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The Injective Chromatic Index of a Claw-Free Subcubic Graph is at Most 6

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Abstract A coloring of edges of a graph G is injective if for any two distinct edges e_1 and e_2, the coloring of e_1 and e_2 are distinct if they are at distance 2 in G or in a common 3-cycle. The injective chromatic index of G is the minimum number of colors needed for an injective edge coloring of G. It was conjectured that the injective chromatic index of any subcubic graph is at most 6. In this paper, we partially confirm this conjecture by showing that the injective chromatic index of any claw-free subcubic graph is less than or equal to 6. The bound 6 is tight and our proof implies a linear-time algorithm for finding an injective edge coloring using at most 6 colors for such graphs.

Keywords injective edge coloring; injective chromatic index; claw-free; subcubic graph

MR(2020) Subject Classification 05C15

1. Introduction

In this paper, we only consider finite simple graphs. Let $G = (V(G), E(G))$ be a graph. For a vertex v of G, denote by $N(v)$ the set of vertices that are adjacent to v. For a subset S of $V(G)$, we denote by $G[S]$ the subgraph of G that is induced by S. Denote by $\Delta(G)$ the maximum degree of G. A subcubic graph is a graph with maximum degree at most 3. We denote by $K_{1,3}$ the complete bipartite graph with one part having 1 vertex and the other 3 vertices. A graph G is called claw-free if no induced subgraph of G is isomorphic to $K_{1,3}$. The study on the properties of claw-free graphs was initiated by Beineke in his study of the properties of line graphs [1]. However, from the late 1970s more scholars paid attention to the matching properties as well as the Hamiltonian properties of claw-free graphs [2–4]. In this paper, we focus on the injective chromatic index of claw-free subcubic graphs.

Let e_1 and e_2 be any two edges of G. The *distance* between e_1 and e_2, $d(e_1, e_2)$, is defined as the distance between the corresponding two vertices in the line graph of G. It is clear that if e_1 and e_2 are at distance 1, then they share at least one end-vertex, and if e_1 and e_2 are at distance 2, then they share no end-vertex and there exists another edge e' such that e' is at distance 1 from e_1 and e_2, respectively. Given a positive integer k, an injective k-edge coloring of G is a

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mapping f from $E(G)$ to $\{1, 2, \ldots, k\}$ such that, $f(e) \neq f(e')$ for any two distinct edges e and e' of G if $d(e, e') = 2$ or they are in a common 3-cycle. The injective chromatic index, denoted by $\chi'_{inj}(G)$, is the minimum integer k such that G has an injective k-edge coloring.

The injective edge coloring is a kind of edge version of the injective coloring. Given a positive integer k, an injective k-edge coloring of G is a mapping f from $V(G)$ to $\{1, 2, \ldots, k\}$ such that any two vertices at distance two apart or any two vertices on a common 3-cycle receive different colors. The injective chromatic number, denoted by $\chi_{inj}(G)$, is the minimum integer k such that G has a k-injective coloring. The injective coloring of graphs was originated from the complexity theory on random access machines, which was proposed by Hahn et al. [5]. It was also applied to the theory of error correcting codes and the designing of computer networks [6]. In [5], the authors proved that for any graph G, $\Delta(G) \leq \chi_{inj}(G) \leq \Delta(G)(\Delta(G) - 1) + 1$, and both bounds are sharp.

In 2015, Cardoso et al. [7] introduced the concept of injective edge coloring of graphs. They proved that it is NP-hard to compute the injective chromatic index of a graph G and determined the injective chromatic indices of paths, cycles, complete bipartite graphs, the Peterson graph. Ferdjallah, Kerdjoudj and Raspaud [8] proved that $\chi'_{inj}(G) \leq 2(\Delta(G) - 1)^2$ for any graph G with $\Delta(G) \geq 3$ and pointed out that $\chi'_{inj}(G) \leq 30$ for any planar graph G while $\chi'_{inj}(G) \leq 9$ for any outerplanar graph G. In particular, they [8] conjectured that $\chi'_{inj}(G) \leq 6$ for any subcubic graph G and confirmed it for subcubic bipartite graph. The conjecture was also proved to be true for subcubic planar graphs [9]. And, in general, the same paper [9] proved that $\chi'_{inj}(G) \leq 7$ for any subcubic graph G. For other results concerning the injective chromatic index of graphs, please refer to [10–12].

In this paper, we partially confirm the above conjecture for claw-free subcubic graphs. Our main result is the following theorem.

