail independent in Assumption 2.1, the paper gives the uniform asymptotics of the tail of the shot noise process (1.1).

Assumption 2.1′ *$\{X_k, k \geq 1\}$ are nonnegative and identically distributed as a generic random variable X with a common distribution F and satisfy the following bivariate upper tail asymptotic independent condition:*

$$\lim_{x\wedge y\to\infty} \frac{\mathbb{P}(X_i > x, X_j > y)}{\mathbb{P}(X_i > x)} = 0 \ \text{ for all } \ i \neq j \geq 1.$$

The upper tail asymptotic independent structure was proposed by Geluk and Tang [16]. This dependence structure has been investigated by many researchers, such as [17–20] and so on. From the definitions in Assumptions 2.1 and 2.1′, the bivariate upper tail asymptotic independence structure is stronger than the bivariate upper tail independence structure.

Under Assumptions 2.1′, 2.2–2.4, the following theorem gives the uniform asymptotics of the tail of the shot noise process (1.1) with the shocks $X_k, k \geq 1$ coming from the class $\mathcal{L} \cap \mathcal{D}$.

Theorem 2.5 *Consider the shot noise process (1.1). Suppose that Assumptions 2.1′, 2.2–2.4 are satisfied and that $F \in \mathcal{L} \cap \mathcal{D}$. Then*

$$\mathbb{P}(S(t) > x) \sim m(t)\mathbb{P}(h(t, Z(t))X > x) \tag{2.1}$$

holds uniformly for $t \in (0, T]$, where $m(t) = \int_0^t \lambda(s)\mathrm{d}s$, $t > 0$.

Remark 2.6 Comparing the above result with [5, Theorem 2.1], we have extended the scope of F in [5, Theorem 2.1] from the class $\mathcal{R}_{-\alpha}$, $0 < \alpha < \infty$ to the class $\mathcal{L} \cap \mathcal{D}$ under Assumptions 2.1′, 2.2–2.4.

We next aim to remove the lower-bound restriction on the shot noise function h in Assumption

2.4. In doing so we need to confine the distribution F to the class $\mathcal{C}$.

Theorem 2.7 *Consider the shot noise process* (1.1). *Suppose that Assumptions 2.1′, 2.2, 2.3 and 2.4′ are satisfied and that* $F \in \mathcal{C}$ *and is continuous. Then* (2.1) *holds uniformly for* $t \in (0, T]$.

Remark 2.8 When the shocks X_k, $k \geq 1$ are bivariate upper tail independent with $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$, under Assumptions 2.2, 2.3 and 2.4′, [5, Theorem 2.2] obtained that (2.1) holds for any fixed $t > 0$, which is not uniform for t. From Theorem 2.7, we find that for the bivariate upper tail asymptotic independent shocks X_k, $k \geq 1$, the equation (2.1) holds uniformly for t in a finite time interval.

In Theorem 2.7, we still need the shot noise function h to have an upper bound. When the shocks X_k, $k \geq 1$ have a pairwise negatively quadrant dependence structure, which is stronger than the bivariate upper tail asymptotic independence structure, the following result removes the upper-bound restriction on the shot noise function h. For this, we firstly give two assumptions.

Assumption 2.1* $\{X_k, k \geq 1\}$ *are nonnegative and identically distributed as a generic random variable* X *with a common distribution* F *and satisfy the following pairwise negative quadrant dependent condition: for all* $x \geq 0$ *and* $y \geq 0$

$$\mathbb{P}(X_i > x, X_j > y) \leq \mathbb{P}(X_i > x)\mathbb{P}(X_j > y) \ \text{ for all } \ i \neq j \geq 1.$$

The negative quadrant dependence structure was introduced by Lehmann [21]. We know that the negative quadrant dependence structure implies the upper tail asymptotic independence structure.

Assumption 2.4* *For a fixed real number* $T > 0$, *the shot noise function* $h(t, s) : [0, \infty) \times [0, \infty) \to [0, \infty)$ *satisfies these conditions:*

(i) $\inf_{t \in (0,T]} h(t, Z(t))$ *is nondegenerate at zero;*

(ii) For some $p > J_F^+$, $\mathbb{E}(\sup_{t \in (0,T]} h(t, Z(t)))^p < \infty$.

Theorem 2.9 *Consider the shot noise process* (1.1). *Suppose that Assumptions 2.1***, 2.2, 2.3 and 2.4** *are satisfied and that* $F \in \mathcal{C}$ *and is continuous. Then* (2.1) *holds uniformly for* $t \in (0, T]$.

In Section 3, the proofs of Theorems 2.5, 2.7 and 2.9 are given.

3. Proofs of main results

We first prove Theorem 2.5.

3.1. Proof of Theorem 2.5

Before giving the proof, we will present some lemmas. The first lemma is a combination of [14, Proposition 2.2.1] and [22, Lemma 3.5].

Lemma 3.1 *If* $V \in \mathcal{D}$, *then for each* $p > J_V^+$, *there exist positive constants* C_1 *and* D_1 *such* *that*

$$\frac{\overline{V}(y)}{\overline{V}(x)} \le C_1(\frac{y}{x})^{-p}$$

for all $x \ge y \ge D_1$ *and*

$$x^{-p} = o(\overline{V}(x)).$$

The following two lemmas correspond to [23, Lemma 3.2] and [5, Lemma A.5], respectively.

