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Half Inverse Riordan Arrays and Their Related Vertical Recurrence Relation

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Abstract We discuss two different procedures to study the half Riordan arrays and their inverses. One of the procedures shows that every Riordan array is the half Riordan array of a unique Riordan array. It is well known that every Riordan array has its half Riordan array. Therefore, this paper answers the converse question: Is every Riordan array the half Riordan array of some Riordan arrays? In addition, this paper shows that the vertical recurrence relation of the column entries of the half Riordan array is equivalent to the horizontal recurrence relation of the original Riordan array's row entries.

Keywords Riordan array; Riordan group; A-sequence; Appell subgroup; Lagrange (associated) subgroup; Bell subgroup; hitting-time subgroup; derivative subgroup

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1. Introduction

Riordan arrays (or Riordan matrices) are infinite, lower triangular matrices defined by the generating function of their columns. With the matrix multiplication, the set of all Riordan arrays form a group, called the Riordan group [1].

More formally, let us consider the set of formal power series ring $\mathcal{F} = \mathbb{K}[\![t]\!]$, where $\mathbb{K}$ is the field $\mathbb{R}$ or $\mathbb{C}$. The *order* of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k$ $(f_k \in \mathbb{K})$, is the minimum number $r \in \mathbb{N}$ such that $f_r \neq 0$, where $\mathbb{N} = \{0, 1, 2 \ldots\}$ is the set of all natural numbers. We denote by $\mathcal{F}_r$ the set of formal power series of order r. Let $g(t) \in \mathcal{F}_0$ and $f(t) \in \mathcal{F}_1$; the pair (g, f) defines the (proper) Riordan array $D = (d_{n,k})_{n,k\in\mathbb{N}} = (g, f)$ having

$$d_{n,k} = [t^n]g(t)f(t)^k \tag{1.1}$$

or, in other words, having gf^k as the generating function of the k-th column of (g, f). The first fundamental theorem of Riordan arrays can be represented as

$$(g(t), f(t))h(t) = g(t)(h \circ f)(t),$$

which can also be simplified to $(g, f)h = gh(f)$. Thus we immediately see that the usual row-by-column product of two Riordan arrays is also a Riordan array:

$$(g_1, f_1)(g_2, f_2) = (g_1 g_2(f_1), f_2(f_1)). \tag{1.2}$$

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The Riordan array $I = (1, t)$ is the identity matrix because its entries are $d_{n,k} = [t^n]t^k = \delta_{n,k}$.

Let $(g(t), f(t))$ be a Riordan array. Then its inverse is

$$(g(t), f(t))^{-1} = (\frac{1}{g(\overline{f}(t))}, \overline{f}(t)), \tag{1.3}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$, i.e., $(f \circ \overline{f})(t) = (\overline{f} \circ f)(t) = t$. In this way, the set $\mathcal{R}$ of all proper Riordan arrays forms a group [1] called the Riordan group.

Here is a list of six important subgroups of the Riordan group [1–3].

- The Appell subgroup $\{(g(t), t) : g \in \mathcal{F}_0\}$.
- The Lagrange (associated) subgroup $\{(1, f(t) : f \in \mathcal{F}_1)\}$.
- The k-Bell subgroup $\{(g(t), t(g(t))^k) : g \in \mathcal{F}_0\}$, where k is a fixed positive integer.
- The hitting-time subgroup $\{(tf'(t)/f(t), f(t)) : f \in \mathcal{F}_1\}$.
- The derivative subgroup $\{(f'(t), f(t)) : f \in \mathcal{F}_1\}$.
- The checkerboard subgroup $\{(g(t), f(t)) : g \in \mathcal{F}_0, f \in \mathcal{F}_1\}$, where g is an even function and f is an odd function.

The 1-Bell subgroup is referred to as the Bell subgroup for short, and the Appell subgroup can be considered as the 0-Bell subgroup if we allow $k = 0$ to be included in the definition of the k-Bell subgroup.