Theorem 1.1 *Let G be a claw-free subcubic. Then $\chi'_{inj}(G) \leq 6$.*

Actually, there are some claw-free subcubic graphs whose injective chromatic indices are 6. For example, the complete graph on 4 vertices K_4 has injective chromatic index 6. Another example is the triangular prism (also called 3-prism). On the one hand, the injective chromatic index of the 3-prism is at least 6 since there are two disjoint 3-cycles in the 3-prism, and any two edges from different 3-cycles are at distance 2. On the other hand, we can easily obtain an injective 6-edge-coloring of the 3-prism. Please see Figure 1 for injective 6-edge-colorings of the 3-prism and K_4.

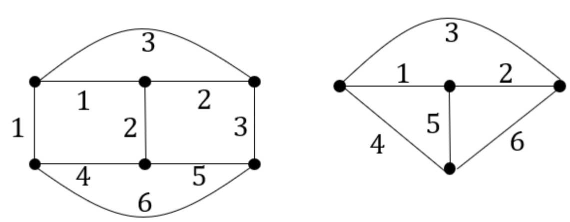

Figure 1 The 3-prism and K_4

In Section 2, we present a greedy algorithm that can produce a partial injective 6-edge-coloring of a claw-free subcubic graph with only a few edges left uncolored. In Section 3, in several different cases, we first apply this algorithm to produce a partial injective 6-edge-coloring of the graph, and then extend it to an injective 6-edge-coloring of the whole graph.

2. Preliminaries

In order to construct a partial injective 6-edge-coloring of a claw-free subcubic graph, we first apply the same method in [13] to order the edges of G, and then use the greedy algorithm to color the edges of G in this order. There will be only a few particular edges left uncolored.

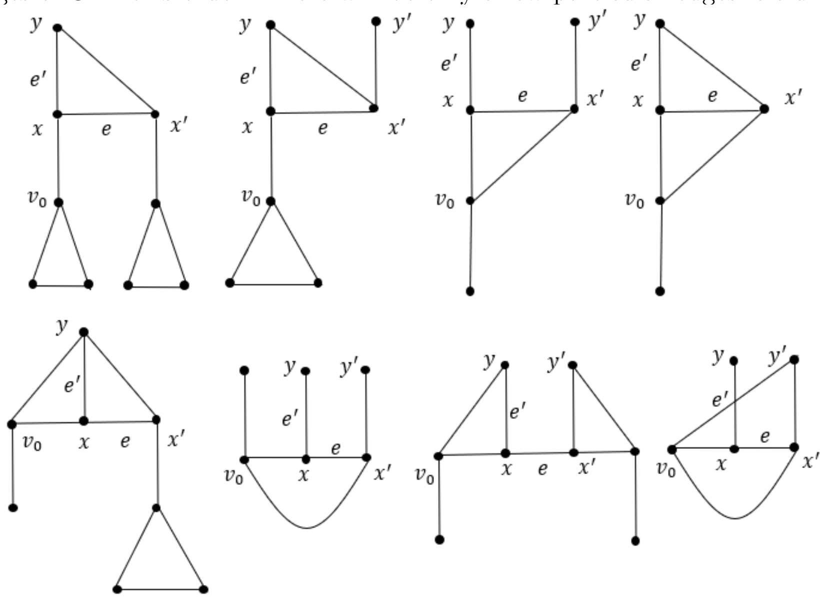

Figure 2 The eight situations of an edge $e = xx'$ with $d_S(x') = d_S(x)$

Suppose S is a subset of $V(G)$. For a vertex $v \in V(G)$, the distance from v to S, denoted by $d_S(v)$, is defined as $\min_{w\in S}\{d(v,w)\}$. Suppose the maximum distance from a vertex of $V(G)$ to S is K. For $i = 0, 1, \ldots, K$, let $D_i = \{v \in V(G) | d_S(v) = i\}$. We define a mapping d_S from $E(G)$ to $\{0, 1, \ldots, K\}$ as: $d_S(e) = \min\{i | e \cap D_i \neq \emptyset,\ 0 \leq i \leq K\}$ for any edge $e \in E(G)$. Note that if $d_S(e) > 0$ then there exists an edge e' sharing one end-vertex with e such that $d_S(e') = d_S(e) - 1$. Let $R = (e_{k_1}, e_{k_2}, \ldots, e_{k_m})$ be an ordering of the edges of G. If, for any two integers i and j in $\{1, 2, \ldots, m\}$, $i < j$ implies $d_S(e_{k_i}) \geq d_S(e_{k_j})$, then we say that the edge ordering R of G is compatible with the mapping d_S.

