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Quasi-Central Semicommutative Rings

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Abstract A ring R is said to be quasi-central semicommutative (simply, a QCS ring) if $ab = 0$ implies $aRb \subseteq Q(R)$ for $a, b \in R$, where $Q(R)$ is the quasi-center of R. It is proved that if R is a QCS ring, then the set of nilpotent elements of R coincides with its Wedderburn radical, and that the upper triangular matrix ring $R = T_n(S)$ for any $n \geq 2$ is a QCS ring if and only if $n = 2$ and S is a duo ring, while $T^k_{2k+2}(R)$ is a QCS ring when R is a reduced duo ring.

Keywords central semicommutative rings; quasi-central semicommutative rings; duo rings

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1. Introduction

Throughout this paper a ring means an associative ring with identity unless otherwise stated. Let R be a ring, $n \geq 2$ an integer, and σ an endomorphism of R. We use $N(R)$, $E(R)$, $Z(R)$, $W(R)$, $N_*(R)$, $N^*(R)$, and $J(R)$ to denote the set of nilpotent elements, the set of idempotents, the center, the Wedderburn radical, the prime radical, the upper nil radical and the Jacobson radical of R, respectively. The symbol $M_n(R)$ ($T_n(R)$) denotes the ring of $n \times n$ matrices (upper triangular matrices) over R, $S_n(R)$ the subring of $T_n(R)$ in which each matrix has the identical principally diagonal elements, E_{ij} the $n \times n$ matrix units, and I_n the $n \times n$ identity matrix. The notation $R[[x;\sigma]]$ ($R[x,\sigma]$) stands for the left skew power series (polynomial) ring over R, and $\mathbb{Z}_n$ for the ring $\mathbb{Z}$ of integers modulo n.

According to Walt [1], an element a of a ring R is said to be left quasi-commutative if for every $r \in R$ there exists $r' \in R$ such that $ra = ar'$. A right quasi-commutative element is defined analogously and a is quasi-commutative if it is left and right quasi-commutative. The set of left quasi-commutative elements, denoted by $Q_l(R)$, is called the left quasi-center of R. The right quasi-center $Q_r(R)$ of R is defined similarly, and $Q(R) = Q_l(R) \bigcap Q_r(R)$ is the quasi-center of R. On the other hand, Feller [2] called a ring R duo if every one-sided ideal of R is an ideal. More precisely, Courter [3] called R left (right) duo if every left (right) ideal of R is an ideal. This is equivalent to saying that $aR \subseteq Ra$ ($Ra \subseteq aR$) for every $a \in R$ (see [3]). Accordingly, a ring R is a left (right) duo ring if and only if $R = Q_r(R)$ ($Q_l(R)$), that is, every element of R is a right (left)-commutative element.

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Let R be a ring and $a,b,c\in R$. A ring R is said to be reduced, abelian, 2-primal if $N(R)=0$, $E(R)\subseteq Z(R)$ and $N_*(R)=N(R)$, respectively. A ring R is symmetric [4] if $abc=0$ implies $acb=0$ (eq., $bac=0$) and reversible [5] if $ab=0$ implies $ba=0$. Due to Bell [6], a ring R is said to satisfy the Insertion-of-Factors-Property (simply, an IFP ring) if $ab=0$ implies $aRb=0$. IFP rings had been studied by other authors under several different names such as SI-rings, ZI-rings, and semicommutative rings [7–9]. In the present paper we choose the term of a semicommutative ring, so as to cohere to the related references. It is known from [10] that reduced ⇒ symmetric ⇒ reversible ⇒ semicommutative, and no reversal holds. For decades, semicommutative rings and various related rings have been studied by numerous authors. A ring R is called central reduced [11], central symmetric [12], central reversible [13], and central semicommutative [14] if $N(R)\subseteq Z(R)$, $abc=0$ implies $bac\in Z(R)$, $ab=0$ implies $ba\in Z(R)$, and $ab=0$ implies $aRb\subseteq Z(R)$, respectively. It can be concluded [15] that central reduced ⇒ central symmetric ⇒ central reversible and central semicommutative, and central reversible or central semicommutative ⇒ abelian and 2-primal. In another direction, a ring R is said to be symmetric-over-center [16] if $abc\in Z(R)$ implies $acb\in Z(R)$. Such a ring is a central symmetric ring by [16, Proposition 2.1] and so is a central semicommutative ring.

In this paper a ring R is said to be quasi-central semicommutative (simply, a QCS ring) if $ab=0$ implies $aRb\subseteq Q(R)$ for $a,b\in R$. Properties of QCS rings and the relationships between such rings and related rings are studied, among others, it is proved that if R is a QCS ring, then RaR is a nilpotent ideal of R for any $a\in N(R)$, so $N(R)=W(R)$ and $J(R[x])=W(R[x])=N(R)[x]$. These generalize some main results on symmetric-over-center rings [17, Theorem 2.2] and improve the existing conclusions on central semicommutative rings. Moreover it is shown that $R=T_n(S)$ is a QCS ring if and only if $n=2$ and S is a duo ring, and that $R=T^k_{2k+2}(S)$ is a QCS ring whenever S is a reduced duo ring.

2. Left (right) quasi-central semicommutative rings

We start this section with the following definition.

Definition 2.1 *A ring R is said to be left (right) quasi-central semicommutative (simply, an LQCS (RQCS) ring) if $ab=0$ implies $aRb\subseteq Q_l(R)$ $(Q_r(R))$ for $a,b\in R$, and a ring is quasi-central semicommutative (simply, a QCS ring) if it is an LQCS ring and an RQCS ring.*

A central semicommutative ring is a QCS ring, but not conversely as we prove soon.

