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A Stabilized Formulation for Linear Elasticity Equation with Weakly Symmetric Stress

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Abstract The linear elastic problem with weak symmetric stress obtained by Lagrange multiplier method is discussed by using the stabilization method. The stress and displacement of the variational problem are approximated by linear element and piecewise constant. By adding stabilization terms $G_1(\cdot,\cdot), G_2(\cdot,\cdot)$ and $G_3(\cdot,\cdot)$, the corresponding mixed discrete variational problem satisfies the weak inf-sup condition. Then the error estimation between the solution of the variational problem and the stabilized mixed finite element solution is studied in detail. Finally, two numerical examples are used to verify the effectiveness of the theoretical analysis.

Keywords mixed finite element method; stabilized formulation; linear elasticity equation; weakly symmetric stress

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1. Introduction

Stabilized method can circumvent the restriction of inf-sup condition without introducing errors. It is known, the coercivity of the bilinear form may be conditional upon the choice of parameters in three different stabilized items. In order to gain the optimal error, one needs to properly choose stabilization parameters. There are many different stabilized methods, such as the Galerkin least-squares method [1], the bubble function method [2], the subgrid scale method [3,4], the pressure gradient projection method [5,6], the local pressure gradient projection method [7,8]. Among these references, [1,7] discuss the elasticity problem based on the displace-pressure formulation. [3] studies the Helmholtz problem and shows the relation of the bubble function methods and the stabilized methods.

There are some researches which use different variational principle to deal with linear elasticity problem. To solve the equation directly based on the Hellinger-Reissner variational formulation by finite element method, the crux could keep the stress space to be symmetric. Some rectangular and simplex elements which can keep the symmetry of stress tensor well have been constructed

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in 2D and 3D. The first finite element formulation for linear elasticity problem was constructed in [9] and after that many other conforming and nonconforming element formulations had been constructed, such as rectangular elements [10–14] and simplex elements [9, 15–17]. In [14], the rectangular element formulations are anisotropic convergent and the number of freedoms tends to least.

The stabilized method allows us to use the simple linear element for stress, which will lead to less degrees of freedom than other elements. Due to these advantages, the stabilized method has wide applications in practical computations. The stabilization was also used to solve Hellinger-Reissner variational formulation in [18–20]. Two classes of mixed finite elements were proposed in [18] for linear elasticity of any order, with interior penalty for nonconforming symmetric stress approximation. [19] proposed mixed finite element spaces using C^0 continuous arbitrary degree polynomial to approximate the stress and displacement. In [20], two classes of stabilized mixed finite element methods were designed on simplified grids. [21] proposed a framework for unified analysis of mixed methods, which was based on a commuting diagram in the weakly symmetric elasticity complex and extends a previous stability result. The stable methods are obtained by combining Stokes stable and elasticity stable finite elements [21].

In this paper, we first employ the Hellinger-Reissner variational formulation with imposed weakly symmetric stress through a Lagrange multiplier which was proposed by Fraeijs de veubeke [22], and find the solution of this variational formulation characterizing as a saddle point of a Lagrangian function involving both displacement and stress imposed weakly symmetric condition through a Lagrange multiplier. We next put three stabilization items $G_1(\cdot,\cdot), G_2(\cdot,\cdot)$ and $G_3(\cdot,\cdot)$ on the either side of the original equation, and introduce the jump value of displacement and the divergence of stress as the new special stable item to make the bilinear form of mixed stabilized discrete variational formulation continuous and coercive. Considering the Lagrange finite element spaces are very popular in the engineering practice, we adopt the $P_1^{2\times 2}$ and P_0^2 polynomial space to approximate stress and displacement, respectively. We investigate the detailed error estimate between the exact solution and the mixed finite element solution, and use two numerical examples to verify the validity of theory analysis, at last.

The remainder of this paper is organized as follows, we introduce some basic concepts and signals used in the paper, and review the variational formulation of plain elasticity equation in Section 2. A weaker form inf-sup stability condition is given in Section 3. We construct a stable mixed method for this formulation and derive its error estimate in Sections 4 and 5, respectively. The numerical examples show the feasibility of this method and coincide with the theoretical analysis well in last section.

2. Notations and preliminaries

Denote by Ω the convex polygonal domain, and T the subdomain of Ω. We define the Sobolev space $H^s(T)$ $(s = 1, 2, \ldots)$ as usual, with semi-norm and norm as

$$|u|_{s,T} = \Big(\sum_{|\alpha|=s} \|D^\alpha u\|_{0,T}^2\Big)^{\frac{1}{2}}, \quad \| u \|_{s,T} = \Big(\sum_{0\le|\alpha|\le s} \|D^\alpha u\|_{0,T}^2\Big)^{\frac{1}{2}},$$

where $D^\alpha u$ is the weak partial derivative of function u. For $T=\Omega$, we write semi-norm and norm simply as $|\cdot|_s$ and $\|\cdot\|_s$, respectively. $L^2(T)$ is the usual square integrable space with norm $\|\cdot\|_0$. The subspace of $H^s(T)$ consisting of functions vanishing on $\partial\Omega$ is denoted by $H_0^s(T)$.

We also use the underline to distinguish between scalar, vector and tensor. For any space X, define $\underline{X}$ and $\underline{\underline{X}}$ to be the two dimensional vector and second-order matrix respectively with components in X. If X is the norm space, the associated norms are defined by

$$\|\underline{v}\|_{\underline{X}}=\Big(\sum_{i=1}^{2}\|v_i\|_X^2\Big)^{\frac12},\quad \|\underline{\underline{\tau}}\|_{\underline{\underline{X}}}=\Big(\sum_{i,j=1}^{2}\|\tau_{ij}\|_X^2\Big)^{\frac12}.$$

We use the same notation $\|\cdot\|_{s,T}$ to denote the norms in $H^s(T)$, $\underline{H}^s(T)$ and $\underline{\underline{H}}^s(T)$. Define $P_k(T)$ $(k=0,1)$ to be the space of the polynomials at most degree k on T. And $\underline{P}_k(T)$, $\underline{\underline{P}}_k(T)$ represent the vector and matrix polynomials space, respectively.

The space $\underline{\underline{H}}(\mathrm{div},T)$ consists of matrix fields with square-integrable divergence, associated with norm $\|\cdot\|_{\underline{\underline{H}}(\mathrm{div},T)}$ as

$$\|\underline{\underline{\tau}}\|^2_{\underline{\underline{H}}(\mathrm{div},T)}=\|\underline{\underline{\tau}}\|^2_{0,T}+\|\underline{\mathrm{div}}\,\underline{\underline{\tau}}\|^2_{0,T}.$$