An infinite lower triangular matrix $[d_{n,k}]_{n,k\in\mathbb{N}}$ is a Riordan array if and only if a unique sequence $A = (a_0 \neq 0, a_1, a_2, \ldots)$ exists such that for every $n, k \in \mathbb{N}$

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + \cdots + a_n d_{n,n}. \tag{1.4}$$

This is equivalent to

$$f(t) = tA(f(t)) \text{ or } t = \bar{f}(t)A(t). \tag{1.5}$$

Here, $A(t)$ is the generating function of the A-sequence. The first formula of (1.5) is also called the second fundamental theorem of Riordan arrays. Moreover, there exists a unique sequence $Z = (z_0, z_1, z_2, \ldots)$ such that every element in column 0 can be expressed as the linear combination

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + \cdots + z_n d_{n,n}, \tag{1.6}$$

or equivalently,

$$g(t) = \frac{1}{1 - tZ(f(t))}, \tag{1.7}$$

in which and thoroughly we always assume $g(0) = g_0 = 1$, a usual hypothesis for proper Riordan arrays. From (1.7), we may obtain

$$Z(t) = \frac{g(\bar{f}(t)) - 1}{\bar{f}(t)g(\bar{f}(t))}.$$

A- and Z-sequence characterizations of Riordan arrays were introduced, developed, and/or studied in Merlini, Rogers, Sprugnoli, and Verri [4], Roger [5], Sprugnoli and the author [6], Jean-Louis and Nkwanta [7], Luzón, Morón, Prieto-Martinez [8, 9], [10], and Shapiro et al. [11] as well as their references. In [6] the expressions of the A- and Z-sequences of the product depending on the analogous sequences of the two factors are given. More precisely, we consider two proper Riordan matrices $D_1 = (g_1, f_1)$ and $D_2 = (g_2, f_2)$ and their product,

$$D_3 = D_1 D_2 = (g_1 g_2(f_1), f_2(f_1)).$$

Denote by $A_i(t)$ and $Z_i(t)$, $i = 1, 2$, and 3, the generating functions of A-sequences and Z-sequences of D_i, $i = 1, 2$, and 3, respectively. Then

$$A_3(t) = A_2(t) A_1(\frac{t}{A_2(t)}) \tag{1.8}$$

and

$$Z_3(t) = (1 - \frac{t}{A_2(t)} Z_2(t)) Z_1(\frac{t}{A_2(t)}) + A_1(\frac{t}{A_2(t)}) Z_2(t). \tag{1.9}$$

Let $A(t)$ and $Z(t)$ be the generating functions of the A- and Z-sequences of a Riordan matrix $D = (g, f)$, and let $A^*(t)$ and $Z^*(t)$ be the generating functions of the A- and Z-sequences of the inverse Riordan matrix $D^{-1} = (1/g(\overline{f}), \overline{f})$. We immediately observe that

$$A^*(\frac{t}{A(t)}) = \frac{1}{A(t)} \tag{1.10}$$

and

$$Z^*(\frac{t}{A(t)}) = \frac{Z(t)}{tZ(t) - A(t)}. \tag{1.11}$$

Yang, Zheng, Yuan, and the author [12] give the following definition of half Riordan arrays (HRAs), which are called vertical half Riordan arrays (VHRA) in Barry [13] and the author [14, 15]. The p-th extensions of HRAs are given in the author [16].

Definition 1.1 *Let $(g, f) = (d_{n,k})_{n,k\in\mathbb{N}}$ be a Riordan array. Its related vertical half Riordan array (VHRA) $(v_{n,k})_{n,k\in\mathbb{N}}$ is defined by*

$$v_{n,k} = d_{2n-k,n}. \tag{1.12}$$

As a pair of half Riordan arrays, Barry [13] defines the following horizontal half Riordan array (HHRA).