Lemma 2.2 ([18, Lemma 2.3]) *Let S be a ring and $R=T_2(S)$.*

(1) For any $0\neq a\in S$, $aE_{22}\notin Q_l(R)$ and $aE_{11}\notin Q_r(R)$.

(2) S is a left (right) duo ring if and only if $SE_{12}\subseteq Q_r(R)$ $(Q_l(R))$.

Lemma 2.3 *Let R be a ring and I an ideal of R. If R/I is a semicommutative ring and $I\subseteq Q_l(R)$ $(Q_r(R))$, then R is an LQCS (RQCS) ring.*

Proof Write $\overline{R}=R/I$. If $a,b\in R$ with $ab=0$, then $\bar a\bar b=\bar 0$ in $\overline{R}$. This implies $\bar a\bar r\bar b=\bar 0$ for all

$r \in R$ by the semicommutativity of $\overline{R}$. It follows that $aRb \subseteq I \subseteq Q_l(R)$ by hypothesis. □

In the sequel, we use the notation $RA = \{rA | r \in R\}$ for any $A \in M_n(R)$.

Theorem 2.4 *Let S be a ring and $R = T_2(S)$. Then R is an LQCS (RQCS) ring if and only if S is a right (left) duo ring.*

Proof Assume that R is an LQCS ring. From $E_{11}E_{22} = 0$, we have $E_{11}sE_{12}E_{22} = sE_{12} \in Q_l(R)$ for any $s \in S$. This means $SE_{12} \subseteq Q_l(R)$, so S is a right duo ring by Lemma 2.2.

Conversely, suppose that S is a right duo ring. Clearly, $I = SE_{12}$ is an ideal of R such that $I \subseteq Q_l(R)$ by Lemma 2.2. Since the direct product of two right duo rings is a right duo ring and any right duo ring is a semicommutative ring [10, p. 494], $R/I \cong S \times S$ is a semicommutative ring. Thus R is an LQCS ring with help of Lemma 2.3. □

A ring R is said to be left (right) quasi-central reduced [18] if $N(R) \subseteq Q_l(R)$ ($Q_r(R)$) and R is quasi-central reduced if it is both left and right quasi-central reduced.

Proposition 2.5 *Any left (right) quasi-central reduced ring R is an LQCS (RQCS) ring, however the converse is not true in general.*

Proof Applying [18, Proposition 2.8], we have $W(R) = N(R)$. This means that $R/W(R)$ is a reduced ring, so it is a semicommutative ring. Meanwhile $W(R) = N(R) \subseteq Q_l(R)$ by hypothesis. It follows that R is an LQCS ring in the light of Lemma 2.3.

Conversely, it is known from [18, Proposition 2.4] that $R = T_2(S)$ is a left quasi-central reduced ring if and only if S is a reduced right duo ring. This implies that $R = T_2(\mathbb{Z}_4)$ is not a left quasi-central reduced ring, but it is a QCS ring by Theorem 2.4. □

Remark 2.6 (1) As just mentioned, $R = T_2(\mathbb{Z}_4)$ is a QCS ring. But R is not abelian, so it is not central semicommutative by [14, Lemma 2.6].

(2) Definition 2.1 is not left-right symmetric. According to [18, Example 2.6], there exists a right duo domain S which is not a left duo ring. This means that $R = T_2(S)$ is an LQCS ring but not an RQCS ring by Theorem 2.4. Moreover S contains a subring S_1 being not a right duo ring, so the subring $R_1 = T_2(S_1)$ of R is not an LQCS ring. □

Proposition 2.7 *(1) The class of LQCS (RQCS) rings is closed under the ring product.*

(2) If R is an LQCS (RQCS) ring, then eRe is an LQCS (RQCS) ring for any $e \in E(R)$.

Proof (1) It is a direct verification.

(2) Let $a, b \in eRe$ with $ab = 0$. There exist $s, t \in R$ such that $a = ese, b = ete$. This means $a = eae$, $b = ebe$ and $eaeebe = 0$. Similarly, any $r \in eRe$ can be written as $r = ere$. From $eaeebe = 0$, we have $eaerebe \in Q_l(R)$ for all $r \in eRe$ by the virtue of R. Thus for any $u = eue \in eRe$, there exists $v \in R$ such that $ueaerebe = eaerebev$. This implies that $eueeaerebe = eaerebeve$, and so $eaerebe \in Q_l(eRe)$. □

The next lemma is crucial for us to obtain the main result of this section.

Lemma 2.8 *Let R be an LQCS (RQCS) ring and $a \in N(R)$. If n is the minimal positive integer such that $a^n = 0$, then $ar_1ar_2\cdots ar_pa = 0$ for any $r_1, r_2, \ldots, r_p \in R$, where $p = n^2 - 2n + 2$.*

Proof It is trivial when $n = 1$, since in this case $a = 0$ and $p = 1$. Thus we may assume that $n \geq 2$. For any positive integer $i < n$, we construct a generating function $\Phi(n, i)$ as follows

$$\begin{aligned}\Phi(n,i) =& ar_{p-in+i}ar_{p-in+i+1}\cdots ar_{p-(i-1)n+i-1}ar_{p-(i-1)n+i}\cdots ar_{p-2n+2}ar_{p-2n+3}\cdots \\ & ar_{p-n}ar_{p-n+1}a^{n-i}r_{p-i+1}\cdots ar_{p-1}ar_pa.\end{aligned}$$

For example,

$$\Phi(n,1) = ar_{p-n+1}a^{n-1}r_pa,\ \Phi(n,2) = ar_{p-2n+2}ar_{p-2n+3}\cdots ar_{p-n+1}a^{n-2}r_{p-1}ar_pa$$

and

$$\Phi(n, n-1) = ar_1ar_2\cdots ar_{p-1}ar_pa.$$

This leads us to prove the validity of the next claim.