Lemma 3.2 *Suppose that* $\{M(t), t \ge 0\}$ *is a renewal counting process with a renewal function* $\mathbb{E}M(t) > 0$ *for all* $t > 0$. *Then it holds for all* $T > 0$ *and all* $v > 0$ *that*

$$\lim_{x\to\infty} \sup_{t\in(0,T]} \frac{1}{\mathbb{E}M(t)} \mathbb{E}((M(t))^v \mathbf{1}_{\{M(t)>x\}}) = 0.$$

Lemma 3.3 *Suppose that Assumptions 2.2 and 2.3 are satisfied. Then* $S(t)$ *defined in* (1.1) *is identically distributed as* $\sum_{k=1}^{N(t)} h(t, Z_k(t))X_k$ *for any* $t \ge 0$.

Lemma 3.4 *Suppose that Assumptions 2.1′ and 2.4 are satisfied. For each real number* $t > 0$, $\{\xi_k(t), k \ge 1\}$ *are nonnegative and i.i.d random variables, which are independent of* $\{X_k, k \ge 1, X\}$. *If* $F \in \mathcal{L} \cap \mathcal{D}$ *then for any* $n \ge 1$,

$$\lim_{x\to\infty} \sup_{t\in(0,T]} \Big|\frac{\mathbb{P}(\sum_{k=1}^n h(t, \xi_k(t))X_k > x)}{n\mathbb{P}(h(t, \xi(t))X > x)} - 1\Big| = 0. \tag{3.1}$$

Proof We will follow the line of the proof of [5, Lemma A.8]. It is obvious that (3.1) holds for $n = 1$. Hereafter, we assume that $n \ge 2$.

Since $F \in \mathcal{L}$, there exists a positive increasing and slowly varying function $l(x) \uparrow \infty$ such that $\frac{l(x)}{x} \to 0$ and for any fixed constant $c_0 > 0$,

$$\overline{F}(x - c_0 l(x)) \sim \overline{F}(x), \tag{3.2}$$

which implies that for any $0 < \varepsilon < 1$, there exists a constant $x_1 > 0$, depending only on F and ε, such that for all $x \ge x_1$

$$l(\frac{x}{b}) \ge (1 - \varepsilon)l(x) \tag{3.3}$$

and

$$\overline{F}(x - \frac{l(x)}{a(1-\varepsilon)}) \le (1 + \varepsilon)\overline{F}(x). \tag{3.4}$$

Since $F \in \mathcal{D}$, there exists a constant $c > 0$ such that for all $x > 0$,

$$\overline{F}(\frac{x}{a}) \ge c\overline{F}(x).$$

Since $\{X_k, 1 \le k \le n\}$ are bivariate upper tail asymptotic independent and $F \in \mathcal{D}$, by Assumption 2.4, there exists a constant $x_2 \ge x_1$, depending only on F, ε and n, such that for all $x \ge x_2$, $1 \le i \ne j \le n$ and $t \in (0, T]$,

$$\mathbb{P}(X_i > \frac{x}{b}, X_j > \frac{x}{b}) \le \mathbb{P}(X_i > \frac{x}{nb}, X_j > \frac{l(x)}{(n-1)b}) \le \varepsilon\overline{F}(x) \tag{3.5}$$

and

$$\mathbb{P}(h(t, \xi(t))X > x) \ge \overline{F}(\frac{x}{a}) \ge c\overline{F}(x). \tag{3.6}$$

We firstly estimate the lower bound of $\mathbb{P}(\sum_{k=1}^{n} h(t,\xi_k(t))X_k > x)$ for $t \in (0,T]$ as $x \to \infty$. By Bonferroni Inequality, Assumption 2.4, (3.5) and (3.6), for all $x \geq x_2$ and $t \in (0,T]$

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{k=1}^{n} h(t,\xi_k(t))X_k > x\Big) \geq \mathbb{P}\Big(\bigcup_{k=1}^{n}\{h(t,\xi_k(t))X_k > x\}\Big)\\
&\geq \sum_{k=1}^{n}\mathbb{P}(h(t,\xi_k(t))X_k > x) - \sum_{1\leq i\neq j\leq n}\mathbb{P}(h(t,\xi_i(t))X_i > x, h(t,\xi_j(t))X_j > x)\\
&\geq \sum_{k=1}^{n}\mathbb{P}(h(t,\xi_k(t))X_k > x) - \sum_{1\leq i\neq j\leq n}\mathbb{P}(bX_i > x, bX_j > x)\\
&\geq n\mathbb{P}(h(t,\xi(t))X > x) - \frac{n\varepsilon}{c}n\mathbb{P}(h(t,\xi(t))X > x).
\end{aligned}$$

Hence,

$$\liminf_{x\to\infty}\inf_{t\in(0,T]}\frac{\mathbb{P}(\sum_{k=1}^{n} h(t,\xi_k(t))X_k > x)}{n\mathbb{P}(h(t,\xi(t))X > x)} \geq 1 - \frac{n\varepsilon}{c}.$$

Letting $\varepsilon \downarrow 0$, we get that

$$\liminf_{x\to\infty}\inf_{t\in(0,T]}\frac{\mathbb{P}(\sum_{k=1}^{n} h(t,\xi_k(t))X_k > x)}{n\mathbb{P}(h(t,\xi(t))X > x)} \geq 1.$$