Definition 1.2 *Let $(g, f) = (d_{n,k})_{n,k\in\mathbb{N}}$ be a Riordan array. Its related horizontal half Riordan array (HHRA) $(h_{n,k})_{n,k\in\mathbb{N}}$ is defined by*

$$h_{n,k} = d_{2n,n+k}. \tag{1.13}$$

Denote $\phi = \overline{t^2/f}$, $\phi' = \phi'(t)$, and $f \in \mathcal{F}_1$. From [14], the VHRA of (g, f) is the Riordan array

$$(v_{n,k})_{n,k\in\mathbb{N}} = (\frac{t\phi' g(\phi)}{\phi}, \phi), \tag{1.14}$$

which has the following decomposition

$$(\frac{t\phi' g(\phi)}{\phi}, \phi) = (\frac{t\phi'}{\phi}, \phi)(g, t). \tag{1.15}$$

Decomposition (1.15) suggests a more general type of half of Riordan array (g, f) defined by

$$(\frac{t\phi' g(\phi)}{\phi}, f(\phi)) = (\frac{t\phi'}{\phi}, \phi)(g, f), \tag{1.16}$$

which is exactly the HHRA of (g, f) (see [14]), namely,

$$(h_{n,k})_{n,k\in\mathbb{N}} = (\frac{t\phi' g(\phi)}{\phi}, f(\phi)). \tag{1.17}$$

In a Riordan array $(g, f) = (d_{n,k})_{n,k\geq 0}$, the first column (0-th column), with its generating function $g(t)$, usually possesses an interesting combinatorial interpretation or represents an important sequence, while the other columns might be considered as the compositions of the first column in the view of the following recurrence relation:

$$d_{n,k} = [t^n]gf^k = \sum_{j=1}^{n} f_j[t^{n-j}]gf^{k-1} = \sum_{j=1}^{n-k+1} f_j d_{n-j,k-1} \tag{1.18}$$

for $n, k \geq 1$, where $f(t) = \sum_{j\geq 1} f_j t^j$. Eq. (1.18) is called the vertical recurrence relation of (g, f). For instance,

$$d_{1,1} = f_1 d_{0,0},$$
$$d_{2.1} = f_2 d_{0,0} + f_1 d_{1,0},\ d_{2,2} = f_1 d_{1,1},$$
$$d_{3,1} = f_3 d_{0,0} + f_2 d_{1,0} + f_1 d_{2.0},\ d_{3,2} = f_2 d_{2,1} + f_1 d_{1,1},\ d_{3,3} = f_1 d_{2,2}, \ldots.$$

The vertical recurrence relation of the column entries of Riordan arrays has many applications. For example, Mao, Mu, and Wang [17] use the relation to give another interesting criterion for the total positivity of Riordan arrays. The relation is also applied in [18] to construct a new Riordan group, called the quasi-Riordan group.

In next section, we present a procedure to study the converse process of finding VHRAs, which shows that every Riordan array is the half Riordan array of a unique Riordan array. In addition, this procedure also shows that the vertical recurrence relation of the column entries of the VHRA is equivalent to the horizontal recurrence relation of the original Riordan array's row entries. Section 3 gives another procedure for studying the inverses of VHRAs.

2. Riordan array as VHRA

Every Riordan array has its VHRA. Is every Riordan array a VHRA of some other Riordan arrays? The answer is yes, and the existence is unique. More precisely, we have the following result.

Theorem 2.1 *Let (g, f) be a Riordan array then it is the VHRA of the unique Riordan array*

$$(\frac{t\overline{f}'(t)g(\overline{f})}{\overline{f}}, \frac{t^2}{\overline{f}}) = (\frac{t\overline{f}'(t)}{\overline{f}}, \overline{f})(g, \frac{f^2}{t}). \tag{2.1}$$

Particularly, we have

(1) A Hitting-time type Riordan array $(tf'/f, f)$ is the VHRA of the Lagrange type Riordan array $(1, t^2/\overline{f})$;

(2) A Appell type Riordan array (g, t) is the VHRA of itself, i.e., (g, t);

(3) A derivative type Riordan array (f', f) is the VHRA of the Bell type Riordan array $(t/\overline{f}, t^2/\overline{f})$;