Claim. $\Phi(n, i) = 0$ for any positive integer $i < n$.

Firstly, $a^{n-1}a = 0$ implies $a^{n-1}r_pa \in Q_l(R)$ by the left quasi-central semicommutativity of R. There exists $r'_{p-n+1} \in R$ such that $r_{p-n+1}a^{n-1}r_pa = a^{n-1}r_par'_{p-n+1}$. It follows that $\Phi(n,1) = ar_{p-n+1}a^{n-1}r_pa = a^nr_par'_{p-n+1} = 0$. This proves the validity of Claim for $n = 2$. In the case $n > 2$, then $a^t\Phi(n,1) = a^{1+t}r_{p-n+1}a^{n-2}ar_pa = 0$ for any integer $t \geq 0$. This gives $a^{1+t}r_{p-n+1}a^{n-2}r_{p-1}ar_pa \in Q_l(R)$ by the virtue of R. Applying this relation repeatedly, then

$$\begin{aligned}\Phi(n,2) &= ar_{p-2n+2}ar_{p-2n+3}\cdots ar_{p-n}(ar_{p-n+1}a^{n-2}r_{p-1}ar_pa)\\ &= ar_{p-2n+2}ar_{p-2n+3}\cdots ar_{p-n-1}(a^2r_{p-n+1}a^{n-2}r_{p-1}ar_pa)r'_{p-n}\\ &= ar_{p-2n+2}ar_{p-2n+3}\cdots ar_{p-n-2}(a^3r_{p-n+1}a^{n-2}r_{p-1}ar_pa)r'_{p-n-1}r'_{p-n}\end{aligned}$$

for some r'_{p-n}, $r'_{p-n-1} \in R$. Note that in the expression of $\Phi(n, 2)$ the occurrence of a on the left of a^{n-2} is exactly n. Continuing this process, there exist $r'_{p-2n+2}, \ldots, r'_{p-n} \in R$ such that

$$\Phi(n,2) = (a^nr_{p-n+1}a^{n-2}r_{p-1}ar_pa)r'_{p-2n+2}\cdots r'_{p-n-1}r'_{p-n} = 0.$$

Thus Claim is valid for $n = 3$ by the previous argument. Assume that $n > 3$, and we already have $\Phi(n, i) = 0$ for all $i < n - 1$. To end the proof, it suffices to show $\Phi(n, i + 1) = 0$. Denote

$$\begin{aligned}\xi(1+t) =& a^{1+t}r_{p-in+i}ar_{p-in+i+1}\cdots ar_{p-(i-1)n+i-1}\cdots ar_{p-2n+1}ar_{p-2n+2}\cdots\\ & ar_{p-n+1}a^{n-i-1}r_{p-i}ar_{p-i+1}\cdots ar_{p-1}ar_pa.\end{aligned}$$

From hypothesis $\Phi(n, i) = 0$, we have $a^t\Phi(n, i) = 0$ for any integer $t \geq 0$. To be more specific,

$$\begin{aligned}& a^{1+t}r_{p-in+i}ar_{p-in+i+1}\cdots ar_{p-(i-1)n+i-1}ar_{p-(i-1)n+i}\cdots ar_{p-2n+2}ar_{p-2n+3}\cdots\\ & ar_{p-n}ar_{p-n+1}(a^{n-i-1}a)r_{p-i+1}\cdots ar_{p-1}ar_pa = 0.\end{aligned}$$

Inserting r_{p-i} between a^{n-i-1} and a, then $\xi(1 + t) \in Q_l(R)$ holds. It follows that

$$\begin{aligned}\Phi(n, i+1) =& ar_{p-(i+1)n+i+1}\cdots ar_{p-in+i-1}(ar_{p-in+i}\cdots ar_{p-2n+2}\cdots\\ & ar_{p-n}ar_{p-n+1}a^{n-i-1}r_{p-i}ar_{p-i+1}\cdots ar_{p-1}ar_pa)\end{aligned}$$

$$=ar_{p-(i+1)n+i+1}\cdots ar_{p-in+i-1}\xi(1)=ar_{p-(i+1)n+i+1}\cdots ar_{p-in+i-2}a\xi(1)r'_{p-in+i-1}$$
$$=ar_{p-(i+1)n+i+1}\cdots ar_{p-in+i-2}\xi(2)r'_{p-in+i-1}$$

for some $r'_{p-in+i-1}\in R$. Continuing this process, there exist $r'_{p-(i+1)n+i+1},\dots,r'_{p-in+i-1}\in R$ such that $\Phi(n,i+1)=a\xi(n-1)r'_{p-(i+1)n+i+1}\cdots r'_{p-in+i-1}=0$, since $\xi(n-1)=\xi(1+n-2)$.

By induction, we have $\Phi(n,i)=0$ for any positive integer $i<n$. In particular, $\Phi(n,n-1)=ar_1ar_2\cdots ar_{p-1}ar_pa=0$. This completes the proof of Lemma 2.8. □

Theorem 2.9 *The following statements are true for an LQCS (RQCS) ring R.*

(1) For $a\in R$, if there exists a positive integer n such that $a^n=0$, then $r_0ar_1\cdots ar_{p+1}=0$ for any $r_0,r_1,\dots,r_{p+1}\in R$, where $p=n^2-2n+2$;

(2) RaR is a nilpotent ideal of R for any $a\in N(R)$;

(3) $W(R)=N_*(R)=N^*(R)=N(R)$;

(4) $J(R[x])=W(R[x])=N_*(R[x])=N^*(R[x])=W(R)[x]=N(R)[x]=N(R[x])$. In particular, $R[x]/J(R[x])$ is a reduced ring.