For function η, vector function $\underline{v}=(v_1,v_2)$ and matrix function $\underline{\underline{\tau}}=(\tau_{ij})_{1\le i,j\le 2}$, we introduce the following differential operators

$$\underline{\mathrm{curl}}\,\eta=(\frac{\partial\eta}{\partial y},\ -\frac{\partial\eta}{\partial x}),\quad \underline{\underline{\mathrm{curl}}}\,\underline{v}=\begin{pmatrix}\frac{\partial v_1}{\partial y} & -\frac{\partial v_1}{\partial x}\\ \frac{\partial v_2}{\partial y} & -\frac{\partial v_2}{\partial x}\end{pmatrix},$$

$$\mathrm{rot}\,\underline{v}=-\frac{\partial v_1}{\partial y}+\frac{\partial v_2}{\partial x},\qquad \underline{\mathrm{div}}\,\underline{\underline{\tau}}=\Big(\frac{\partial\tau_{11}}{\partial x}+\frac{\partial\tau_{12}}{\partial y},\ \frac{\partial\tau_{21}}{\partial x}+\frac{\partial\tau_{22}}{\partial y}\Big),$$

$$\underline{\underline{\mathrm{grad}}}\,\underline{v}=\begin{pmatrix}\frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y}\\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y}\end{pmatrix},\quad \underline{\underline{\varepsilon}}(\underline{v})=\begin{pmatrix}\frac{\partial v_1}{\partial x} & \frac12(\frac{\partial v_1}{\partial y}+\frac{\partial v_2}{\partial x})\\ \frac12(\frac{\partial v_1}{\partial y}+\frac{\partial v_2}{\partial x}) & \frac{\partial v_2}{\partial y}\end{pmatrix}.$$

Let the scalar product of tensor be $\underline{\underline{\tau}}:\underline{\underline{\sigma}}=\sum_{i,j=1}^2\tau_{ij}\sigma_{ij}$. The trace and asymmetry of $\underline{\underline{\tau}}$ are denoted by $\mathrm{tr}(\underline{\underline{\tau}})=\tau_{11}+\tau_{22}$ and $\mathrm{as}(\underline{\underline{\tau}})=\underline{\underline{\tau}}:\underline{\underline{\chi}}$ with $\underline{\underline{\chi}}=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$.

The linear elasticity problem considered in this paper is

$$\begin{cases}\underline{\mathrm{div}}\,\underline{\underline{\sigma}}=\underline{f}, & \text{in}\,\Omega,\\ \underline{\underline{A}}(\underline{\underline{\sigma}})-\underline{\underline{\varepsilon}}(\underline{u})=\underline{\underline{0}}, & \text{in}\,\Omega,\\ \underline{u}=\underline{0}, & \text{on}\,\partial\Omega,\end{cases}\tag{2.1}$$

where the displacement is the vector function $\underline{u}:\Omega\to\mathbb{R}^2$ and $\underline{u}\in\underline{H}^1(\Omega)$. The stress is denoted by $\underline{\underline{\sigma}}:\Omega\to\mathbb{S}$ and $\underline{\underline{\sigma}}\in\underline{\underline{H}}^1(\Omega,\mathbb{S})$. $\mathbb{S}$ denotes the space of symmetric matrices on $\mathbb{R}^2$.

The compliance tensor is denoted by

$$\underline{\underline{A}}(\underline{\underline{\sigma}})=\frac{1}{2\mu}(\underline{\underline{\sigma}}-\frac{\lambda}{2\mu+2\lambda}\mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{\delta}})\ \text{ with }\ \underline{\underline{\delta}}=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

which is a bounded, symmetric, positive definite tensor over Ω. The given load is denoted by the vector function $\underline{f}:\Omega\to\mathbb{R}^2$.

Setting $\phi=\frac{1}{2}\mathrm{rot}\,\underline{u}$, and noting that $\underline{\underline{\varepsilon}}(\underline{u})=\underline{\underline{\mathrm{grad}}}\,\underline{u}-\phi\underline{\underline{\chi}}$. Taking $\underline{\underline{\varepsilon}}(\underline{u})$ into the second equation of (2.1), we can get

$$\frac{1}{2\mu}\underline{\underline{\sigma}}-\frac{\lambda}{4\mu(\mu+\lambda)}\mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{\delta}}-\underline{\underline{\mathrm{grad}}}\,\underline{u}+\phi\underline{\underline{\chi}}=\underline{\underline{0}}. \tag{2.2}$$

Supplementing this equation with the equilibrium condition of (2.1), the symmetric condition of $\underline{\underline{\sigma}}$ and the fixed boundary condition, we get

$$\begin{cases}\underline{\mathrm{div}}\,\underline{\underline{\sigma}}=\underline{f}, & \text{in}\,\Omega,\\ \mathrm{as}(\underline{\underline{\sigma}})=0, & \text{in}\,\Omega,\\ \underline{u}=\underline{0}, & \text{on}\,\partial\Omega.\end{cases} \tag{2.3}$$

The systems of (2.2) and (2.3) are equivalent to the following weak formulation [23] that is to find a triple $(\underline{\underline{\sigma}},\,\underline{u},\,\gamma)\in\underline{\underline{H}}(\mathrm{div},\Omega)\times\underline{L}^2(\Omega)\times L^2(\Omega)$ such that

$$\begin{cases}a(\underline{\underline{\sigma}},\,\underline{\underline{\tau}})+b(\underline{\underline{\tau}};\,(\underline{u},\,\gamma))=0, & \forall\,\underline{\underline{\tau}}\in\underline{\underline{H}}(\mathrm{div},\Omega),\\ \displaystyle\int_\Omega\underline{\mathrm{div}}\,\underline{\underline{\sigma}}\cdot\underline{v}\mathrm{d}x=(\underline{f},\underline{v}), & \forall\,\underline{v}\in\underline{L}^2(\Omega),\\ \displaystyle\int_\Omega\mathrm{as}(\underline{\underline{\sigma}})\eta\mathrm{d}x=0, & \forall\,\eta\in L^2(\Omega),\end{cases} \tag{2.4}$$

where

$$a(\underline{\underline{\sigma}},\,\underline{\underline{\tau}})=\int_\Omega[\frac{1}{2\mu}\underline{\underline{\sigma}}:\underline{\underline{\tau}}-\frac{\lambda}{4\mu(\mu+\lambda)}\mathrm{tr}(\underline{\underline{\sigma}})\mathrm{tr}(\underline{\underline{\tau}})]\mathrm{d}x,$$

$$b(\underline{\underline{\tau}};\,(\underline{u},\,\gamma))=\int_\Omega\mathrm{div}\,\underline{\underline{\tau}}\cdot\underline{u}\mathrm{d}x+\int_\Omega\mathrm{as}(\underline{\underline{\tau}})\gamma\mathrm{d}x.$$

3. The weaker Inf-sup condition

Suppose the bilinear $a(\cdot,\,\cdot)$ is continuous and coercive, which means there exist two constants $\alpha_1,\,\alpha_2>0$ such that

$$a(\underline{\underline{\sigma}},\,\underline{\underline{\tau}})\le\alpha_1\|\underline{\underline{\sigma}}\|_0\,\|\underline{\underline{\tau}}\|_0,\quad a(\underline{\underline{\tau}},\,\underline{\underline{\tau}})\ge\alpha_2\|\underline{\underline{\tau}}\|_0^2. \tag{3.1}$$

Theorem 3.1 *Let $0<\mu_0<\mu_1$, $\mu\in[\mu_0,\,\mu_1]$, $\lambda\in[0,\,\infty)$, and $\underline{f}\in\underline{L}^2(\Omega)$. Then there exists a unique triple $(\underline{\underline{\tau}};\,(\underline{u},\,\gamma))\in\underline{\underline{H}}^1(\Omega)\times(\underline{H}^2(\Omega)\cap\underline{H}_0^1(\Omega))\times H^1(\Omega)$ satisfying (2.4). Moreover, there exists a constant C depending only on Ω, μ_0 and μ_1 such that*

$$\|\underline{\underline{\sigma}}\|_1+\|\underline{u}\|_2+\|\gamma\|_1\le C\|\underline{f}\|_0. \tag{3.2}$$

Note that the constant C in the above theorem is independent of λ. For the case of $\lambda\to\infty$, it corresponds to a nearly incompressible material. For a proof of this aspect of the theorem see [24, 25].