Let G be a claw-free subcubic graph. Suppose we already have a partial injective 6-edge-coloring ϕ of G. Let b be a color in $\{1, 2, \ldots, 6\}$ and e an uncolored edge. If $\phi(e') \neq b$ for any colored edge e' that is distance two apart from e or lies on a common 3-cycle with e, we say that the color b is available for e. We use $A(e)$ to denote the set of colors available for e. And denote by $F(e)$ the set of colors unavailable for e. It is obvious that $F(e) = \{1, 2, \ldots, 6\} \setminus A(e)$.

Lemma 2.1 *Let G be a claw-free subcubic graph. Let S be any subset of $V(G)$. The greedy algorithm coloring the edges of G in an ordering R compatible with the mapping d_S will produce a partial injective 6-edge-coloring of G where only edges e with $d_S(e) = 0$ are left uncolored.*

Proof Let $e = xx'$ be an edge with $d_S(e) > 0$. Without loss of generality, let $e' = xy$ be a neighbor of e with $d_S(e') < d_S(e)$. Then, at the stage of the greedy algorithm when e is to be colored, no edges incident with y have yet been colored.

If $d(x) = 2$ or x has another neighbor z with $d_S(z) < d_S(x)$ then it is easy to see that $|F(e)| \leq 3$ and we are done. Thus we assume $d(x) = 3$ and y is the only neighbor of x with $d_S(y) < d_S(x)$. Let another neighbor of x be v_0. There will be two cases:

Case 1. $d_S(x) = d_S(x') \geq 1$. In this case, x' has a neighbor y' with $d_S(y') = d_S(y)$ (It is possible that $y = y'$). Notice that G is claw-free and $\Delta(G) = 3$, the possible situations of this case are described in Figure 2. It is now straightforward to check that $|F(e)| \leq 4$. And so e can be colored properly.

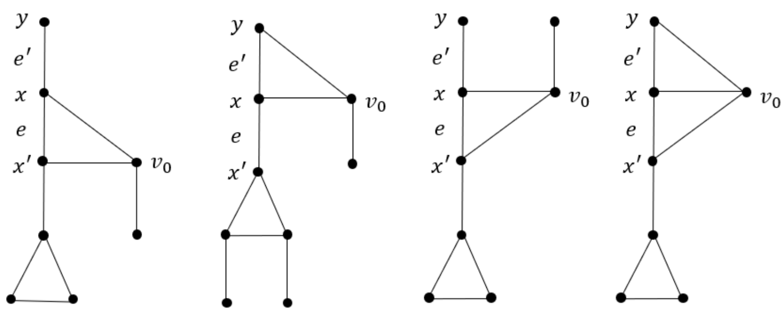

Figure 3 The four situations of an edge $e = xx'$ with $d_S(x') = d_S(x) + 1$

Case 2. $d_S(x') = d_S(x)+1 \geq 2$. Then $d_S(v_0) = d_S(x)$ or $d_S(v_0) = d_S(x)+1$. And all possible situations of this case are described in Figure 3. It is easy to check that in all these situations $|F(e)| \leq 5$ and so the edge e can be colored properly. □

3. The proof of Theorem 1.1

The proof of Theorem 1.1 consists of a series of lemmas. In this section, we only focus on connected claw-free subcubic graphs.

Lemma 3.1 *If G has a vertex of degree 1, then $\chi'_{inj}(G) \leq 6$.*

Proof Let v_0 be a vertex of degree 1 and let e_0 be the edge incident with v_0. Put $S = \{v_0\}$. Then, by Lemma 2.1, all edges except e_0 can be colored properly. Since there are at most three edges that are distance two apart from e_0, it can also be colored properly. □

Lemma 3.2 *If G has a vertex of degree 2, then $\chi'_{inj}(G) \leq 6$.*

Proof Let v_0 be a vertex of degree 2, and v_1 and v_2 the two neighbors of v_0. Let $e_1 = v_0v_1$ and $e_2 = v_0v_2$. Put $S = \{v_0\}$. By Lemma 2.1, G has a partial injective 6-edge-coloring with only e_1

and e_2 left uncolored. If $v_1v_2 \in E(G)$, then it is clear that $|F(e_1)| \leq 4$ and $|F(e_2)| \leq 4$, implying that e_1 and e_2 can also be colored properly. If $v_1v_2 \notin E(G)$, then it is straightforward to check that $|F(e_1)| \leq 5$ and $|F(e_2)| \leq 5$. Since $d(e_1, e_2) = 1$, they can also be colored properly. □

From now on, we assume that G is a connected claw-free cubic graph.