Proof (1) It is a direct consequence of Lemma 2.8.

(2) There exists a positive integer n such that $a^n=0$ for any $a\in N(R)$. We show that $(RaR)^{p+1}=0$, where $p=n^2-2n+2$. If $a_1,a_2,\dots,a_{p+1}\in RaR$, then a_i can be written as $a_i=r_{i1}as_{i1}+r_{i2}as_{i2}+\cdots+r_{im_i}as_{im_i}$ for some $r_{ik},s_{ik}\in R$, $i=1,2,\dots,p+1$, and $k=1,2,\dots,m_i$. It turns out that $a_1a_2\cdots a_{p+1}=0$ by Lemma 2.8, and so $(RaR)^{p+1}=0$.

(3) On the one hand, $W(R)\subseteq N_*(R)\subseteq N^*(R)\subseteq N(R)$ is well known. On the other hand, $N(R)\subseteq W(R)$ with help of (2). Consequently, $W(R)=N_*(R)=N^*(R)=N(R)$.

(4) It is known that $J(R[x])=I[x]$ for some nil ideal I of R and that $N_*(R[x])=N_*(R)[x]$ from [19, Theorems 1 and 3]. This means $J(R[x])\subseteq N(R)[x]=W(R)[x]$ by (3). Combining this with $W(R)[x]\subseteq N_*(R[x])=N_*(R[x])\subseteq J(R[x])$, we obtain $J(R[x])=W(R)[x]$. With help of [16, Lemma 2.3], we have $W(R[x])=W(R)[x]$. It turns out that $J(R[x])=W(R[x])=N_*(R[x])=N^*(R[x])=W(R)[x]=N(R)[x]$. Moreover R is a 2-primal ring by (3), so is $R[x]$ duo to [20, Proposition 2.6]. This implies $J(R[x])=N(R[x])$, proving the equalities of (4). Finally from $R[x]/J(R[x])=R[x]/N(R[x])$, we conclude that $R[x]/J(R[x])$ is a reduced ring. □

Corollary 2.10 *The conclusions of Theorem 2.9 are true for left (right) quasi-central reduced rings, central semicommutative rings, and symmetric-over-center rings.*

Proof It is known from [16, Proposition 2.1] that a symmetric-over-center ring is central symmetric in the sense of [12]. So the conclusions hold by Proposition 2.5 and Theorem 2.9. □

Corollary 2.11 *For any ring R, $M_n(R)$ is neither an LQCS ring nor an RQCS ring.*

Proof Assume on the contrary, then $E_{1n},E_{n1}\in N(M_n(R))$ implies $E_{1n}+E_{n1}\in N(R)$. However $(E_{1n}+E_{n1})^2=E_{11}+E_{nn}$ is a nonzero idempotent, this contradicts Theorem 2.9. □

Corollary 2.12 *A ring R is an LQCS (RQCS) ring if and only if $ab=0$ implies $aRb\subseteq$*

$Q_l(R)\bigcap W(R)$ $(Q_r(R)\bigcap W(R))$ *for* $a,b\in R$.

Proof It suffices to show $ab=0$ implies $aRb\subseteq W(R)$. From $ab=0$, we have $ba\in N(R)=W(R)$ by Theorem 2.9. Hence $bar\in W(R)$ for any $r\in R$, and so $arb\in N(R)=W(R)$. $\Box$

Corollary 2.13 *A semiprime LQCS (RQCS) ring is a reduced ring, and a prime LQCS (RQCS) ring is a domain.*

Proof The validity is clearly from Corollary 2.12 and Theorem 2.9. $\Box$

Definition 2.14 *A ring* R *is said to be left (right) quasi-central symmetric if* $abc=0$ *implies* $bac\in Q_l(R)$ $(Q_r(R))$ *for* $a,b,c\in R$, *and* R *is quasi-central symmetric if it is left and right quasi-central symmetric.*

Definition 2.15 *A ring* R *is said to be left (right) quasi-central reversible if* $ab=0$ *implies* $ba\in Q_l(R)$ $(Q_r(R))$ *for* $a,b\in R$, *and* R *is quasi-central reversible if it is left and right quasi-central reversible.*

Clearly, a left (right) quasi-central symmetric ring is left (right) quasi-central reversible.

Lemma 2.16 *Let* R *be a ring and* I *an ideal. If* R/I *is symmetric (reversible) ring such that* $I\subseteq Q_l(R)$ $(Q_r(R))$, *then* R *is a left (right) quasi-central symmetric (reversible) ring.*

Proof It is similar to the proof of Lemma 2.3. $\Box$

Proposition 2.17 *If* R *is a left (right) quasi-central reduced ring, then* R *is a left (right) quasi-central symmetric ring.*

Proof Since R is a left quasi-central reduced ring, we have $N(R)=W(R)\subseteq Q_l(R)$ with help of [18, Proposition 2.8]. Thus $\overline{R}=R/W(R)$ is a reduced ring, so is a symmetric ring. If $a,b,c\in R$ satisfy $abc=0$, then $\bar{a}\bar{b}\bar{c}=\bar{0}$ in $\overline{R}$. This implies $\bar{b}\bar{a}\bar{c}=\bar{0}$ by the symmetry of $\overline{R}$. It turns out that $bac\in N(R)\subseteq Q_l(R)$, and so we are done. $\Box$

Proposition 2.18 *Any left (right) quasi-central symmetric ring* R *is an LQCS (RQCS) ring.*

Proof Let $a,b\in R$ with $ab=0$. Then we have $rab=0$ for all $r\in R$. This implies $arb\subseteq Q_l(R)$ by the left quasi-central symmetry of R. It can be concluded that $aRb\subseteq Q_l(R)$. $\Box$

Theorem 2.19 *The following conclusions are true for a ring* S *and* $R=T_2(S)$.