Throughout this paper, we denote by c generic positive constants not necessarily identical at different places but always independent of the discretization parameters of interest (such as mesh size h).

Let $\mathcal{T}_h$ be a shape regular decomposition of Ω. For each $T \in \mathcal{T}_h$, T is a triangular or rectangular element. We set the finite element space as follows

$$\begin{aligned}\Sigma_h &= \{\underline{\underline{\tau}} \in \underline{\underline{H}}^1(\Omega)|\underline{\underline{\tau}}|_T \in \underline{\underline{P}}_1(T)\},\\ V_h &= \{\underline{v} \in \underline{L}^2(\Omega)|\underline{v}|_T \in \underline{P}_0(T)\},\\ S_h &= \{\eta \in H^1(\Omega)|\eta|_T \in P_1(T)\}.\end{aligned} \tag{3.3}$$

On the stress space Σ_h, we define the Clément interpolation $j_h : \underline{\underline{H}}^1(\Omega) \to \Sigma_h$ which has the following properties

$$\|\underline{\underline{\tau}} - j_h\underline{\underline{\tau}}\|_{0,T} \le ch\|\underline{\underline{\tau}}\|_{1,T}, \quad \|j_h\underline{\underline{\tau}}\|_{1,T} \le c\|\underline{\underline{\tau}}\|_{1,T}, \quad \|j_h\underline{\underline{\tau}}\|_{\underline{\underline{H}}(\mathrm{div},T)} \le c\|\underline{\underline{\tau}}\|_{1,T}. \tag{3.4}$$

Let $I_h : \underline{L}^2(\Omega) \to V_h$ and $P_h : L^2(\Omega) \to S_h$ be the L^2 projection operators.

Then we have

$$\begin{aligned}&\|I_h\underline{u}\|_0 \le C\|\underline{u}\|_1, \quad \forall\, \underline{u} \in \underline{H}^1(\Omega),\\ &\|P_h\gamma\|_0 \le C\|\gamma\|_1, \quad \forall\, \gamma \in H^1(\Omega),\\ &\|u - I_h\underline{u}\|_0 \le Ch\|\underline{u}\|_1, \quad \forall\, \underline{u} \in \underline{H}^1(\Omega),\\ &\|\gamma - P_h\gamma\|_0 \le Ch\|\gamma\|_1, \quad \forall\, \gamma \in H^1(\Omega).\end{aligned} \tag{3.5}$$

Let e be the boundary of the element and E_h be the set of e on Ω. Define

$$\|[\underline{v}_h]\|_{E_h} = \Big(\sum_{E_h}\int_e [\underline{v}_h]^2 \mathrm{d}s\Big)^{\frac{1}{2}}. \tag{3.6}$$

Lemma 3.2 *Let Σ_h, V_h, S_h be the finite element spaces defined by (3.3). Then for some $(\underline{v}_h, \eta_h) \in V_h \times S_h$, there are positive constants k_1, k_2 and k_3 satisfying*

$$\sup_{\forall \underline{\underline{\tau}}_h \in \Sigma_h} \frac{b(\underline{\underline{\tau}}_h;\, (\underline{v}_h,\, \eta_h))}{\|\underline{\underline{\tau}}_h\|_{\underline{\underline{H}}(\mathrm{div},\Omega)}} \ge k_1(\|\underline{v}_h\|_0 + \|\eta_h\|_0) - k_2 h^{\frac{1}{2}}\|[\underline{v}_h]\|_{E_h} - k_3 h\|\eta_h\|_0. \tag{3.7}$$

Proof For a given $(\underline{v}_h, \eta_h) \in V_h \times S_h$, there exists $\underline{\underline{\tau}}^1 \in \underline{\underline{H}}^1(\Omega)$, which satisfy $\underline{\mathrm{div}}\,\underline{\underline{\tau}}^1 = \underline{v}_h$. And there holds

$$\|\underline{\underline{\tau}}^1\|_1 \le c\|\underline{v}_h\|_0. \tag{3.8}$$

Thus, we have

$$\|\underline{\underline{\tau}}^1\|_{\underline{\underline{H}}(\mathrm{div},\Omega)} \le \|\underline{\underline{\tau}}^1\|_1 \le c\|\underline{v}_h\|_0. \tag{3.9}$$

Choosing $K = \frac{1}{|T|}\int_T [\eta_h - \mathrm{as}(\underline{\underline{\tau}}^1)]\mathrm{d}x$ gives

$$\|K\|_0 \le c(\|\eta_h\|_0 + \|\underline{\underline{\tau}}^1\|_0). \tag{3.10}$$

Let $\beta = \eta_h - \mathrm{as}(\underline{\underline{\tau}}^1) - K$. It is obvious that the mean value of β is zero. And it is easy to find $\underline{q} \in \underline{H}_0^1(\Omega)$ such that $\mathrm{div}\,\underline{q} = \beta$. Take

$$\underline{\underline{\tau}}^2 = \underline{\underline{\tau}}^1 + \underline{\underline{\mathrm{curl}}}\,\underline{q} + \frac{K}{2}\underline{\underline{\chi}}. \tag{3.11}$$

It is easy to conclude that

$$\underline{\mathrm{div}}\,\underline{\underline{\tau}}^2 = \underline{\mathrm{div}}\,\underline{\underline{\tau}}^1 = \underline{v}_h. \tag{3.12}$$

From the regularity and the norm of $\|\cdot\|_{\underline{\underline{H}}(\mathrm{div},\,\Omega)}$, we have

$$\|\underline{\underline{\tau}}^2\|_{\underline{\underline{H}}(\mathrm{div},\,\Omega)} \le \|\underline{\underline{\tau}}^2\|_1 \le c(\|\underline{v}_h\|_0 + \|\eta_h\|_0). \tag{3.13}$$

Then, for any $\alpha \in S_h$,

$$\begin{aligned}\int_\Omega \mathrm{as}(\underline{\underline{\tau}}^2)\alpha \mathrm{d}x &= \int_\Omega \mathrm{as}(\underline{\underline{\tau}}^1 + \underline{\underline{\mathrm{curl}}}\,\underline{q} + \frac{K}{2}\underline{\underline{\chi}})\alpha \mathrm{d}x \\ &= \int_\Omega (\mathrm{as}(\underline{\underline{\tau}}^1) + \mathrm{div}\,\underline{q} + K)\alpha \mathrm{d}x = \int_\Omega \eta_h \alpha \mathrm{d}x.\end{aligned} \tag{3.14}$$

From (3.11)–(3.14), we deduce that

$$\begin{aligned}&\int_\Omega \underline{\mathrm{div}}\,\underline{\underline{\tau}}^2 \cdot \underline{v}_h \mathrm{d}x + \int_\Omega \mathrm{as}(\underline{\underline{\tau}}^2)\eta_h \mathrm{d}x \\ &= \int_\Omega \underline{v}_h^2 \mathrm{d}x + \int_\Omega \eta_h^2 \mathrm{d}x = \|\underline{v}_h\|_0^2 + \|\eta_h\|_0^2 \\ &\ge c(\|\underline{v}_h\|_0 + \|\eta_h\|_0)(\|\underline{v}_h\|_0 + \|\eta_h\|_0) \\ &\ge c\|\underline{\underline{\tau}}^2\|_1(\|\underline{v}_h\|_0 + \|\eta_h\|_0),\end{aligned} \tag{3.15}$$

which derives that

$$\frac{\int_\Omega \underline{\mathrm{div}}\,\underline{\underline{\tau}}^2 \cdot \underline{v}_h \mathrm{d}x + \int_\Omega \mathrm{as}(\underline{\underline{\tau}}^2)\eta_h \mathrm{d}x}{\|\underline{\underline{\tau}}^2\|_1} \ge k_1(\|\underline{v}_h\|_0 + \|\eta_h\|_0). \tag{3.16}$$