Now we estimate the upper bound of $\mathbb{P}(\sum_{k=1}^{n} h(t,\xi_k(t))X_k > x)$ for $t \in (0,T]$ as $x \to \infty$. Since $F \in \mathcal{L}$, we use the $l(x)$ in (3.2) to deal with the upper bound. By Assumption 2.4, for all $x > 0$ and $t \in (0,T]$, it holds that

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{k=1}^{n} h(t,\xi_k(t))X_k > x\Big) \leq \mathbb{P}\Big(\bigcup_{k=1}^{n}\{h(t,\xi_k(t))X_k > x - l(x)\}\Big) +\\
&\quad \mathbb{P}\Big(\sum_{k=1}^{n} h(t,\xi_k(t))X_k > x, \max_{1\leq k\leq n} h(t,\xi_k(t))X_k \leq x - l(x)\Big)\\
&\leq \sum_{k=1}^{n}\mathbb{P}(h(t,\xi_k(t))X_k > x - l(x)) +\\
&\quad \mathbb{P}\Big(\sum_{k=1}^{n} h(t,\xi_k(t))X_k > x, \frac{x}{n} < \max_{1\leq k\leq n} h(t,\xi_k(t))X_k \leq x - l(x)\Big)\\
&\leq \sum_{k=1}^{n}\mathbb{P}(h(t,\xi_k(t))X_k > x - l(x)) +\\
&\quad \sum_{i=1}^{n}\mathbb{P}\Big(\sum_{k=1}^{n} h(t,\xi_k(t))X_k > x, \frac{x}{n} < h(t,\xi_i(t))X_i \leq x - l(x)\Big)\\
&\leq \sum_{k=1}^{n}\mathbb{P}(h(t,\xi_k(t))X_k > x - l(x)) +\\
&\quad \sum_{i=1}^{n}\mathbb{P}\Big(\sum_{1\leq k\neq i\leq n} h(t,\xi_k(t))X_k > l(x), h(t,\xi_i(t))X_i > \frac{x}{n}\Big)\\
&\leq n\mathbb{P}(h(t,\xi(t))X > x - l(x)) + \sum_{i=1}^{n}\sum_{1\leq k\neq i\leq n}\mathbb{P}(X_k > \frac{l(x)}{(n-1)b}, X_i > \frac{x}{nb})
\end{aligned}$$

$$=: n\mathbb{P}(h(t,\xi(t))X > x - l(x)) + J(x). \tag{3.7}$$

Since $l(x)$ is increasing, by Assumption 2.4, (3.3) and (3.4), it holds that for all $x \geq \max\{b,1\}x_1$ and $t \in (0,T]$,

$$\begin{aligned}
\mathbb{P}(h(t,\xi(t))X > x - l(x)) &= \int_a^b \overline{F}(\frac{x-l(x)}{u})\mathbb{P}(h(t,\xi(t)) \in \mathrm{d}u) \\
&\leq \int_a^b \overline{F}(\frac{x}{u} - \frac{l(x/b)}{a(1-\varepsilon)})\mathbb{P}(h(t,\xi(t)) \in \mathrm{d}u) \\
&\leq \int_a^b \overline{F}(\frac{x}{u} - \frac{l(x/u)}{a(1-\varepsilon)})\mathbb{P}(h(t,\xi(t)) \in \mathrm{d}u) \\
&\leq (1+\varepsilon)\int_a^b \overline{F}(\frac{x}{u})\mathbb{P}(h(t,\xi(t)) \in \mathrm{d}u) \\
&= (1+\varepsilon)\mathbb{P}(h(t,\xi(t))X > x).
\end{aligned} \tag{3.8}$$

For $J(x)$, by (3.5) and (3.6), for all $x \geq x_2$ and $t \in (0,T]$, we have that

$$J(x) \leq \frac{n(n-1)\varepsilon}{c}\mathbb{P}(h(t,\xi(t))X > x). \tag{3.9}$$

By (3.7)–(3.9), for all $x \geq \max\{bx_1, x_2\}$ and $t \in (0,T]$, we get that

$$\mathbb{P}\Big(\sum_{k=1}^{n} h(t,\xi_k(t))X_k > x\Big) \leq (1+\varepsilon+\frac{(n-1)\varepsilon}{c})n\mathbb{P}(h(t,\xi(t))X > x).$$

Letting $\varepsilon \downarrow 0$, it holds that

$$\limsup_{x\to\infty}\sup_{t\in(0,T]}\frac{\mathbb{P}(\sum_{k=1}^{n} h(t,\xi_k(t))X_k > x)}{n\mathbb{P}(h(t,\xi(t))X > x)} \leq 1.$$

This completes the proof of Lemma 3.4. □

Proof of Theorem 2.5 By Lemma 3.3, we know that $S(t)$ is identically distributed as $\sum_{k=1}^{N(t)} h(t,Z_k(t))X_k$ for any $t \geq 0$. For any integer $m \geq \frac{1}{b}$, $t \in (0,T]$ and $x > 0$, we divide the tail probability $\mathbb{P}(S(t) > x)$ into two parts:

$$\begin{aligned}
\mathbb{P}(S(t) > x) &= \Big(\sum_{n=1}^{m} + \sum_{n=m+1}^{\infty}\Big)\mathbb{P}\Big(\sum_{k=1}^{n} h(t,Z_k(t))X_k > x\Big)\mathbb{P}(N(t)=n) \\
&=: I_1(x,t) + I_2(x,t).
\end{aligned} \tag{3.10}$$