Lemma 3.3 *Let G be a connected claw-free cubic graph. If G has a cut vertex, then $\chi'_{inj}(G) \leq 6$.*

Proof Let v_0 be a cut vertex of G. And let u_0, v_1 and v_2 be the three neighbors of v_0. Since G is claw-free and v_0 is a cut vertex, $G[\{u_0, v_1, v_2\}]$ contains exactly one edge. Without loss of generality, assume $v_1v_2 \in E(G)$. Please see Figure 4 for the names of vertices and edges of G.

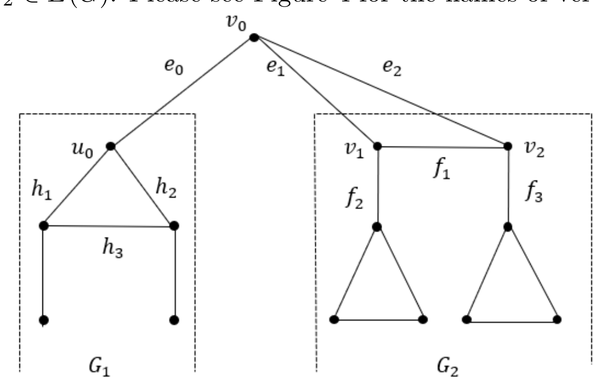

Figure 4 A claw-free cubic graph with a cut vertex

Let G_1 and G_2 be the two components of $G - v_0$. Since both G_1 and G_2 have a vertex of degree less than 3, by Lemmas 3.1 and 3.2, the union of G_1 and G_2 has a injective 6-edge-coloring, denoted by φ, which is also a partial injective 6-edge-coloring of G.

Notice that h_1, h_2, h_3 receive three different colors and f_2, f_3 receive two different colors. W.l.o.g., we may assume that $\varphi(f_1) \neq \varphi(f_2)$. We can permute the colors among the edges of G_1 so that $\{\varphi(f_1), \varphi(f_2), \varphi(f_3)\} \subseteq \{\varphi(h_1), \varphi(h_2), \varphi(h_3)\}$, $\varphi(h_1) = \varphi(f_1)$ and $\varphi(h_2) = \varphi(f_2)$. Now it is straightforward to check that $|A(e_1)| \geq 1$, $|A(e_2)| \geq 2$, and $|A(e_0)| \geq 1$. So the remaining three edges e_1, e_2, e_0 in this order can be greedily colored without introducing new colors. □

As G is a claw-free cubic graph, each vertex must be contained in a 3-cycle. Choose a 3-cycle $C = v_0v_1v_2v_0$. Let u_0, u_1, u_2 be the third neighbors of v_0, v_1, v_2, respectively. Let $e_1 = v_0v_1$, $e_2 = v_1v_2$, $e_3 = v_0v_2$, $e_4 = v_0u_0$, $e_5 = v_1u_1$, $e_6 = v_2u_2$. We shall frequently use these names of vertices and edges in the following lemma.

Lemma 3.4 *If G is a 2-connected claw-free cubic graph, then $\chi'_{inj}(G) \leq 6$.*

Proof Set $S = V(C)$. Let φ be the partial injective 6-edge-coloring of G produced by the greedy algorithm described in Lemma 2.1. Then the only uncolored edges are $e_1, e_2, \ldots, e_6$.

Note that if $u_0 = u_1 = u_2$ then G is isomorphic to K_4 and we can easily get an injective edge coloring of G using exactly 6 colors. Thus we assume that $|\{u_0, u_1, u_2\}| \geq 2$. If $|\{u_0, u_1, u_2\}| = 2$, w.l.o.g., we assume $u_0 = u_1 \neq u_2$. Please see Figure 5 for the names of vertices and edges of G. Since $|A(e_3)| \geq 3$ and $|A(e_4)| \geq 4$, $A(e_3) \cap A(e_4) \neq \emptyset$. Let $b \in A(e_3) \cap A(e_4)$. Notice that

$d(e_3, e_4) = 1$, we can color e_3 and e_4 with the same color b. At the moment after e_3 and e_4 have been colored, we have $|A(e_2)|, |A(e_6)| \geq 2$, $|A(e_5)| \geq 3$ and $|A(e_1)| \geq 4$. By greedily coloring the uncolored edges in the order e_2, e_6, e_5, e_1, we extend the partial injective edge coloring φ to an injective 6-edge-coloring of G.