Recalling the definition of j_h ahead, we have

$$\begin{aligned}&\sup_{\forall \underline{\underline{\tau}}_h \in \Sigma_h} \frac{b(\underline{\underline{\tau}}_h, (\underline{v}_h, \eta_h))}{\|\underline{\underline{\tau}}\|_{\underline{\underline{H}}(\mathrm{div},\,\Omega)}} = \sup_{\forall \underline{\underline{\tau}}_h \in \Sigma_h} \frac{\int_\Omega \underline{\mathrm{div}}\,\underline{\underline{\tau}}_h \cdot \underline{v}_h \mathrm{d}x + \int_\Omega \mathrm{as}(\underline{\underline{\tau}}_h)\eta_h \mathrm{d}x}{\|\underline{\underline{\tau}}\|_{\underline{\underline{H}}(\mathrm{div},\,\Omega)}} \\ &\ge \frac{\int_\Omega \underline{\mathrm{div}}\, j_h\underline{\underline{\tau}}^2 \cdot \underline{v}_h \mathrm{d}x + \int_\Omega \mathrm{as}(j_h\underline{\underline{\tau}}^2)\eta_h \mathrm{d}x}{\|j_h\underline{\underline{\tau}}^2\|_{\underline{\underline{H}}(\mathrm{div},\,\Omega)}} \\ &\ge \frac{\int_\Omega \underline{\mathrm{div}}\,\underline{\underline{\tau}}^2 \cdot \underline{v}_h \mathrm{d}x + \int_\Omega \mathrm{as}(\underline{\underline{\tau}}^2)\eta_h \mathrm{d}x}{\|\underline{\underline{\tau}}^2\|_1} - \\ &\quad \frac{\int_\Omega \underline{\mathrm{div}}\,(\underline{\underline{\tau}}^2 - j_h\underline{\underline{\tau}}^2) \cdot \underline{v}_h \mathrm{d}x + \int_\Omega \mathrm{as}(\underline{\underline{\tau}}^2 - j_h\underline{\underline{\tau}}^2)\eta_h \mathrm{d}x}{\|\underline{\underline{\tau}}^2\|_1}.\end{aligned} \tag{3.17}$$

Let $\underline{n} = (n_1, n_2)$ be the unit normal vector with respect to the edge e of T. By using the Green formula, we get

$$\begin{aligned}&\int_\Omega \underline{\mathrm{div}}(\underline{\underline{\tau}}^2 - j_h\underline{\underline{\tau}}^2) \cdot \underline{v}_h \mathrm{d}x \\ &= -\int_\Omega (\underline{\underline{\tau}}^2 - j_h\underline{\underline{\tau}}^2) : \underline{\underline{\varepsilon}}(\underline{v}_h)\mathrm{d}x + \sum_{E_h} \int_e (\underline{\underline{\tau}}^2 - j_h\underline{\underline{\tau}}^2)\underline{n} \cdot \underline{v}_h \mathrm{d}s \qquad (3.18) \\ &\le \sum_{E_h} \|\underline{\underline{\tau}}^2 - j_h\underline{\underline{\tau}}^2\|_{0,\,e} \|[\underline{v}_h]\|_{0,\,e} \qquad (3.19) \\ &\le \|\underline{\underline{\tau}}^2 - j_h\underline{\underline{\tau}}^2\|_{E_h} \|[\underline{v}_h]\|_{E_h}. \qquad (3.20)\end{aligned}$$

Furthermore, according to the trace theorem and (3.4), we get

$$\|\underline{\underline{\tau}}^2 - j_h\underline{\underline{\tau}}^2\|_{E_h}^2 \le c\|\underline{\underline{\tau}}^2 - j_h\underline{\underline{\tau}}^2\|_0\, \|\underline{\underline{\tau}}^2 - j_h\underline{\underline{\tau}}^2\|_1 \le ch\|\underline{\underline{\tau}}^2\|_1^2,$$

which means

$$\|\underline{\underline{\tau}}^2 - j_h\underline{\underline{\tau}}^2\|_{E_h} \le k_2 h^{\frac{1}{2}}\|\underline{\underline{\tau}}^2\|_1, \tag{3.21}$$

then

$$\int_\Omega \mathrm{as}(\underline{\underline{\tau}}^2 - j_h\underline{\underline{\tau}}^2)\eta_h \mathrm{d}x \le \|\eta_h\|_0\, \|\underline{\underline{\tau}}^2 - j_h\underline{\underline{\tau}}^2\|_0 \le k_3 h\|\eta_h\|_0\|\underline{\underline{\tau}}^2\|_1. \tag{3.22}$$

Combining (3.17)–(3.22), we can get the result of (3.7). □

Lemma 3.3 *Let* $\Pi_1 : \underline{L}^2(\Omega) \to \underline{P}_1 \cap \underline{H}_0^1(\Omega)$ *and* $\Pi_1\underline{q}_h \in \underline{C}^0(\Omega)$*. Then there holds*

$$ch^{\frac{1}{2}}\|[\underline{q}_h]\|_{E_h} \le \|\underline{q}_h - \Pi_1\underline{q}_h\|_0 \le c\|\underline{q}_h\|_0, \quad \forall\, \underline{q}_h \in \underline{P}_0. \tag{3.23}$$

Proof From the definition of interpolation and inverse inequality, it is easy to know $[\Pi_1\underline{q}_h]|_{\partial T} = \underline{0}$, and

$$\begin{aligned} ch\|[\underline{q}_h]\|_{E_h}^2 &= ch\sum_{E_h}\|[\underline{q}_h - \Pi_1\underline{q}_h]\|_e^2 \\ &\le ch\|[\underline{q}_h - \Pi_1\underline{q}_h]\|_{E_h}^2 \le \|\underline{q}_h - \Pi_1\underline{q}_h\|_0^2. \end{aligned} \tag{3.24}$$

Considering the operator Π_1 is continuous, we get

$$\|(I - \Pi_1)\underline{q}_h\|_0 = \|\underline{q}_h - \Pi_1\underline{q}_h\|_0 \le c\|\underline{q}_h\|_0. \tag{3.25}$$

Combining (3.24) with (3.25) derives (3.23) immediately. □

4. The stabilized method

From the mixed variation formulation of elasticity problem, we rewrite Eq. (2.4) as follows. Find $(\underline{\underline{\sigma}}; (\underline{u}, \gamma)) \in \underline{\underline{H}}(\mathrm{div}, \Omega) \times \underline{L}^2(\Omega) \times L^2(\Omega)$, such that

$$Q((\underline{\underline{\sigma}}; (\underline{u}, \gamma)), (\underline{\underline{\tau}}; (\underline{v}, \eta))) = (\underline{f}, \underline{v}), \tag{4.1}$$

where

$$Q((\underline{\underline{\sigma}}; (\underline{u}, \gamma)), (\underline{\underline{\tau}}; (\underline{v}, \eta))) = a(\underline{\underline{\sigma}}, \underline{\underline{\tau}}) + b(\underline{\underline{\tau}}; (\underline{u}, \gamma)) + b(\underline{\underline{\sigma}}; (\underline{v}, \eta)). \tag{4.2}$$