Since $F \in \mathcal{D}$, by Lemma 3.1 for some $p > J_F^+$, there exist $C_1 > 0$ and $D_1 > 0$, such that

$$\frac{\overline{F}(y)}{\overline{F}(x)} \leq C_1(\frac{y}{x})^{-p} \tag{3.11}$$

for all $x \geq y \geq D_1$. Therefore, for $I_2(x,t)$, by (3.11) and Markov's Inequality, it holds for sufficiently large x and uniformly for all $t \in (0,T]$ that

$$\begin{aligned}
I_2(x,t) &\leq \Big(\sum_{m<n\leq\frac{x}{D_1 b}} + \sum_{n>\frac{x}{D_1 b}}\Big)\mathbb{P}\Big(\sum_{k=1}^{n} X_k > \frac{x}{b}\Big)\mathbb{P}(N(t)=n) \\
&\leq \sum_{m<n\leq\frac{x}{D_1 b}} n\overline{F}(\frac{x}{nb})\mathbb{P}(N(t)=n) + \mathbb{P}(N(t) > \frac{x}{D_1 b})
\end{aligned}$$

$$\le C_1\overline{F}(x)\sum_{m<n\le\frac{x}{D_1b}} n(nb)^p\mathbb{P}(N(t)=n)+(\frac{x}{D_1b})^{-(p+1)}\mathbb{E}(N(t))^{p+1}\mathbf{1}_{\{N(t)>\frac{x}{D_1b}\}}$$
$$\le \max\{C_1b^p\overline{F}(x),(D_1b)^{p+1}x^{-(p+1)}\}\mathbb{E}(N(t))^{p+1}\mathbf{1}_{\{N(t)>m\}}.$$

Thus, by (3.6), Lemmas 3.1 and 3.2,

$$\begin{aligned}
&\lim_{m\to\infty}\limsup_{x\to\infty}\sup_{t\in(0,T]}\frac{I_2(x,t)}{m(t)\mathbb{P}(h(t,Z(t))X>x)}\\
&\le \lim_{m\to\infty}\limsup_{x\to\infty}\sup_{t\in(0,T]}\max\{\frac{C_1b^p}{c},\frac{(D_1b)^{p+1}}{c}\cdot\frac{x^{-(p+1)}}{\overline{F}(x)}\}\frac{1}{m(t)}\mathbb{E}(N(t))^{p+1}\mathbf{1}_{\{N(t)>m\}}\\
&=\limsup_{x\to\infty}\max\{\frac{C_1b^p}{c},\frac{(D_1b)^{p+1}}{c}\cdot\frac{x^{-(p+1)}}{\overline{F}(x)}\}\lim_{m\to\infty}\sup_{t\in(0,T]}\frac{1}{m(t)}\mathbb{E}(N(t))^{p+1}\mathbf{1}_{\{N(t)>m\}}\\
&=0. \qquad (3.12)
\end{aligned}$$

We next deal with $I_1(x,t)$. Since $m(t)=\sum_{n=1}^{\infty}n\mathbb{P}(N(t)=n)$, $t\ge 0$, by Lemmas 3.2 and 3.4, we get that

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{x\to\infty}\sup_{t\in(0,T]}|\frac{I_1(x,t)}{m(t)\mathbb{P}(h(t,Z(t))X>x)}-1|\\
&\le \lim_{m\to\infty}\lim_{x\to\infty}\sup_{t\in(0,T]}|\frac{\sum_{n=1}^{m}[\mathbb{P}(\sum_{k=1}^{n}h(t,Z_k(t))X_k>x)-n\mathbb{P}(h(t,Z(t))>x)]\mathbb{P}(N(t)=n)}{m(t)\mathbb{P}(h(t,Z(t))X>x)}|+\\
&\quad\lim_{m\to\infty}\sup_{t\in(0,T]}\frac{1}{m(t)}\mathbb{E}N(t)\mathbf{1}_{\{N(t)>m\}}\\
&\le \lim_{m\to\infty}\lim_{x\to\infty}\sup_{t\in(0,T]}\sum_{n=1}^{m}|\frac{\mathbb{P}(\sum_{k=1}^{n}h(t,Z_k(t))X_k>x)}{n\mathbb{P}(h(t,Z(t))X>x)}-1|\\
&\le \lim_{m\to\infty}\sum_{n=1}^{m}\lim_{x\to\infty}\sup_{t\in(0,T]}|\frac{\mathbb{P}(\sum_{k=1}^{n}h(t,Z_k(t))X_k>x)}{n\mathbb{P}(h(t,Z(t))X>x)}-1|\\
&=0. \qquad (3.13)
\end{aligned}$$

By (3.10), (3.12) and (3.13), we have that

$$\mathbb{P}(S(t)>x)\sim m(t)\mathbb{P}(h(t,Z(t))X>x)$$

holds uniformly for $t\in(0,T]$. This completes the proof of Theorem 2.5. □

3.2. Proof of Theorem 2.7

Before giving the proof of Theorem 2.7, we give some lemmas.