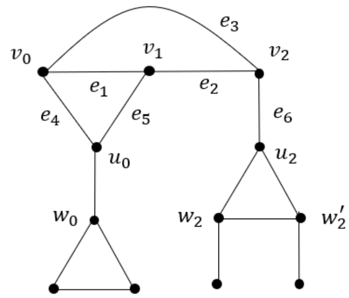

Figure 5 The case when $|\{u_0, u_1, u_2\}| = 2$

We now suppose that $|\{u_0, u_1, u_2\}| = 3$. If the induced subgraph $G[\{u_0, u_1, u_2\}]$ contains three edges, then G is isomorphic to the 3-prism and we can obtain an injective edge coloring of G using 6 colors. If the induced subgraph $G[\{u_0, u_1, u_2\}]$ contains two edges. Without loss of generality, let $u_0u_1 \in E(G)$ and $u_1u_2 \in E(G)$. Then u_1 and its three neighbors induce a $K_{1,3}$ in G. Thus this case will not happen.

We first deal with the case that $G[\{u_0, u_1, u_2\}]$ contains exactly one edge. Without loss of generality, let $u_0u_1 \in E(G)$. As G is a claw-free cubic graph, u_0 and u_1 must have a common neighbor, say w_0 (see Figure 6). In this case, it is easy to check that $|A(e_i)| \geq 2$ for $i = 2, 3$, $|A(e_1)| \geq 3$, $|A(e_6)| \geq 3$ and $|A(e_j)| \geq 4$ for $j = 4, 5$. By greedily coloring the remaining edges in the order $e_2, e_3, e_1, e_6, e_4, e_5$, one extends the partial injective edge coloring φ to an injective 6-edge-coloring of G.

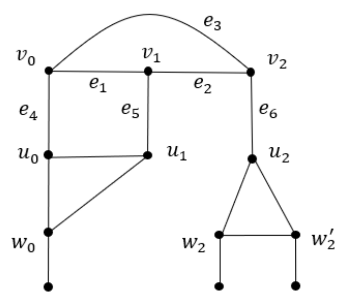

Figure 6 The case when $|\{u_0, u_1, u_2\}| = 3$ and $e(G[\{u_0, u_1, u_2\}]) = 1$

We next deal with the case that $G[\{u_0, u_1, u_2\}]$ is empty. Let $N(u_i) = \{v_i, w_i, w_i'\}$ for $i = 0, 1, 2$. Since G is a claw-free cubic graph, $w_iw_i' \in E(G)$ for $i = 0, 1, 2$. And if $N(u_0) \cap N(u_1) \neq \emptyset$ then v_2 is a cut vertex of G, which contradicts the assumption that G is 2-connected. Thus we assume that $N(u_0) \cap N(u_1) = \emptyset$. By symmetry, we can also assume that $N(u_0) \cap N(u_2) = N(u_1) \cap N(u_2) = \emptyset$. Please see Figure 7 for the structure of G. In this case, we have $|A(e_i)| \geq 2$ for $i = 1, 2, 3$ and $|A(e_j)| \geq 3$ for $j = 4, 5, 6$.

Suppose there are two integers i and j in $\{1, 2, 3\}$ such that $|A(e_i) \cup A(e_j)| = 2$. Without

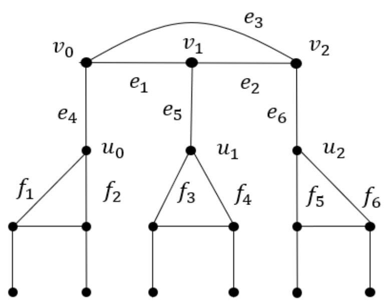

Figure 7 One drawing of G with three vertices of degree 3 in S

loss of generality, assume that $|A(e_1) \cup A(e_2)| = 2$. Then $A(e_1) = A(e_2)$, $|A(e_1)| = |A(e_2)| = 2$ and $|A(e_3)| = 4$. As $|A(e_4)| \geq 3$, there is a color $b \in A(e_4)\backslash A(e_2)$. After coloring the edge e_4 with the color b, it is clear that that $|A(e_i)| = 2$ for $i = 1, 2$ and $|A(e_j)| \geq 2$ for $j = 5, 6$. Notice that $d(e_2, e_5) = 1$, $d(e_2, e_6) = 1$ and $b \notin A(e_2)$, by greedily coloring the remaining five edges e_5, e_6, e_1, e_2, e_3 in this order, we can extend the partial injective edge coloring φ to an injective 6-edge-coloring of G. Hence, we assume that $|A(e_i) \cup A(e_j)| \geq 3$ for any two integers $i, j \in \{1, 2, 3\}$. We next prove a claim which is essential to the remaining proofs.