Let

$$\begin{aligned} &Q((\underline{\underline{\sigma}}_h; (\underline{u}_h, \gamma_h)), (\underline{\underline{\tau}}_h; (\underline{v}_h, \eta_h))) \\ &\quad = a(\underline{\underline{\sigma}}_h, \underline{\underline{\tau}}_h) + b(\underline{\underline{\tau}}_h; (\underline{u}_h, \gamma_h)) + b(\underline{\underline{\sigma}}_h; (\underline{v}_h, \eta_h)). \end{aligned} \tag{4.3}$$

By (3.1) the discrete bilinear form satisfies

$$\begin{aligned} &a(\underline{\underline{\sigma}}_h, \underline{\underline{\tau}}_h) \le \alpha_1\|\underline{\underline{\sigma}}_h\|_0\, \|\underline{\underline{\tau}}_h\|_0, \\ &a(\underline{\underline{\tau}}_h, \underline{\underline{\tau}}_h) \ge \alpha_2\|\underline{\underline{\tau}}_h\|_0^2. \end{aligned} \tag{4.4}$$

Based on the analysis of Lemma 3.2, we introduce the stabilization items as

$$G_1(\underline{u}_h, \underline{v}_h) = -\gamma_1\sum_{E_h}\int_e h[\underline{u}_h]\cdot[\underline{v}_h]\mathrm{d}s,$$

$$G_2(\gamma_h,\eta_h) = -\gamma_2\int_\Omega h^2\gamma_h\eta_h \mathrm{d}x,$$
$$G_3(\underline{\sigma}_h,\underline{\tau}_h) = \gamma_3\int_\Omega \underline{\operatorname{div}}\,\underline{\sigma}_h\cdot\underline{\operatorname{div}}\,\underline{\tau}_h \mathrm{d}x. \tag{4.5}$$

And we set

$$\tilde{Q}_h((\underline{\sigma}_h;(\underline{u}_h,\gamma_h)),(\underline{\tau}_h;(\underline{v}_h,\eta_h)))$$
$$= Q((\underline{\sigma}_h;(\underline{u}_h,\gamma_h)),(\underline{\tau}_h;(\underline{v}_h,\eta_h))) + G_1(\underline{u}_h,\underline{v}_h) + G_2(\gamma_h,\eta_h) + G_3(\underline{\sigma}_h,\underline{\tau}_h).$$
$$F_h(\underline{v}_h) = (\underline{f},\,\underline{v}_h) + \gamma_3\int_\Omega \underline{f}\cdot\underline{\operatorname{div}}\,\underline{\tau}_h \mathrm{d}x. \tag{4.6}$$

The stabilized discrete equation of (4.1) is to find $(\underline{\sigma}_h;(\underline{u}_h,\gamma_h))\in\Sigma_h\times V_h\times S_h$ such that

$$\tilde{Q}_h((\underline{\sigma}_h;(\underline{u}_h,\gamma_h)),(\underline{\tau}_h;(\underline{v}_h,\eta_h))) = F_h(\underline{v}_h). \tag{4.7}$$

Define the norm on the space $\Sigma_h\times V_h\times S_h$ as

$$|||\,(\underline{\tau}_h;(\underline{v}_h,\,\eta_h))|||_h^2 = \|\underline{\tau}_h\|^2_{\underline{H}(\mathrm{div},\,\Omega)} + \|\underline{v}_h\|_0^2 + \|\eta_h\|_0^2 + h\|[\underline{v}_h]\|^2_{E_h}. \tag{4.8}$$

From the definition of (3.6), we derive the inverse inequality

$$\|[\underline{q}_h]\|_{E_h} \le ch^{-\frac{1}{2}}\|\underline{q}_h\|_0,\quad \forall\,\underline{q}_h\in\underline{P}_0(T). \tag{4.9}$$

Next we discus the continuity of $\tilde{Q}_h$ defined by (4.6). For any $(\underline{\tau}_h;(\underline{v}_h,\eta_h))\in\Sigma_h\times V_h\times S_h$,

$$\begin{aligned}
&\tilde{Q}_h((\underline{\sigma}_h;(\underline{u}_h,\gamma_h)),(\underline{\tau}_h;(\underline{v}_h,\eta_h)))\\
&\le \alpha_1\|\underline{\sigma}_h\|_0\|\underline{\tau}_h\|_0 + \|\underline{\operatorname{div}}\,\underline{\tau}_h\|_0\|\underline{u}_h\|_0 +\\
&\quad \|\gamma_h\|_0\|as(\underline{\tau})_h\|_0 + \|\underline{\operatorname{div}}\,\underline{\sigma}_h\|_0\|\underline{v}_h\|_0 + \|\underline{\sigma}_h\|_0\|\eta_h\|_0 +\\
&\quad \gamma_1 h\|[\underline{u}_h]\|_{E_h}\|[\underline{v}_h]\|_{E_h} + \gamma_2 h^2\|\gamma_h\|_0\|\eta_h\|_0 + \gamma_3\|\underline{\operatorname{div}}\,\underline{\sigma}_h\|_0\|\underline{\operatorname{div}}\,\underline{\tau}_h\|_0\\
&\le C(\|\underline{\sigma}_h\|_0^2 + \|\underline{\operatorname{div}}\,\underline{\sigma}_h\|_0^2 + \|\underline{u}_h\|_0^2 + \|\gamma_h\|_0^2 + h\|[\underline{u}_h]\|^2_{E_h})^{\frac{1}{2}}\cdot\\
&\quad (\|\underline{\tau}_h\|_0^2 + \|\underline{\operatorname{div}}\,\underline{\tau}_h\|_0^2 + \|\underline{v}_h\|_0^2 + \|\eta_h\|_0^2 + h\|[\underline{v}_h]\|^2_{E_h})^{\frac{1}{2}}\\
&\le C\,|||\,(\underline{\sigma}_h;(\underline{u}_h,\,\gamma_h))\,|||_h\,|||(\underline{\tau}_h;(\underline{v}_h,\,\eta_h))\,|||_h\,.
\end{aligned} \tag{4.10}$$

The following theorem shows the coercive of the bilinear form $\tilde{Q}_h$.

Theorem 4.1 *Let Σ_h, V_h, S_h be the finite element spaces defined by (3.3). Then for any $(\underline{\sigma}_h;(\underline{u}_h,\gamma_h))\in\Sigma_h\times V_h\times S_h$, there holds*

$$\sup_{\forall(\underline{\tau}_h;(\underline{v}_h,\eta_h))\in\Sigma_h\times V_h\times S_h}\frac{\tilde{Q}_h((\underline{\sigma}_h;(\underline{u}_h,\gamma_h)),(\underline{\tau}_h;(\underline{v}_h,\eta_h)))}{|||(\underline{\tau}_h;(\underline{v}_h,\,\eta_h))|||_h} \ge C\,|||\,(\underline{\sigma}_h;(\underline{u}_h,\,\gamma_h))\,|||_h\,. \tag{4.11}$$

Proof For given $(\underline{u}_h,\gamma_h)\in V_h\times S_h$, considering Lemma 3.2, we can choose $\bar{\underline{\rho}}_h\in\Sigma_h$ to match (3.7). Then, taking $\underline{\rho}_h = \frac{\|\underline{u}_h\|_0+\|\gamma_h\|_0}{\|\bar{\underline{\rho}}_h\|_{\underline{H}(\mathrm{div},\Omega)}}\bar{\underline{\rho}}_h$, and from the norm definition of the divergence space, we have

$$\|\underline{\rho}_h\|_{\underline{H}(\mathrm{div},\Omega)} = \|\underline{u}_h\|_0 + \|\gamma_h\|_0. \tag{4.12}$$

Then for any $(\underline{\underline{\sigma}}_h;(\underline{u}_h,\gamma_h))\in\Sigma_h\times V_h\times S_h$, we can look for special $(\underline{\underline{\tau}}_h;(\underline{v}_h,\eta_h))$ to satisfy (4.11).