(4) A Lagrange type Riordan array $(1,f)$ is the VHRA of the product of the Hitting-time type Riordan array $(t\overline{f}'(t)/\overline{f},\overline{f})$ and another Lagrange type Riordan array $(1,f^2/t)$, i.e.,

$$(\frac{t\overline{f}'(t)}{\overline{f}},\overline{f})(1,\frac{f^2}{t})=(\frac{t\overline{f}'(t)}{\overline{f}},\frac{t^2}{\overline{f}}); \tag{2.2}$$

(5) a Bell type Riordan array $(f/t,f)$ is the VHRA of the product of the Hitting-time type Riordan array $(t\overline{f}'(t)/\overline{f},\overline{f})$ and another Bell type Riordan array $(f/t,f^2/t)$, i.e.,

$$(\frac{t\overline{f}'(t)}{\overline{f}},\overline{f})(\frac{f}{t},\frac{f^2}{t})=(\frac{t^2\overline{f}'(t)}{\overline{f}^2},\frac{t^2}{\overline{f}}). \tag{2.3}$$

Proof Let (g,f) be a Riordan array. We assume that it is the VHRA of the Riordan array (h,v) to check the existence of (h,v) by using the following constructive process.

Due to the definition of a VHRA, we set $f(t)=\overline{t^2/v(t)}$ from which we may solve $v(t)=t^2/\overline{f}$. Based on the definition of a VHRA we also set

$$g(t)=\frac{tf'(t)h(f)}{f},$$

from which we may solve

$$h(t)=\frac{tg(\overline{f})}{\overline{f}f'(\overline{f})}=\frac{t\overline{f}'(t)g(\overline{f})}{\overline{f}},$$

where we use $\overline{f}'(t)=1/f'(\overline{f})$ in the last step. Hence, (h,v) exists and has expression

$$(h,v)=(\frac{t\overline{f}'(t)g(\overline{f})}{\overline{f}},\frac{t^2}{\overline{f}})=(\frac{t\overline{f}'(t)}{\overline{f}},\overline{f})(g,\frac{f^2}{t}).$$

We now consider some special cases.

(1) If $(g,f)=(tf'/f,f)$, then from (2.1) we obtain

$$(h,v)=(\frac{t\overline{f}'(t)}{\overline{f}}\frac{\overline{f}f'(\overline{f})}{t},\frac{t^2}{\overline{f}})=(1,\frac{t^2}{\overline{f}}),$$

where we use the fact $\overline{f}'(t)f'(\overline{f})=1$ in the last step.

(2) If $(g,f)=(g,t)$, then $\overline{f}=t$ and (2.1) yields the desired result.

(3) If $(g,f)=(f',f)$, then (2.1) suggests Inverse of VHRA

$$(h,v)=(\frac{t\overline{f}'(t)f'(\overline{f})}{\overline{f}},\frac{t^2}{\overline{f}})=(\frac{t}{\overline{f}},\frac{t^2}{\overline{f}}).$$

(4) If $(g,f)=(1,f)$, then from (2.1) we have

$$(h,v)=(\frac{t\overline{f}'(t)}{\overline{f}},\overline{f})(1,\frac{f^2}{t})=(\frac{t\overline{f}'(t)}{\overline{f}},\frac{t^2}{\overline{f}}).$$

(5) If $(g,f)=(f/t,f)$, then (h,v) can be expressed as (2.3). □

Theorem 2.1 shows a Riordan array (g,f) is the VHRA of the unique Riordan array

$$E=(e_{n,k})_{n,k\in\mathbb{N}}=(\frac{t\overline{f}'(t)g(\overline{f})}{\overline{f}},\frac{t^2}{\overline{f}})=:(d,h), \tag{2.4}$$

where $d = \frac{t\overline{f}'(t)g(\overline{f})}{\overline{f}}$ and $h = \frac{t^2}{\overline{f}}$. Based on the vertical recurrence relation of column entries of the Riordan array (g, f) defined by (2.1), we now present the recurrence relation of the column entries of E.