Claim A. If there is some $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$ such that e_i and e_j are adjacent and $A(e_i) \cap A(e_j) \neq \emptyset$, then the partial injective edge coloring φ can be extended to an injective 6-edge-coloring of G.

Proof Without loss of generality, assume $A(e_3) \cap A(e_6) \neq \emptyset$. Let $b \in A(e_3) \cap A(e_6)$. After coloring the two edges e_3 and e_6 with the same color b, it is clear that that $|A(e_i)| \geq 1$ for $i = 1, 2$, $|A(e_j)| \geq 2$ for $j = 4, 5$. Moreover, we have $|A(e_1) \cup A(e_2)| \geq 2$ because $|A(e_1) \cup A(e_2)| \geq 3$ before we color the two edges e_3 and e_6. Now, we can greedily color the remaining four edges e_1, e_2, e_4, e_5 in this order to obtain an injective 6-edge-coloring of G.

There are four cases to be considered.

Case 1. There is some integer i in $\{1, 2, 3\}$ such that $|A(e_i)| = 2$. W.l.o.g., assume $|A(e_1)| = 2$. Then $|\{\varphi(f_1), \varphi(f_2), \varphi(f_3), \varphi(f_4)\}| = 4$ and so $F(e_2) \cap F(e_3) = \{\varphi(f_5), \varphi(f_6)\}$. Since $A(e_2) \cup A(e_3) = \overline{F(e_2) \cap F(e_3)}$, we have $|A(e_2) \cup A(e_3)| = 4$. As $|A(e_6)| \geq 3$, either $A(e_6) \cap A(e_2) \neq \emptyset$ or $A(e_6) \cap A(e_3) \neq \emptyset$. By Claim A, we are done.

Case 2. $|A(e_i)| \geq 3$ for $i \in \{1, 2, 3\}$ and there is some integer i in $\{1, 2, 3\}$ such that $|A(e_i)| = 4$. Without loss of generality, assume $|A(e_1)| = 4$. As $|A(e_4)| \geq 3$, it is clear that $A(e_1) \cap A(e_4) \neq \emptyset$. And we are done by Claim A.

Case 3. $|A(e_i)| = 3$ for $i \in \{1, 2, 3\}$ and there are two integers i and j in $\{1, 2, 3\}$ such that $A(e_i) = A(e_j)$. In this case, it is obvious that $|F(e_1)| = |F(e_2)| = |F(e_3)| = 3$. Without loss of generality, we assume $A(e_1) = A(e_2)$. Then $F(e_1) = F(e_2)$. Since $F(e_3) \subseteq F(e_1) \cup F(e_2)$, $F(e_3) \subseteq F(e_1)$. It follows that $F(e_1) = F(e_2) = F(e_3)$ and so $A(e_1) = A(e_2) = A(e_3)$. According to Claim A, we may assume that $A(e_i) \cap A(e_j) = \emptyset$ for any $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$. And it is easy to see that the partial injective edge coloring φ can be extended to an injective 6-edge-coloring of G.

Case 4. $|A(e_i)| = 3$ for $i \in \{1,2,3\}$ and $A(e_i) \neq A(e_j)$ for any two integers i and j in $\{1,2,3\}$. In this case, it is clear that $|A(e_2) \cup A(e_3)| \geq 4$, which implies that either $A(e_6) \cap A(e_2) \neq \emptyset$ or $A(e_6) \cap A(e_3) \neq \emptyset$. By Claim A, we can extend the partial injective edge coloring φ to an injective 6-edge-coloring of G. $\square$

Theorem 1.1 is proved by the above lemmas. Our proof implies a linear-time algorithm for finding an injective edge coloring using at most 6 colors for any claw-free subcubic graph. Recall that there are graphs with their injective chromatic indices attaining the upper bound 6. We end our paper by asking the following question.

Question Are there infinitely many claw-free subcubic graphs with injective chromatic indices equal to 6? Could we characterize all claw-free subcubic graphs whose injective chromatic indices attain the upper bound 6?

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