Taking $(\underline{\underline{\tau}}_h^1;(\underline{v}_h^1,\eta_h^1))=(\underline{\underline{\sigma}}_h;(-\underline{u}_h,-\gamma_h))$ and from (4.4), we have

$$\begin{aligned}&\tilde{Q}_h^1((\underline{\underline{\sigma}}_h;(\underline{u}_h,\gamma_h)),(\underline{\underline{\tau}}_h^1;(\underline{v}_h^1,\eta_h^1)))\\&=\tilde{Q}_h((\underline{\underline{\sigma}}_h;(\underline{u}_h,\gamma_h)),(\underline{\underline{\sigma}}_h;(-\underline{u}_h,-\gamma_h)))\\&=a(\underline{\underline{\sigma}}_h,\underline{\underline{\sigma}}_h)+\gamma_1\sum_{E_h}\int_e h[\underline{u}_h]^2\mathrm{d}s+\gamma_2\int_\Omega h^2\gamma_h^2dx+\gamma_3\int_\Omega(\underline{\mathrm{div}}\,\underline{\underline{\sigma}}_h)^2\mathrm{d}x\\&\ge\alpha_2\|\underline{\underline{\sigma}}_h\|_0^2+\gamma_1 h\|[\underline{u}_h]\|_{E_h}^2+\gamma_2h^2\|\gamma_h\|_0^2+\gamma_3\|\underline{\mathrm{div}}\,\underline{\underline{\sigma}}_h\|_0^2\\&\ge C_1\|\underline{\underline{\sigma}}_h\|^2_{\underline{\underline{H}}(\mathrm{div},\Omega)}+\gamma_1h\|[\underline{u}_h]\|_{E_h}^2+\gamma_2h^2\|\gamma_h\|_0^2,\end{aligned}\tag{4.13}$$

where $C_1=\min\{\alpha_2,\gamma_3\}$.

Taking $(\underline{\underline{\tau}}_h^2;(\underline{v}_h^2,\eta_h^2))=(\underline{\underline{\rho}}_h;(\underline{0},0))$ and from Lemma 3.2 and (4.12), we have

$$\begin{aligned}b(\underline{\underline{\rho}}_h,(\underline{u}_h,\gamma_h))&=\int_\Omega\underline{u}_h\cdot\underline{\mathrm{div}}\,\underline{\underline{\rho}}_h\mathrm{d}x+\int_\Omega\gamma_h\cdot\mathrm{as}(\underline{\underline{\rho}}_h)\mathrm{d}x\\&\ge\|\underline{\underline{\rho}}_h\|_{\underline{\underline{H}}(\mathrm{div},\Omega)}(k_1(\|\underline{u}_h\|_0+\|\gamma_h\|_0)-k_2h^{\frac12}\|[\underline{u}_h]\|_{E_h}-k_3h\|\gamma_h\|_0)\\&=(\|\underline{u}_h\|_0+\|\gamma_h\|_0)(k_1(\|\underline{u}_h\|_0+\|\gamma_h\|_0)-k_2h^{\frac12}\|[\underline{u}_h]\|_{E_h}-k_3h\|\gamma_h\|_0)\\&\ge k_1(\|\underline{u}_h\|_0+\|\gamma_h\|_0)^2-\frac{l_1}{2}(\|\underline{u}_h\|_0+\|\gamma_h\|_0)^2-\frac{1}{2l_1}(k_2h^{\frac12}\|[\underline{u}_h]\|_{E_h}+k_3h\|\gamma_h\|_0)^2\\&\ge(k_1-l_1)(\|\underline{u}_h\|_0^2+\|\gamma_h\|_0^2)-\frac{1}{l_1}(k_2^2h\|[\underline{u}_h]\|_{E_h}^2+k_3^2h^2\|\gamma_h\|_0^2)\triangleq M,\end{aligned}\tag{4.14}$$

which leads to

$$\begin{aligned}&\tilde{Q}_h^2((\underline{\underline{\sigma}}_h;(\underline{u}_h,\gamma_h)),(\underline{\underline{\rho}}_h;(\underline{0},0)))\\&=a(\underline{\underline{\sigma}}_h,\underline{\underline{\rho}}_h)+\int_\Omega\underline{u}_h\cdot\underline{\mathrm{div}}\,\underline{\underline{\rho}}_h\mathrm{d}x+\int_\Omega\gamma_h\cdot\mathrm{as}(\underline{\underline{\rho}}_h)\mathrm{d}x+\gamma_3\int_\Omega\underline{\mathrm{div}}\,\underline{\underline{\sigma}}_h\cdot\underline{\mathrm{div}}\,\underline{\underline{\rho}}_h\mathrm{d}x\\&\ge-\alpha_1\|\underline{\underline{\sigma}}_h\|_0\|\underline{\underline{\rho}}_h\|_0-\gamma_3\|\underline{\mathrm{div}}\,\underline{\underline{\sigma}}_h\|_0\|\underline{\mathrm{div}}\,\underline{\underline{\rho}}_h\|_0+M\\&\ge-\theta(\|\underline{\underline{\sigma}}_h\|_0\|\underline{\underline{\rho}}_h\|_0+\|\underline{\mathrm{div}}\,\underline{\underline{\sigma}}_h\|_0\|\underline{\mathrm{div}}\,\underline{\underline{\rho}}_h\|_0)+M\\&\ge-\theta(\frac{1}{2l_2}\|\underline{\underline{\sigma}}_h\|^2_{\underline{\underline{H}}(\mathrm{div},\Omega)}+2l_2(\|\underline{u}_h\|_0^2+\|\gamma_h\|_0^2))+M\\&=-\frac{\theta}{2l_2}\|\underline{\underline{\sigma}}_h\|^2_{\underline{\underline{H}}(\mathrm{div},\Omega)}+m(\|\underline{u}_h\|_0^2+\|\gamma_h\|_0^2)-\frac{k_2^2h}{l_1}\|[\underline{u}_h]\|_{E_h}^2-\frac{k_3^2h^2}{l_1}\|\gamma_h\|_0^2,\end{aligned}\tag{4.15}$$

with $\theta=\max\{\alpha_1,\gamma_3\}$, we can choose proper l_1,l_2 such that $m=k_1-l_1-2l_2\theta\ge0$. Let

$$(\underline{\underline{\tau}}_h;(\underline{v}_h,\eta_h))=(\underline{\underline{\tau}}_h^1;(\underline{v}_h^1,\eta_h^1))+\delta(\underline{\underline{\tau}}_h^2;(\underline{v}_h^2,\eta_h^2))=(\underline{\underline{\sigma}}_h+\delta\underline{\underline{\rho}}_h;(-\underline{u}_h,-\gamma_h)).$$