Theorem 2.2 *Let $(g, f) = (d_{n,k})_{n,k\in\mathbb{N}}$ be a Riordan array with A-sequence $A = (a_0, a_1, \ldots)$. Then, (g, f) is the VHRA of E shown in (2.4), which has the recurrence relation for its column entries as*

$$e_{n,k} = [t^n]dh^k = \sum_{j=1}^{n} a_{j-1}[t^{n-j}]dh^{k-1} = \sum_{j=1}^{n-k+1} a_{j-1}e_{n-j,k-1}. \tag{2.5}$$

In short, the horizontal recurrence relation for the row entries of (g, f) is the vertical recurrence relation for the column entries of $E = (d, h)$, where E is defined by (2.4).

Proof Let $A(t)$ be the generating function of the A-sequence of the Riordan array (g, f). From the second equation of (1.5), we have

$$A(t) = \frac{t}{\overline{f}}.$$

By (2.4),

$$h = \frac{t^2}{\overline{f}} = tA(t),$$

which implies

$$h_j = [t^j]h(t) = [t^{j-1}]A(t) = a_{j-1}.$$

Thus, by using (1.18) we obtain (2.5). □

Corollary 2.3 *Let $E = (e_{n,k})_{n,k\geq 0}$ be the Riordan array defined by (2.4), and let $(g, f) = (d_{n,k})_{n,k\in\mathbb{N}}$ be the VHRA of E with A-sequence $A = (a_0, a_1, \ldots)$. Then we have the following vertical recurrence relations of the column entries of E in terms of A-sequence of (g, f).*

(1) Appell type Riordan array $(g(t), t) = (d_{n,k})_{n,k\geq 0}$ is the VHRA of $E = (g(t), t) = (e_{n,k})_{n,k\geq 0}$, where the vertical recurrence relation for the column entries of E is

$$e_{n,k} = e_{n-1,k-1}. \tag{2.6}$$

(2) Bell type Riordan array $(f(t)/t, f(t))$ is the VHRA of $E = (\frac{t^2\overline{f}'(t)}{\overline{f}^2}, \frac{t^2}{\overline{f}}) = (e_{n,k})_{n,k\geq 0}$, where the vertical recurrence relation for the column entries of E is

$$e_{n,k} = \sum_{j=1}^{n-k+1} a_{j-1}e_{n-j,k-1}, \; n, k \geq 1, \text{ and } e_{n,0} = [t^n]\frac{t^2\overline{f}'(t)}{\overline{f}^2}, \; n \geq 0. \tag{2.7}$$

(3) Hitting-time type Riordan array $(tf'(t)/f(t), f(t))$ is the VHRA of $E = (1, t^2/\overline{f}) = (e_{n,k})_{n,k\geq 0}$, where the vertical recurrence relation for the column entries of E is

$$e_{n,k} = \sum_{j=1}^{n-k+1} a_{j-1}e_{n-j,k-1}, \; n, k \geq 1, \text{ and } e_{n,0} = 1, \; n \geq 0. \tag{2.8}$$

(4) Derivative type Riordan array $(f'(t), f(t))$ *is the VHRA of* $E = (t/\overline{f}, t^2/\overline{f}) = (A(t), tA(t)) = (e_{n,k})_{n,k\geq 0}$*, where the vertical recurrence relation for the column entries of* E *is*

$$e_{n,k} = \sum_{j=1}^{n-k+1} a_{j-1}e_{n-j,k-1}, \; n,k \geq 1, \text{ and } e_{n,0} = a_n, \; n \geq 0. \tag{2.9}$$

The proof is straightforward and is omitted.