Lemma 3.5 *Let η be a nonnegative random variable with a continuous distribution V. $\{\xi(t),t\ge 0\}$ is a nonnegative stochastic process, which is independent of η. Let*

$$f(t,s):[0,\infty)\times[0,\infty)\longmapsto(0,\infty)$$

be a function. If $V\in\mathcal{C}$, then for any $T>0$,

$$\lim_{v\uparrow 1}\lim_{x\to\infty}\sup_{t\in(0,T]}\frac{\mathbb{P}(vx<f(t,\xi(t))\eta\le x)}{\mathbb{P}(f(t,\xi(t))\eta>x)}=0. \qquad (3.14)$$

Proof Since $V \in \mathcal{C}$ and is continuous, by the result of [24] (or the note after [11, Definition 3.2]), we know that $\log \overline{V}(e^s)$ is uniformly continuous for $s \in [0, \infty)$ and continuous elsewhere. Thus, for any $\varepsilon > 0$, there exists a sufficiently small constant $\delta > 0$ such that for all $x > 0$ and $|1 - v| < \delta$,

$$|\frac{\overline{V}(vx)}{\overline{V}(x)} - 1| \leq \varepsilon. \tag{3.15}$$

Therefore, for the above ε and δ, by (3.15) for all $x > 0$, $|1 - v| < \delta$ and $t \in (0, T]$, it holds that

$$\begin{aligned}\mathbb{P}(vx < f(t,\xi(t))\eta \leq x) &= \int_0^\infty (\overline{V}(\frac{vx}{y}) - \overline{V}(\frac{x}{y}))\mathbb{P}(f(t,\xi(t)) \in \mathrm{d}y) \\ &\leq \varepsilon \int_0^\infty \overline{V}(\frac{x}{y})\mathbb{P}(f(t,\xi(t)) \in \mathrm{d}y) = \varepsilon\mathbb{P}(f(t,\xi(t))\eta > x).\end{aligned}$$

By the arbitrariness of ε, we know that (3.14) holds. □

When the distributions of shocks X_k, $k \geq 1$ belong to the class $\mathcal{C}$, the following lemma removes the lower-bound restriction on the shot noise function h in Lemma 3.4.

Lemma 3.6 *Suppose that Assumptions 2.1′ and 2.4′ are satisfied. For each real number $t > 0$, $\{\xi_k(t), k \geq 1, \xi(t)\}$ are nonnegative and i.i.d random variables, which are independent of $\{X_k, k \geq 1, X\}$. If $F \in \mathcal{C}$ and is continuous, then for any $n \geq 1$, (3.1) holds.*

Proof We will use the line of the proof of Lemma 3.4. It is obvious that (3.1) holds for $n = 1$. Hereafter, we assume that $n \geq 2$. We firstly estimate the lower bound of $\mathbb{P}(\sum_{k=1}^n h(t,\xi_k(t))X_k > x)$ for $t \in (0, T]$ as $x \to \infty$. By Assumption 2.1′, for any $\varepsilon > 0$ there exists a constant $x_3 > x_2$ such that for all $1 \leq i \neq j \leq n$, $x > x_3$ and $y > x_3$,

$$\mathbb{P}(X_i > x, X_j > y) \leq \varepsilon\mathbb{P}(X_i > x). \tag{3.16}$$

For all $x > 0$ and $t \in (0, T]$,

$$\begin{aligned}&\mathbb{P}\Big(\sum_{k=1}^n h(t,\xi_k(t))X_k > x\Big) \\ &\geq n\mathbb{P}(h(t,\xi(t))X > x) - \sum_{1 \leq i \neq j \leq n} \mathbb{P}(h(t,\xi_i(t))X_i > x, h(t,\xi_j(t))X_j > x) \\ &=: n\mathbb{P}(h(t,\xi(t))X > x) - I_3(x,t).\end{aligned} \tag{3.17}$$

By (3.16), for all $x > bx_3$ and $t \in (0, T]$,

$$\begin{aligned}I_3(x,t) &\leq \sum_{1 \leq i \neq j \leq n} \mathbb{P}(h(t,\xi_i(t))X_i > x, X_j > \frac{x}{b}) \\ &= \sum_{1 \leq i \neq j \leq n} \int_0^b \mathbb{P}(X_i > \frac{x}{y}, X_j > \frac{x}{b})\mathbb{P}(h(t,\xi_i(t)) \in \mathrm{d}y) \\ &\leq \varepsilon \sum_{1 \leq i \neq j \leq n} \int_0^b \mathbb{P}(X_i > \frac{x}{y})\mathbb{P}(h(t,\xi_i(t)) \in \mathrm{d}y) \\ &\leq n^2\varepsilon\mathbb{P}(h(t,\xi(t))X > x).\end{aligned}$$

By the arbitrariness of ε, we have that

$$\limsup_{x\to\infty}\sup_{t\in(0,T]}\frac{I_3(x,t)}{n\mathbb{P}(h(t,\xi(t))X>x)}=0. \tag{3.18}$$

Thus, we get that

$$\liminf_{x\to\infty}\inf_{t\in(0,T]}\frac{\mathbb{P}(\sum_{k=1}^{n}h(t,\xi_k(t))X_k>x)}{n\mathbb{P}(h(t,\xi(t))X>x)}\geq 1.$$