(1) *R is left (right) quasi-central symmetric if and only if S is symmetric right (left) duo.*

(2) *R is left (right) quasi-central reversible if and only if S is reversible right (left) duo.*

Proof (1) Assume that R is a left quasi-central symmetric ring and $a,b,c\in S$ with $abc=0$. Let $A=aE_{22},B=bE_{22}$, $C=cE_{22}\in R$. Then we have $ABC=abcE_{22}=0$. It yields that $BAC=bacE_{22}\in Q_l(R)$ by the virtue of R. This implies $bac=0$ by Lemma 2.2 (1). In view of Lemma 2.2 (2), we need to show $SE_{12}\subseteq Q_l(R)$. For any $a\in S$, then $aE_{12}E_{11}E_{22}=0$ gives $E_{11}aE_{12}E_{22}=aE_{12}\in Q_l(R)$ by the left quasi-central symmetry of R and so we are done.

(2) It is very similar to the proof of (1). □

Remark 2.20 The condition one-sided duo property and that of reversibility do not imply each other. For any field F, the ring $T = F\langle x, y\rangle/(x^3, y^3, yx, xy - x^2, xy - y^2)$ is duo but not reversible by [10, Example 3.9 and Remark 1]. Conversely, if R is a domain which is neither a right nor a left Ore ring, then R is reversible ring and not one-sided duo ring with help of [10, Example 3.2]. Moreover $R = T_2(\mathbb{Z})$ is a quasi-central reversible ring by Theorem 2.19, but R is not a central reversible ring by [13, Lemma 2.13], since it is not abelian.

Remark 2.21 It is known from Theorem 2.19 and [18, Proposition 2.4] that $R = T_2(\mathbb{Z}_4)$ is a quasi-central symmetric ring which is neither left nor right quasi-central reduced. Let Q_8 be the quaternion group of order 8, $S = \mathbb{Z}_2 Q_8$ the group algebra, and $R = T_2(S)$. It is proved in [10, Example 3.8] that S is a reversible duo ring but not a symmetric ring. Thus $R = T_2(S)$ is a quasi-central reversible ring which is neither a left nor a right quasi-central symmetric ring by Theorem 2.19. Moreover if T is the ring in Remark 2.20, then $R = T_2(T)$ is a QCS ring which is not one-sided quasi-central reversible with help of Theorems 2.4 and 2.19.

Example 2.22 ([21, Example 2.1]) There exists a central (hence a quasi-central) reversible ring which is neither an LQCS ring nor an RQCS ring.

Proof Let $A = F[a, b, c]$ be the free algebra of polynomials with zero constant terms in non-commuting identerminates a, b, c over $\mathbb{Z}_2$. Then A is a ring without identity. Let I be an ideal of $\mathbb{Z}_2 + A$, generated by $ab, ba^2, b^2a, bca, bac + cba, r_1r_2r_3r_4r_5$, where $r_1, r_2, r_3, r_4, r_5 \in A$ and let $R = (\mathbb{Z}_2 + A)/I$. We call each product of the indeterminates a, b, c a monomial and say that α is a monomial of degree n if it is a product of exactly n number of indetermintes. Let H_n be the set of all linear combinations of monomials of degree n over $\mathbb{Z}_2$. Note that H_n is finite for any n and that the ideal I of R is homogeneous, i.e., if $\sum_{i=1}^{s} \alpha_i \in I$ with $\alpha_i \in H_i$ then each $\alpha_i \in I$. It is proved in [21, Example 2.1] that R is a central reversible ring (so is a quasi-central reversible ring) which is not a central semicommutative ring. Firstly we show that R is not an LQCS ring. By the definition of I, it yields that $ab \in I$ and $acb \notin I$. We claim that $Racb \nsubseteq acbR$. It suffices to show $aacb + acb\alpha \notin I$ for any $\alpha \in A$ (eq., $\alpha \in \mathbb{Z}_2 + A$). We may rite $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + h$, where $\alpha_i \in H_i$ and $h \in I$, since $A^5 \subseteq I$. It follows that $aacb + acb\alpha = aacb + acb\alpha_1 + h'$ for some $h' \in I$. Thus $aacb + acb\alpha \notin I$ if and only if $aacb + acb\alpha_1 \notin I$. Note that $\alpha_1 = k_1a + k_2b + k_3c$ for some $k_i \in \mathbb{Z}_2$. From $ab \in I$ and $bac + cba \in I$, we have $acba \in I$. It follows that $acb\alpha_1 = acb(k_2b + k_3c) + h''$ for some $h'' \in I$. Therefore $aacb + acb\alpha \notin I$ if and only if $aacb + acb(k_2b + k_3c) \notin I$ for any $k_2, k_3 \in \mathbb{Z}_2$.

Case 1. If $k_2 = 0$, $k_3 = 0$, then $aacb + acb(k_2b + k_3c) = aacb$.

Case 2. If $k_2 = 1$, $k_3 = 0$, then $aacb + acb(k_2b + k_3c) = aacb + acbb$.