As an example of Corollary 2.3, from (2.7) or (2.8), the 1-Bell type Riordan array and the hitting-time type Riordan array

$$(g, f) = (\frac{1}{1-t}, \frac{t}{1-t}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 2 & 1 & 0 & 0 & 0 & \\ 1 & 3 & 3 & 1 & 0 & 0 & \\ 1 & 4 & 6 & 4 & 1 & 0 & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix}$$

is the VHRA of

$$(1, t(1+t)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 2 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 3 & 1 & 0 & \\ 0 & 0 & 0 & 3 & 4 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix}.$$

From (2.9), the derivative type Riordan array $(g, f) = (1/(1-t)^2, t/(1-t)$ is the VHRA of

$$(\frac{t}{\overline{f}}, \frac{t^2}{\overline{f}}) = (1+t, t(1+t)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 2 & 1 & 0 & 0 & 0 & \\ 0 & 1 & 3 & 1 & 0 & 0 & \\ 0 & 0 & 3 & 4 & 1 & 0 & \\ 0 & 0 & 1 & 6 & 5 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix}.$$

3. Inverse of VHRA

We may consider the vertical half Riordan array as the result of applying an operator, called the half Riordan array operator (Φ), which is defined by

$$\Phi : (g, f) \to (\frac{z\phi'}{\phi}, \phi)(g, z), \tag{3.1}$$

where $\phi(z) = \overline{z^2/f(z)}$. The inverse operator applied on a Riordan array (g, f) is defined by

$$\Psi : (g, f) \to \left(\frac{1}{g(\bar{f})}, \bar{f}\right). \tag{3.2}$$

Then being motivated by

$$\phi(t) = \overline{\frac{t^2}{f(t)}} \Longleftrightarrow \overline{f}(t) = \overline{\frac{t^2}{\overline{\phi}(t)}},$$

we may establish the following mapping chain.

Theorem 3.1 *Let (g, f) be a Riordan array, and let $\phi = \overline{t^2/f(t)}$. Then the inverse of its VHRA is*

$$\left(\frac{t}{\overline{\phi}\phi'(\overline{\phi})g}, \overline{\phi}\right) = \left(\frac{t\overline{\phi}'(t)}{\overline{\phi}}, \overline{\phi}\right)\left(\frac{1}{g(\phi)}, 1\right),$$

where VHRA is

$$\begin{aligned}\left(\frac{t\overline{f}'(t)}{\overline{\phi}(\overline{f})\phi'(\overline{\phi}(\overline{f}))g(\overline{f})}, \overline{f}\right) =& \left(\frac{t\overline{f}'(t)\overline{\phi}'(\overline{f})}{\overline{\phi}(\overline{f})}, \overline{f}\right)\left(\frac{1}{g}, 1\right)\\ =& \left(\frac{t(d/dt)\overline{\phi}(\overline{f})}{\overline{\phi}(\overline{f})}, \overline{f}\right)\left(\frac{1}{g}, 1\right).\end{aligned}$$

Consequently, there exists the following mapping chain, where $\widehat{\Psi}$ is referred to as VHRA-Ψ(Inverse)-VHRA mapping.

$$\begin{array}{ccc} (g, f) & -\ \Phi\ \to & \left(\frac{t\phi'(t)g(\phi)}{\phi}, \phi\right) = \Phi(g, f) \\ | & & | \\ \widehat{\Psi} & & \Psi \\ \downarrow & & \downarrow \\ \left(\frac{t\overline{f}'(t)}{\overline{f}}, \overline{f}\right)\left(\frac{t\overline{\phi}'(t)}{\overline{\phi}}, \overline{\phi}\right)\times & & \\ (1, \phi)\left(\frac{1}{g}, t\right) & & \left(\frac{t}{\overline{\phi}\phi'(\overline{\phi})g}, \overline{\phi}\right) \\ = \left(\frac{t(d/dt)(\overline{\phi}(\overline{f}))}{\overline{\phi}(\overline{f})g(\overline{f})}, \overline{f}\right) & & \\ = \left(\frac{\overline{f}\,\overline{f}'(t)(2\overline{f} - f'(\overline{f})\overline{\phi}(\overline{f}))}{\overline{f}\phi(\overline{f})g(\overline{f})}, \overline{f}\right) & \leftarrow\ \Phi\ - & = \left(\frac{t(2t - f'(t)\overline{\phi})}{f\overline{\phi}g}, \overline{\phi}\right), \end{array} \tag{3.3}$$