We next estimate the upper bound of $\mathbb{P}(\sum_{k=1}^{n}h(t,\xi_k(t))X_k>x)$ for $t\in(0,T]$ as $x\to\infty$. Similarly to (3.7), for any $0<v<1$, $x>0$ and $t\in(0,T]$,

$$\begin{aligned}
&\mathbb{P}(\sum_{k=1}^{n}h(t,\xi_k(t))X_k>x)\\
&\leq n\mathbb{P}(h(t,\xi(t))X>vx)+\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mathbb{P}(h(t,\xi_i(t))X_i>\frac{x}{n},h(t,\xi_j(t))X_j>\frac{(1-v)x}{n-1})\\
&\leq n\mathbb{P}(h(t,\xi(t))X>vx)+\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mathbb{P}(h(t,\xi_i(t))X_i>\frac{(1-v)x}{n},h(t,\xi_j(t))X_j>\frac{(1-v)x}{n-1})\\
&=: I_4(x,t)+I_5(x,t).
\end{aligned} \tag{3.19}$$

By Lemma 3.5, we know that

$$\lim_{v\uparrow 1}\lim_{x\to\infty}\sup_{t\in(0,T]}\frac{I_4(x,t)}{n\mathbb{P}(h(t,\xi(t))X>x)}=1. \tag{3.20}$$

Note that, by $F\in\mathcal{C}\subset\mathcal{D}$, we have that for any $\omega>0$,

$$\begin{aligned}
&\limsup_{x\to\infty}\sup_{t\in(0,T]}\frac{\mathbb{P}(h(t,\xi(t))X>\omega x)}{\mathbb{P}(h(t,\xi(t))X>x)}\\
&=\limsup_{x\to\infty}\sup_{t\in(0,T]}\frac{\int_0^b\overline{F}(\frac{\omega x}{y})\mathbb{P}(h(t,\xi(t))X\in \mathrm{d}y)}{\int_0^b\overline{F}(\frac{x}{y})\mathbb{P}(h(t,\xi(t))X\in \mathrm{d}y)}\\
&\leq\limsup_{x\to\infty}\sup_{z\geq x/b}\frac{\overline{F}(\omega z)}{\overline{F}(z)}<\infty,
\end{aligned} \tag{3.21}$$

which, together with (3.18), implies that for any $0<v<1$,

$$\limsup_{x\to\infty}\sup_{t\in(0,T]}\frac{I_5(x,t)}{n\mathbb{P}(h(t,\xi(t))X>x)}=0. \tag{3.22}$$

Plugging (3.20) and (3.22) into (3.19), we obtain that

$$\limsup_{x\to\infty}\sup_{t\in(0,T]}\frac{\mathbb{P}(\sum_{k=1}^{n}h(t,\xi_k(t))X_k>x)}{n\mathbb{P}(h(t,\xi(t))X>x)}\leq 1.$$

This completes the proof of Lemma 3.6. □

Proof of Theorem 2.7 The proof of Theorem 2.7 is analogous to that of Theorem 2.5 by replacing Lemma 3.4 by Lemma 3.6. We omit the details. □

3.3. Proof of Theorem 2.9

We firstly present a lemma before giving the proof of Theorem 2.9.

Lemma 3.7 *Suppose that Assumptions 2.1* and 2.4* are satisfied. For each real number $t > 0$, $\{\xi_k(t), k \geq 1\}$ are nonnegative and i.i.d random variables with the same distribution as $Z(t)$, which are independent of $\{X_k, k \geq 1, X, Z(t)\}$. If $F \in \mathcal{C}$ and is continuous, then for any $n \geq 1$, (3.1) holds for $\xi(t) = Z(t)$.*

Proof Similarly to the proof of Lemma 3.6, we only need to estimate $I_i(x,t)$, $i = 3, 4, 5$ in (3.17) and (3.19).

For $I_3(x,t)$, since $\mathbb{E}(\sup_{t\in(0,T]} h(t, Z(t)))^p < \infty$ for some $p > J_F^+$, we know that

$$\sup_{t\in(0,T]} h(t, Z(t)) < \infty \ \text{ a.s.}$$

By Assumption 2.1* and Markov's Inequality we get that

$$\begin{aligned}
&\limsup_{x\to\infty}\sup_{t\in(0,T]}\frac{I_3(x,t)}{n\mathbb{P}(h(t,Z(t))X > x)}\\
&= \limsup_{x\to\infty}\sup_{t\in(0,T]}\frac{\sum_{1\leq i\neq j\leq n}\int_0^\infty\int_0^\infty \mathbb{P}(X_i > \frac{x}{u}, X_j > \frac{x}{v})\mathbb{P}(h(t,\xi_i(t))\in \mathrm{d}u)\mathbb{P}(h(t,\xi_j(t))\in \mathrm{d}v)}{n\mathbb{P}(h(t,Z(t))X > x)}\\
&\leq \limsup_{x\to\infty}\sup_{t\in(0,T]}\frac{\sum_{1\leq i\neq j\leq n}\int_0^\infty \mathbb{P}(X_i > \frac{x}{u})\mathbb{P}(h(t,\xi_i(t))\in \mathrm{d}u)\int_0^\infty \mathbb{P}(X_j > \frac{x}{v})\mathbb{P}(h(t,\xi_j(t))\in \mathrm{d}v)}{n\mathbb{P}(h(t,Z(t))X > x)}\\
&\leq \limsup_{x\to\infty}\sup_{t\in(0,T]} n\mathbb{P}(h(t,Z(t))X > x) \leq \limsup_{x\to\infty} n\mathbb{P}(\sup_{t\in(0,T]} h(t,Z(t))X > x) = 0. \qquad (3.23)
\end{aligned}$$