Case 3. If $k_2 = 0$, $k_3 = 1$, then $aacb + acb(k_2b + k_3c) = aacb + acbc$.

Case 4. If $k_2 = 1$, $k_3 = 1$, then $aacb + acb(k_2b + k_3c) = aacb + acbb + acbc$.

Obviously, we have $aacb \notin I$, $aacb + acbb \notin I$, $aacb + acbc \notin I$, and $aacb + acbb + acbc \notin I$ by the definition of I. This means $aacb + acb\alpha \notin I$ for any $\alpha \in \mathbb{Z}_2 + A$ from the previous argument.

Thus R is not an LQCS ring. Similarly, it can be proved that R is not an RQCS ring by taking into account $acbb+\beta acb\notin I$ for any $\beta\in\mathbb{Z}_2+A$. □

Remark 2.23 Similar to the proof of Remark 2.6, it can be proved that neither Definition 2.14 nor Definition 2.15 is left-right symmetric, and that the subring of a one-sided quasi-central symmetric (reversible) ring need not be the same ring. Also note that if $R[x]$ is a one-sided duo ring, then R is a commutative ring by [22, Lemma 9]. Thus the polynomial ring over a left (right) quasi-central reduced ring need not be neither a left (right) quasi-central reversible ring nor an LQCS (RQCS) ring. Let $\mathbb{H}$ be the real Hamilton quaternions ring and $R=T_2(\mathbb{H})$. Then R is a quasi-central reduced ring by [18, Proposition 2.4]. Observing that $R\cong T_2(\mathbb{H}[x])$ and $\mathbb{H}[x]$ is not a one-sided duo ring, $R[x]$ satisfies our requirement.

A ring R is said to be left (right) quasi-central Armendariz [18] if $f(x)=\sum_{i=0}^{m}a_ix^i, g(x)=\sum_{j=0}^{n}b_jx^j\in R[x]$ satisfy $f(x)g(x)=0$, then $a_ib_j\in Q_l(R)$ $(Q_r(R))$ for all i and j.

Remark 2.24 The class of left (right) quasi-central Armendariz rings and that of LQCS (RQCS) rings are independent of each other. It is known from [23, Example 3.2] that the commutative ring $S=S_2(\mathbb{Z}_8)$ is not Armendariz. This means that $R=T_2(S)$ is a QCS ring which is neither a left nor a right quasi-central Armendariz ring by [18, Theorem 2.13]. On the other hand, $R=F\langle a,b|a^2=0\rangle$ is an Armendariz ring and so is a quasi-central Armendariz ring for any field F. However R is neither an LQCS nor an RQCS ring, since R is not a 2-primal ring with help of [24, Example 4.8].

3. Examples of left (right) quasi-central semicommutative rings

Let R be a ring, k and n positive integers such that $k<n$. We write $V=\sum_{i=1}^{n-1}E_{i,i+1}$, $V_n(R)=RI_n+RV+\cdots+RV^{n-1}$ and $T_n^k(R)=V_k(R)+\sum_{i=1}^{k+1}\sum_{j=k+i}^{n}RE_{ij}$. In particular, $V_2(R)$ is the trivial extension $T(R,R)$ of R. Moreover we write the set of all $n\times 1$ matrices over R by $R^n=\{(a_1,a_2,\ldots,a_n)^{\mathrm{T}}|a_i\in R\}$.

Proposition 3.1 *Let S be a ring and n a positive integer. Then $R_1=T_n(S)$ for $n\geq 3$ and $R_2=S_n(R)$ for $n\geq 5$ is neither an LQCS ring nor an RQCS ring.*

Proof (1) For any $n\geq 3$, clearly $E_{13}=E_{12}E_{23}\in R_1E_{23}$, and $E_{13}\notin E_{23}R_1$. This means $R_1E_{23}\nsubseteq E_{23}R_1$ and so $E_{23}\notin Q_l(R_1)$. From $E_{22}E_{33}=0$ and $E_{22}E_{23}E_{33}=E_{23}\notin Q_l(R_1)$, we conclude that R_1 is not an LQCS ring. Similarly, since $E_{13}\in E_{12}R_1$ and $E_{13}\notin R_1E_{12}$, we have $E_{12}\notin Q_r(R_1)$. Combining $E_{11}E_{22}=0$ with $E_{11}E_{12}E_{22}=E_{12}\notin Q_r(R_1)$, we can conclude that R_1 is not an RQCS ring.

(2) Consider $R_2=S_n(S)$ for $n\geq 5$. Since $E_{15}=E_{12}E_{25}\in R_2E_{25}$ and $E_{15}\notin E_{25}R_2$, we get $E_{25}\notin Q_l(R_2)$. Thus $E_{23}E_{45}=0$ and $E_{23}E_{34}E_{45}=E_{25}\notin Q_l(R_2)$ imply that R_2 is not an LQCS ring. Similarly, as $E_{15}\in E_{14}R_2$ and $E_{15}\notin R_2E_{14}$, we have $E_{14}\notin Q_r(R_2)$. From $E_{12}E_{34}=0$ and $E_{12}E_{23}E_{34}=E_{14}\notin Q_r(R_2)$, we conclude that R_2 is not an RQCS ring. □

Theorem 2.4 and [14, Corollary 2.14] imply that $T_2(\mathbb{Z})$ and $S_4(\mathbb{Z})$ are QCS rings.