From (4.13) and (4.15) it follows

$$\begin{aligned}&\tilde{Q}_h((\underline{\underline{\sigma}}_h;(\underline{u}_h,\gamma_h)),(\underline{\underline{\tau}}_h;(\underline{v}_h,\eta_h)))=\tilde{Q}_h^1+\delta\tilde{Q}_h^2\\&\ge C_1\|\underline{\underline{\sigma}}_h\|^2_{\underline{\underline{H}}(\mathrm{div},\Omega)}+\gamma_1h\|[\underline{u}_h]\|_{E_h}^2+\gamma_2h^2\|\gamma_h\|_0^2+\delta(-\frac{\theta}{2l_2}\|\underline{\underline{\sigma}}_h\|^2_{\underline{\underline{H}}(\mathrm{div},\Omega)}+\end{aligned}$$

$$m(\|\underline{u}_h\|_0^2+\|\gamma_h\|_0^2)-\frac{k_2^2h}{l_1}\|[\underline{u}_h]\|_{E_h}^2-\frac{k_3^2h^2}{l_1}\|\gamma_h\|_0^2)$$
$$=(C_1-\delta\frac{\theta}{2l_2})\|\underline{\underline{\sigma}}_h\|^2_{\underline{\underline{H}}(\mathrm{div},\Omega)}+\delta m(\|\underline{u}_h\|_0^2+\|\gamma_h\|_0^2)+(\gamma_1-\delta\frac{k_2^2}{l_1})h\|[\underline{u}_h]\|_{E_h}^2+$$
$$h^2(\gamma_2-\delta\frac{k_3^2}{l_1})\|\gamma_h\|_0^2. \tag{4.16}$$

We can take the proper δ to keep the coefficient of (4.16) positive.

Take $C_2=\min\{C_1-\delta\frac{\theta}{2l_2},\delta m,\gamma_1-\delta\frac{k_2^2}{l_1}\}$, then the (4.16) becomes

$$\tilde{Q}_h((\underline{\underline{\sigma}}_h;(\underline{u}_h,\gamma_h)),(\underline{\underline{\tau}}_h;(\underline{v}_h,\eta_h)))\geq C_2\,|||\,(\underline{\underline{\sigma}}_h;(\underline{u}_h,\gamma_h))\,|||_h^2\,. \tag{4.17}$$

At last, by the definition of $\underline{\underline{\rho}}_h$, we have $\|\underline{\underline{\rho}}_h\|_0\leq\|\underline{u}_h\|_0+\|\gamma_h\|_0$, then

$$|||\,(\underline{\underline{\tau}}_h;(\underline{v}_h,\eta_h))|||_h=|||(\underline{\underline{\sigma}}_h+\delta\underline{\underline{\rho}}_h;(-\underline{u}_h,-\gamma_h))|||_h$$
$$=(\|\underline{\underline{\sigma}}_h+\delta\underline{\underline{\rho}}_h\|^2_{\underline{\underline{H}}(\mathrm{div},\Omega)}+\|\underline{u}_h\|_0^2+\|\gamma_h\|_0^2+h\|[\underline{u}_h]\|_{E_h}^2)^{\frac{1}{2}}$$
$$\leq(2\|\underline{\underline{\sigma}}_h\|^2_{\underline{\underline{H}}(\mathrm{div},\Omega)}+(2\delta^2+1)\|\underline{u}_h\|_0^2+(2\delta^2+1)\|\gamma_h\|_0^2+h\|[\underline{u}_h]\|_{E_h}^2)^{\frac{1}{2}}$$
$$\leq\frac{1}{C_3}\,|||\,(\underline{\underline{\sigma}}_h;(\underline{u}_h,\gamma_h))|||_h \tag{4.18}$$

with $\frac{1}{C_3}=\max\{2,\,2\delta^2+1)$.

Let $C=C_2C_3$. Combining (4.17) and (4.18) derives (4.11) directly. $\square$

5. Error estimate

In this section, we will give the error between the exact solution and the mixed finite element solution.

Theorem 5.1 *Let* $(\underline{\underline{\sigma}};(\underline{u},\gamma))$ *and* $(\underline{\underline{\sigma}}_h;(\underline{u}_h,\gamma_h))$ *be the solution of* (4.1) *and* (4.7)*, respectively. Then there holds*

$$\|\underline{\underline{\sigma}}-\underline{\underline{\sigma}}_h\|_H+\|\underline{u}-\underline{u}_h\|_0+\|\gamma-\gamma_h\|_0\leq ch\|\underline{f}\|_0. \tag{5.1}$$

Proof Since the finite element space is conforming, we use $(\underline{\underline{\tau}}_h;(\underline{v}_h,\eta_h))$ to replace $(\underline{\underline{\tau}};(\underline{v},\eta))$ in (4.1) and subtract (4.6). In addition from the (2.1) and $\underline{\mathrm{div}}\,\underline{\underline{\tau}}_h\in\underline{L}^2(\Omega)$, we have

$$(\underline{f},\,\underline{\mathrm{div}}\,\underline{\underline{\tau}}_h)=(\underline{\mathrm{div}}\,\underline{\underline{\sigma}},\underline{\mathrm{div}}\,\underline{\underline{\tau}}_h). \tag{5.2}$$

The error equation is as follows

$$\tilde{Q}_h((\underline{\underline{\sigma}}-\underline{\underline{\sigma}}_h;(\underline{u}-\underline{u}_h,\gamma-\gamma_h)),(\underline{\underline{\tau}}_h;(\underline{v}_h,\eta_h)))=\gamma_1\sum_{E_h}\int_e h[\underline{u}]\cdot[\underline{v}_h]\mathrm{d}s+\gamma_2\int_\Omega h^2\gamma\eta_h\mathrm{d}x. \tag{5.3}$$

From the conclusion of Theorem 4.1, we can get

$$|||\,(\underline{\underline{\sigma}}_h-J_h\underline{\underline{\sigma}};(\underline{u}_h-I_h\underline{u},\gamma_h-P_h\gamma))|||_h$$
$$\leq\sup_{\forall(\underline{\underline{\tau}}_h;(\underline{v}_h,\eta_h))\in\Sigma_h\times V_h\times S_h}\frac{\tilde{Q}_h((\underline{\underline{\sigma}}_h-J_h\underline{\underline{\sigma}};(\underline{u}_h-I_h\underline{u},\gamma_h-P_h\gamma)),(\underline{\underline{\tau}}_h;(\underline{v}_h,\eta_h)))}{|||(\underline{\underline{\tau}}_h;(\underline{v}_h,\eta_h))|||_h}$$

$$= \sup_{\forall(\underline{\tau}_h;(\underline{v}_h,\eta_h))\in\Sigma_h\times V_h\times S_h} \frac{\tilde{Q}_h((\underline{\sigma}-J_h\underline{\sigma};(\underline{u}-I_h\underline{u},\gamma-P_h\gamma)),(\underline{\tau}_h;(\underline{v}_h,\eta_h)))}{|||(\underline{\tau}_h;(\underline{v}_h,\eta_h))|||_h} - \frac{\gamma_1\sum\limits_{E_h}\int_e h[\underline{u}]\cdot[\underline{v}_h]\mathrm{d}s+\gamma_2\int_\Omega h^2\gamma\eta_h\mathrm{d}x}{|||(\underline{\tau}_h;(\underline{v}_h,\eta_h))|||_h}$$