Figure 1 Mapping chain for (g, f)

where

$$\left(\frac{t\overline{f}'(t)}{\overline{f}}, \overline{f}\right)\left(\frac{t\overline{\phi}'(t)}{\overline{\phi}}, \overline{\phi}\right)(1, \phi)\left(\frac{1}{g}, t\right)$$

is the multiplication of two hitting-time type Riordan arrays, a Lagrange type Riordan array, and a Appell type Riordan array.

Proof Since $\phi = \overline{t^2/f(t)}$ or $\overline{\phi} = t^2/f(t)$, we have

$$\overline{f} = \overline{t^2/\overline{\phi}}.$$

Noting the above relationship between $\overline{\phi}$ and $\overline{f}$, the proof of the theorem follows from (3.1), (3.2), and the formula

$$\phi'(t) = \frac{f(\phi)}{2\phi - tf'(\phi)}. \tag{3.4}$$

Here (3.4) comes from $\phi^2 = tf(\phi)$ by taking a derivative on its both sides. Thus,

$$2\phi\phi' = f(\phi) + tf'(\phi)\phi',$$

which implies (3.4). □

Considering $(g, f) = (g, t/(1-t))$, from Theorem 3.1, we obtain the mapping chain shown in Figure 2. Particularly, for $g = 1/(1-t)$, we have the result shown in Figure 3.

$$\begin{array}{ccc} (g, \frac{t}{1-t}) & -- \quad \Phi \quad \longrightarrow & (\frac{g(tC)}{1-2tC(t)}, tC(t)) \\ | & & | \\ \widehat{\Psi} & & \Psi \\ \downarrow & & \downarrow \\ (\frac{1-t}{(1+t)g(t/(1+t))}, \frac{t}{1+t}) & \longleftarrow \quad \Phi \quad -- & (\frac{1-2t}{(1-t)g}, t(1-t)). \end{array} \tag{3.5}$$

Figure 2 Mapping chain for $(g, t/(1-t))$

$$\begin{array}{ccc} (\frac{1}{1-t}, \frac{t}{1-t}) & -- \quad \Phi \quad \longrightarrow & (\frac{1}{1-2tC(t)}, tC(t)) \\ | & & | \\ \widehat{\Psi} & & \Psi \\ \downarrow & & \downarrow \\ (\frac{1-t}{(1+t)^2}, \frac{t}{1+t}) & \longleftarrow \quad \Phi \quad -- & (1-2t, t(1-t)). \end{array} \tag{3.6}$$

Figure 3 Mapping chain for $(1/(1-t), t/(1-t))$

The first few entries of three Riordan arrays shown in (3.6) are presented below.

$$(\frac{1}{1-2tC(t)}, tC(t)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \\ 6 & 3 & 1 & 0 & 0 & 0 & \\ 20 & 10 & 4 & 1 & 0 & 0 & \\ 70 & 35 & 15 & 5 & 1 & 0 & \\ 252 & 126 & 56 & 21 & 6 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix}.$$

$$(1-2t, t(1-t)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 1 & 0 & 0 & 0 & 0 & \\ 0 & -3 & 1 & 0 & 0 & 0 & \\ 0 & 2 & -4 & 1 & 0 & 0 & \\ 0 & 0 & 5 & -5 & 1 & 0 & \\ 0 & 0 & -2 & 9 & -6 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix}.$$

$$(\frac{1-t}{(1+t)^2}, \frac{t}{1+t}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -3 & 1 & 0 & 0 & 0 & 0 & \\ 5 & -4 & 1 & 0 & 0 & 0 & \\ -7 & 9 & -5 & 1 & 0 & 0 & \\ 9 & -16 & 14 & -6 & 1 & 0 & \\ -11 & 25 & -30 & 20 & -7 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix}.$$

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