Corollary 3.2 *Let S be a ring and $n \geq 2$ an integer. Then $R = T_n(S)$ is an LQCS (RQCS) ring if and only if $n = 2$ and S is a right (left) duo ring.*

Proof It is a direct consequence of Proposition 3.1 and Theorem 2.4. □

Lemma 3.3 *Let S be a ring and $R = V_n(S)$ for $n \geq 2$. If S is a left (right) quasi-central reduced ring, then $AB = 0$ implies $ARB \subseteq V_n(W(S))$ for any $A, B \in R$.*

Proof Write $\overline{S} = S/W(S)$ and $\overline{R} = V_n(\overline{S})$. The canonical ring homomorphism from S onto $\overline{S}$ induces a ring surjective homomorphism from R onto $\overline{R}$. Since S is a left quasi-central reduced ring, $W(S) = N(S)$ by [18, Proposition 2.8] and so $\overline{S}$ is a reduced ring. This implies that $\overline{R}$ is a semicommutative ring with help of [25, Theorem 2.5 and Lemma 1.4]. Now $AB = 0$ implies $\overline{A}\,\overline{B} = \bar{0}$ in $\overline{R}$. It follows that $\overline{A}\,\overline{C}\,\overline{B} = \bar{0}$ for all $C \in R$ by the semicommutativity of $\overline{R}$. Accordingly we have $ARB \subseteq V_n(W(S))$. □

Corollary 3.4 *Let S be a left (right) quasi-central reduced ring and $R = T(S, S)$. If for any $r, s \in S$ and $a, b \in W(S)$, there exist $u, v \in S$ such that $ra = au$, $rb + sa = bu + av$ ($ar = ua$, $br + as = ub + va$), then R is an LQCS (RQCS) ring.*

Proof Let $A, B \in R$ with $AB = 0$. There exist $a, b \in W(S)$ such that $ACB = aI_2 + bE_{12}$ for all $C \in R$ with help of Lemma 3.3. For any $M = rI_2 + sE_{12} \in R$, then we have $MACB = raI_2 + (rb + sa)E_{12}$. By hypothesis, there exist $u, v \in S$ such that $ra = au$, $rb + sa = bu + av$. Let $M_1 = uI_2 + vE_{12}$. A simple computation gives $MACB = ACBM_1$. This shows $ACB \in Q_l(R)$, and so R is an LQCS ring. □

Of course, a central reduced ring S satisfies the conditions stated in Corollary 3.4.

Theorem 3.5 *Let S be a reduced left (right) duo ring and $R = T_2(S)$. Then $W = T(R, R)$ is an LQCS (RQCS) ring if and only if for any $r, s, a, b \in S$ there exist $u, v \in S$ such that $ra = au, rb + sa = bu + av$ ($ar = ua, br + as = ub + va$).*

Proof Clearly, $R = T_2(S)$ is a left (right) quasi-central reduced ring by [18, Proposition 2.4].

Assume that the element-wise condition stated in Theorem 3.5 holds. For $\mathscr{A}, \mathscr{B} \in W$ with $\mathscr{A}\mathscr{B} = 0$ and $\mathscr{C} \in W$, then $\mathscr{A}\mathscr{C}\mathscr{B} \in T(W(R), W(R))$, so there exist $A, B \in W(R)$ such that

$$\mathscr{A}\mathscr{C}\mathscr{B} = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}$$

with help of Lemma 3.3. Observing that S is a reduced ring, there exist $a, b \in S$ such that

$$\mathscr{A}\mathscr{C}\mathscr{B} = \begin{pmatrix} aE_{12} & bE_{12} \\ 0 & aE_{12} \end{pmatrix}.$$

For any $\mathscr{D} \in W$, there exist $D_1 = rE_{11} + r_1E_{12} + r_2E_{22}$, $D_2 = sE_{11} + s_1E_{12} + s_2E_{22} \in R$ with

$$\mathscr{D} = \begin{pmatrix} D_1 & D_2 \\ 0 & D_1 \end{pmatrix},$$

where $r, s, , r_1, r_2, s_1, s_2 \in S$. Through a simple computation, it yields the following equality

$$\mathscr{D}\mathscr{A}\mathscr{C}\mathscr{B} = \begin{pmatrix} raE_{12} & (rb+sa)E_{12} \\ 0 & raE_{12} \end{pmatrix}.$$

Take $\mathscr{D}' = \left(\begin{smallmatrix} uE_{22} & vE_{22} \\ 0 & uE_{22} \end{smallmatrix}\right)$. It is easily checked that $\mathscr{D}\mathscr{A}\mathscr{C}\mathscr{B} = \mathscr{A}\mathscr{C}\mathscr{B}\mathscr{D}'$, showing the validity of $\mathscr{A}\mathscr{C}\mathscr{B} \in Q_l(W)$.

For the converse, suppose that W is an LQCS ring and $r, s, a, b \in S$. Let us consider

$$\mathscr{A} = \begin{pmatrix} E_{11} & 0 \\ 0 & E_{11} \end{pmatrix}, \ \mathscr{B} = \begin{pmatrix} E_{22} & 0 \\ 0 & E_{22} \end{pmatrix}, \ \mathscr{C} = \begin{pmatrix} aE_{12} & bE_{12} \\ 0 & aE_{12} \end{pmatrix} \in W.$$