Hence, by (3.17), we get that

$$\liminf_{x\to\infty}\inf_{t\in(0,T]}\frac{\mathbb{P}(\sum_{k=1}^n h(t,\xi_k(t))X_k > x)}{n\mathbb{P}(h(t,Z(t))X > x)} \geq 1.$$

For $I_4(x,t)$, by Lemma 3.5 we know that (3.20) still holds. We next deal with $I_5(x,t)$. We will firstly prove (3.21) holds. By Markov's Inequality and Lemma 3.1, for some $v > 1$ such that $\frac{p}{v} > J_F^+$, we get that for sufficiently large x

$$\begin{aligned}
\mathbb{P}\Big(\sup_{t\in(0,T]} h(t,Z(t)) > x\Big) &\leq x^{-p}\mathbb{E}\Big(\sup_{t\in(0,T]} h(t,Z(t))\Big)^p\\
&= (x^v)^{-\frac{p}{v}}\mathbb{E}\Big(\sup_{t\in(0,T]} h(t,Z(t))\Big)^p = o(\overline{F}(x^v)). \qquad (3.24)
\end{aligned}$$

Since $\inf_{t\in(0,T]} h(t,Z(t))$ is nondegenerate at zero, there exists some $\Delta > 0$ such that

$$\mathbb{P}\Big(\inf_{t\in(0,T]} h(t,Z(t)) > \Delta\Big) > 0. \qquad (3.25)$$

Then, for any $\omega > 0$, by $F \in \mathcal{D}$, (3.24) and (3.25), it holds that

$$\begin{aligned}
&\limsup_{x\to\infty}\sup_{t\in(0,T]}\frac{\mathbb{P}(h(t,Z(t))X > \omega x)}{\mathbb{P}(h(t,Z(t))X > x)}\\
&\leq \limsup_{x\to\infty}\Big(\sup_{t\in(0,T]}\frac{\int_0^{x^{\frac{1}{v}}}\overline{F}(\frac{\omega x}{u})\mathbb{P}(h(t,Z(t))\in \mathrm{d}u)}{\int_0^{x^{\frac{1}{v}}}\overline{F}(\frac{x}{u})\mathbb{P}(h(t,Z(t))\in \mathrm{d}u)} + \sup_{t\in(0,T]}\frac{\int_{x^{\frac{1}{v}}}^\infty \overline{F}(\frac{\omega x}{u})\mathbb{P}(h(t,Z(t))\in \mathrm{d}u)}{\int_\Delta^\infty \overline{F}(\frac{x}{u})\mathbb{P}(h(t,Z(t))\in \mathrm{d}u)}\Big)
\end{aligned}$$

$$\le \limsup_{x\to\infty}\Big(\sup_{z\ge x^{1-\frac{1}{v}}}\frac{\overline{F}(\omega z)}{\overline{F}(z)}+\sup_{t\in(0,T]}\frac{\mathbb{P}(h(t,Z(t))>x^{\frac{1}{v}})}{\overline{F}(\frac{x}{\Delta})\mathbb{P}(h(t,Z(t))>\Delta)}\Big)$$

$$\le \limsup_{x\to\infty}\Big(\sup_{z\ge x^{1-\frac{1}{v}}}\frac{\overline{F}(\omega z)}{\overline{F}(z)}+\frac{\mathbb{P}(\sup_{t\in(0,T]}h(t,Z(t))>x^{\frac{1}{v}})}{\overline{F}(\frac{x}{\Delta})\mathbb{P}(\inf_{t\in(0,T]}h(t,Z(t))>\Delta)}\Big)$$

$$=: \limsup_{x\to\infty}\Big(\sup_{z\ge x^{1-\frac{1}{v}}}\frac{\overline{F}(\omega z)}{\overline{F}(z)}+I_6(x,t)\Big) \tag{3.26}$$

$$= \limsup_{x\to\infty}\sup_{z\ge x^{1-\frac{1}{v}}}\frac{\overline{F}(\omega z)}{\overline{F}(z)}<\infty. \tag{3.27}$$

Thus, by (3.23) and (3.27), we get that (3.22) holds. Using (3.19), (3.20) and (3.22), we obtain that

$$\limsup_{x\to\infty}\sup_{t\in(0,T]}\frac{\mathbb{P}(\sum_{k=1}^{n}h(t,\xi_k(t))X_k>x)}{n\mathbb{P}(h(t,Z(t))X>x)}\le 1.$$