$$:= \sup_{\forall(\underline{\tau}_h;(\underline{v}_h,\eta_h))\in\Sigma_h\times V_h\times S_h} \frac{\sum_{i=1}^{10}L_i}{|||(\underline{\tau}_h;(\underline{v}_h,\eta_h))|||_h}, \tag{5.4}$$

where

$$L_1 = a(\underline{\sigma}-J_h\underline{\sigma},\underline{\tau}_h), \qquad L_2 = \int_\Omega(\underline{u}-I_h\underline{u})\cdot\underline{\mathrm{div}}\,\underline{\tau}_h\mathrm{d}x,$$

$$L_3 = \int_\Omega(\gamma-P_h\gamma)\mathrm{as}(\underline{\tau}_h)\mathrm{d}x, \qquad L_4 = \int_\Omega\underline{\mathrm{div}}\,(\underline{\sigma}-J_h\underline{\sigma})\cdot\underline{v}_h\mathrm{d}x,$$

$$L_5 = \int_\Omega\mathrm{as}(\underline{\sigma}-J_h\underline{\sigma})\eta_h\mathrm{d}x, \qquad L_6 = \gamma_3\int_\Omega\underline{\mathrm{div}}\,(\underline{\sigma}-J_h\underline{\sigma})\cdot\underline{\mathrm{div}}\,\underline{\tau}_h\mathrm{d}x,$$

$$L_7 = -\gamma_2\int_\Omega h^2(\gamma-P_h\gamma)\eta_h\mathrm{d}x, \qquad L_8 = -\gamma_1\sum_{E_h}\int_e h[\underline{u}-I_h\underline{u}]\cdot[\underline{v}_h]\mathrm{d}s,$$

$$L_9 = -\gamma_2\int_\Omega h^2\gamma\eta_h\mathrm{d}x, \qquad L_{10} = -\gamma_1\sum_{E_h}\int_e h[\underline{u}]\cdot[\underline{v}_h]\mathrm{d}s.$$

Next, we estimate the bound of L_i $(i=1,\dots,10)$ one by one

$$\begin{aligned}
|L_1| &= \int_\Omega A(\underline{\sigma}-J_h\underline{\sigma}):\underline{\tau}_h\mathrm{d}x \le \alpha_1\|\underline{\sigma}-J_h\underline{\sigma}\|_0\|\underline{\tau}_h\|_0 \le Ch\|\underline{\sigma}\|_1\|\underline{\tau}_h\|_0,\\
|L_2| &\le \|\underline{u}-I_h\underline{u}\|_0\|\underline{\mathrm{div}}\underline{\tau}_h\|_0 \le Ch\|\underline{u}\|_1\|\underline{\mathrm{div}}\underline{\tau}_h\|_0,\\
|L_3| &\le \|\gamma-P_h\gamma\|_0\|\underline{\tau}_h\|_0 \le Ch\|\gamma\|_1\|\underline{\tau}_h\|_0,\\
|L_4| &\le \|\underline{\sigma}-J_h\underline{\sigma}\|_0\|\underline{v}_h\|_0 \le Ch\|\underline{\sigma}\|_1\|\underline{v}_h\|_0,\\
|L_5| &\le \|\underline{\sigma}-J_h\underline{\sigma}\|_0\|\eta_h\|_0 \le Ch\|\underline{\sigma}\|_1\|\eta_h\|_0,\\
|L_6| &\le \gamma_3\|\underline{\mathrm{div}}(\underline{\sigma}-J_h\underline{\sigma})\|_0\|\underline{\mathrm{div}}\underline{\tau}_h\|_0 \le Ch\gamma_3\|\underline{\sigma}\|_2\|\underline{\mathrm{div}}\underline{\tau}_h\|_0,\\
|L_7| &\le \gamma_2h^2\|\gamma-P_h\gamma\|_0\|\eta_h\|_0 \le C\gamma_2h^3\|\gamma\|_1\|\eta_h\|_0.
\end{aligned}$$

Using the inverse inequality (4.9) and Lemma 3.3 successively, we have

$$\begin{aligned}
|L_8| &\le \gamma_1h\|[\underline{u}-I_h\underline{u}]\|_{E_h}\|[\underline{v}]_h\|_{E_h}\\
&\le Ch^{\frac12}\gamma_1\|[\underline{u}-I_h\underline{u}]\|_0\|[\underline{v}]_h\|_{E_h} \le Ch\gamma_1\|\underline{u}\|_1\|\underline{v}_h\|_0,\\
|L_9| &\le C\gamma_2h^2\|\gamma\|_1\|\eta_h\|_0.
\end{aligned}$$

For any $\underline{u}\in\underline{H}^1(\Omega)$, the jump value of displacement is zero, i.e., $|L_{10}|=0$. So we have

$$|||(\underline{\sigma}_h-J_h\underline{\sigma};(\underline{u}_h-I_h\underline{u},\gamma_h-P_h\gamma))|||_h \le Ch(\|\underline{\sigma}\|_1+\|\underline{u}\|_1+\|\gamma\|_1),$$

from the conclusion of Theorem 3.1, we can get the desired result. □

6. Numerical example

We consider the elasticity problem in 2D for $[0,1]^2$. The equations of linear elasticity can be written as a system of equations of the form

$$\begin{cases} \underline{\operatorname{div}\,\underline{\sigma}} = \underline{f}, & \text{in } \Omega, \\ A(\underline{\underline{\sigma}}) = \underline{\underline{\varepsilon}}(\underline{u}), & \text{in } \Omega, \\ \underline{u} = \underline{0}, & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where the Lamé constants are $\mu = 1/2$ and $\lambda = 1$.

For numerical simulation, we take the displacement of the vector function $\underline{u}$ in (6.1), as

$$\underline{u}_1 = \begin{pmatrix} 4x(1-x)y(1-y) \\ -4x(1-x)y(1-y) \end{pmatrix} \quad \text{and} \quad \underline{u}_2 = \begin{pmatrix} e^{(x-y)}(1-x)y(1-y) \\ \sin \pi x \sin \pi y \end{pmatrix}$$

and set Error $= |||(\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h; (\underline{u} - \underline{u}_h, \gamma - \gamma_h))|||_h$.

Mesh $n \times n$	4×4	8×8	16×16	32×32	64×64
Error	0.403297	0.19841	0.0818765	0.0324603	0.0135391
Order of convergence	-	1.0234	1.2770	1.3348	1.2615

Table 1 The error and the order of convergence for $\underline{u}_1$

Mesh $n \times n$	4×4	8×8	16×16	32×32	64×64
Error	0.178351	0.164165	0.0822141	0.0344859	0.0137398
Order of convergence	-	0.1196	0.9977	1.2534	1.3276

Table 2 The error and the order of convergence for $\underline{u}_2$

We construct a sequence of $n \times n$ meshes with n uniform subintervals in the x-axis direction and the y-axis direction, respectively. Here the $(\underline{\underline{\sigma}}; (\underline{u}, \gamma))$ and $(\underline{\underline{\sigma}}_h; (\underline{u}_h, \gamma_h))$ are the original solution and the stabilized mixed finite element solution, respectively. Since γ is of higher convergence order and its absolute error is bigger than that of $\underline{u}$ and $\underline{\underline{\sigma}}$, we find that the order of convergence is higher than that in the conclusion of Theorem 5.1 from the results showed in Tables 1 and 2.

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