This completes the proof of Lemma 3.7. □

Proof of Theorem 2.9 We will use the line of the proof of Theorem 2.5 and we only need to estimate $I_i(x,t)$, $i=1,2$ in (3.10). By using Lemmas 3.2 and 3.7, similarly to the estimation of $I_1(x,t)$ in (3.13), we can get that

$$\lim_{m\to\infty}\lim_{x\to\infty}\sup_{t\in(0,T]}\Big|\frac{I_1(x,t)}{m(t)\mathbb{P}(h(t,Z(t))X>x)}-1\Big|=0. \tag{3.28}$$

For $I_2(x,t)$, since $F\in\mathcal{C}\subset\mathcal{D}$, (3.11) still holds. For some $v>1$ such that $\frac{p}{v}>J_F^+$, using (3.26) for $\omega=\frac{1}{n}$, by (3.11) and Markov's Inequality, it holds that for all $x>0$ and $t\in(0,T]$

$$\begin{aligned}
I_2(x,t) &\le \sum_{n=m+1}^{\infty}\mathbb{P}\Big(\bigcup_{k=1}^{n}\{h(t,Z_k(t))X_k>\frac{x}{n}\}\Big)\mathbb{P}(N(t)=n)\\
&\le \sum_{m<n\le x^{1-\frac{1}{v}}/D_1} n\mathbb{P}(h(t,Z(t))X>\frac{x}{n})\mathbb{P}(N(t)=n)+\mathbb{P}(N(t)>x^{1-\frac{1}{v}}/D_1)\\
&\le \sum_{m<n\le x^{1-\frac{1}{v}}/D_1} n\mathbb{P}(N(t)=n)\mathbb{P}(h(t,Z(t))X>x)\Big(\sup_{z\ge x^{1-\frac{1}{v}}}\frac{\overline{F}(\frac{z}{n})}{\overline{F}(z)}+I_6(x,t)\Big)+\\
&\quad (x^{1-\frac{1}{v}}/D_1)^{-\frac{p}{1-v^{-1}}}\mathbb{E}(N(t))^{\frac{p}{1-v^{-1}}}\mathbf{1}_{\{N(t)>x^{1-\frac{1}{v}}/D_1\}}\\
&\le \sum_{m<n\le x^{1-\frac{1}{v}}/D_1} n\mathbb{P}(N(t)=n)\mathbb{P}(h(t,Z(t))X>x)(C_1n^p+I_6(x,t))+\\
&\quad x^{-p}D_1^{\frac{p}{1-v^{-1}}}\mathbb{E}(N(t))^{\frac{p}{1-v^{-1}}}\mathbf{1}_{\{N(t)>m\}}\\
&\le \mathbb{P}(h(t,Z(t))X>x)\mathbb{E}(C_1(N(t))^{p+1}+I_6(x,t)N(t))\mathbf{1}_{\{N(t)>m\}}+\\
&\quad x^{-p}D_1^{\frac{p}{1-v^{-1}}}\mathbb{E}(N(t))^{\frac{p}{1-v^{-1}}}\mathbf{1}_{\{N(t)>m\}}\\
&=: I_{21}(x,t)+I_{22}(x,t).
\end{aligned} \tag{3.29}$$

By Lemma 3.2, (3.24) and (3.25), we get that

$$\lim_{m\to\infty}\limsup_{x\to\infty}\sup_{t\in(0,T]}\frac{I_{21}(x,t)}{m(t)\mathbb{P}(h(t,Z(t))X>x)}$$
$$\leq \lim_{m\to\infty}\sup_{t\in(0,T]} C_1\frac{1}{m(t)}\mathbb{E}(N(t))^{p+1}\mathbf{1}_{\{N(t)>m\}}+$$
$$\limsup_{x\to\infty}\sup_{t\in(0,T]} I_6(x,t)\cdot\lim_{m\to\infty}\sup_{t\in(0,T]}\frac{1}{m(t)}\mathbb{E}N(t)\mathbf{1}_{\{N(t)>m\}}=0. \tag{3.30}$$

For $I_{22}(x,t)$, by Lemma 3.3 (ii) of Yang et al. (2012),

$$\liminf_{x\to\infty}\inf_{t\in(0,T]}\frac{\mathbb{P}(h(t,Z(t))X>x)}{\overline{F}(x)}\geq\liminf_{x\to\infty}\frac{\mathbb{P}(\inf_{t\in(0,T]}h(t,Z(t))X>x)}{\overline{F}(x)}>0. \tag{3.31}$$

Thus, by Lemmas 3.1 and 3.2 and (3.31),

$$\lim_{m\to\infty}\limsup_{x\to\infty}\sup_{t\in(0,T]}\frac{I_{22}(x,t)}{m(t)\mathbb{P}(h(t,Z(t))X>x)}$$
$$\leq D_1^{\frac{p}{1-v^{-1}}}\lim_{x\to\infty}\frac{x^{-p}}{\overline{F}(x)}\cdot\limsup_{x\to\infty}\sup_{t\in(0,T]}\frac{\overline{F}(x)}{\mathbb{P}(h(t,Z(t))X>x)}\cdot$$
$$\lim_{m\to\infty}\sup_{t\in(0,T]}\frac{1}{m(t)}\mathbb{E}(N(t))^{\frac{p}{1-v^{-1}}}\mathbf{1}_{\{N(t)>m\}}=0. \tag{3.32}$$

By (3.10), (3.28)–(3.30) and (3.32), we get that (2.1) holds uniformly for $t\in(0,T]$. This completes the proof of Theorem 2.9. □

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