Clearly, we have $\mathscr{A}\mathscr{B} = 0$. This implies $\mathscr{A}\mathscr{C}\mathscr{B} = \left(\begin{smallmatrix} aE_{12} & bE_{12} \\ 0 & aE_{12} \end{smallmatrix}\right) \in Q_l(W)$ by hypothesis. So for $\mathscr{D} = \left(\begin{smallmatrix} rE_{11} & sE_{11} \\ 0 & rE_{11} \end{smallmatrix}\right) \in W$, there exists $\mathscr{D}' = \left(\begin{smallmatrix} D_1' & D_2' \\ 0 & D_1' \end{smallmatrix}\right) \in W$ such that $\mathscr{D}\mathscr{A}\mathscr{C}\mathscr{B} = \mathscr{A}\mathscr{C}\mathscr{B}\mathscr{D}'$. We may write $D_1' = r_1'E_{11} + s_1'E_{12} + uE_{22}$ and $D_2' = r_2'E_{11} + s_2'E_{12} + vE_{22} \in R$ for some $r_1', r_2', s_1', s_2', u, v \in S$. It follows that $ra = au$, $rb + sa = bu + av$ by comparing the elements on two sides of $\mathscr{D}\mathscr{A}\mathscr{C}\mathscr{B} = \mathscr{A}\mathscr{C}\mathscr{B}\mathscr{D}'$. This completes the proof of Theorem 3.5. □

Corollary 3.6 *Let R be a ring. If $V_n(R)$ is an LQCS (RQCS) ring for some integer $n \geq 3$, then $V_2(R)$ is an LQCS (RQCS) ring.*

Proof Let $S = RI_n + RV^{n-1}$. We have $V_2(R) = RI_2 + RV \cong S$ by a direct verification. Now it suffices to show that S is an LQCS ring. If $A, B \in S$ satisfy $AB = 0$, then $ACB = a_0I + a_{n-1}V^{n-1} \in Q_l(V_n(R))$ for all $C \in S$. Thus for any $D = r_0I_n + r_{n-1}V^{n-1} \in S$, there exists $D' = r_0'I_n + r_1'V + \cdots + r_{n-1}'V^{n-1} \in V_n(R)$ such that $DACB = ACBD'$. This gives $r_0a_0 = a_0r_0'$, $r_0a_{n-1} + r_{n-1}a_0 = a_0r_{n-1}' + a_{n-1}r_0'$. Let $D'' = r_0'I_n + r_{n-1}'V^{n-1} \in S$. Then we have $DACB = ACBD''$ by a simple computation. This implies $ACB \in Q_l(S)$. □

It is known from [25, Proposition 1.6] and [21, Theorem 2.3] that if S is a (central) reduced ring, then $R = T(S, S)$ is a (central) semicommutative ring. One may naturally ask whether $R = T(S, S)$ is a QCS ring whenever S is a quasi-central reduced ring.

Example 3.7 There exists a quasi-central reduced ring S such that $R = T(S, S)$ is neither an LQCS ring nor an RQCS ring.

Proof Let K be any field, $F = K(t)$ the field of rational functions in a variable t over K, σ an automorphism of F satisfying $\sigma(f(t)) = f(t^{-1})$ for any $f(t) \in F$. Thus we have $\sigma(t^{-1}) = t$, $\sigma^{-1}(t) = t^{-1}$, and $\sigma^{-1}(t^{-1}) = t$. Let $S = F[[x; \sigma]]$ be the left skew power series ring over F. It follows from [18, Example 2.6] that S is a left duo ring. Applying the fact that σ is a surjective endomorphism, it is easily checked that S is also a right duo ring. Thus $R = T_2(S)$ is a quasi-central reduced ring by [18, Proposition 2.4]. We claim that $W = T(R, R)$ is not an LQCS ring. On the contrary, for $r = t$, $s = 1$, $a = x^2, b = x \in S$, there exist $u, v \in S$ such that $ra = au, rb + sa = bu + av$ by Theorem 3.5. This means $tx^2 = x^2u$, $tx + a = bu + av$. Clearly, u can be written as $u = l_0 + l_1x + \cdots + l_px^p$ for some $l_0, l_1, \ldots, l_p \in F$. Comparing the coefficients on two sides of $tx^2 = x^2u$, we must have $u = l_0 \in F$. Thus $tx^2 = x^2u$ gives $\sigma^2(u) = t$. It yields that $u = \sigma^{-2}(t) = \sigma^{-1}(t^{-1}) = t$. Meanwhile from $tx + a = bu + av$, we have $tx + x^2 = xt + x^2v$. This implies that $tx + x^2 = \sigma(t)x + x^2v = t^{-1}x + x^2v$. It turns out that $t = t^{-1}$. This is a contradiction. Therefore, W is not an LQCS ring by Theorem 3.5. Similarly, retaking $r = t$, $s = 1$, $a = x^2$, $b = x \in S$, it can be proved that W is not an RQCS ring with help of Theorem 3.5. □

A ring R is strongly (von Neumann) regular if for any $a \in R$, there exists $b \in R$ such that $a = aba$ and $ab = ba$. It is known that such a ring is reduced and duo [10, 26].

Theorem 3.8 *Let S be a ring and $R = T_2(S)$. If S is a strongly regular ring or a commutative reduced ring, then $U = V_n(R)$ is a QCS ring.*

Proof It is known from [18, Propositions 2.4 and 2.8] that R is a quasi-central reduced ring such that $W(R) = N(R)$. If $\mathscr{A}, \mathscr{A} \in U$ with $\mathscr{A}\mathscr{B} = 0$, then we have $\mathscr{A}\mathscr{C}\mathscr{B} \in V_n(W(R))$ by Lemma 3.3. Since S is a reduced ring, $W(R) = N(R) = S\varepsilon_{12}$, where ε_{ij} is the matrix unit of R. It turns out that $\mathscr{A}\mathscr{C}\mathscr{B} \in V_n(S\varepsilon_{12})$. There exist $a_0, a_1, \ldots, a_{n-1} \in S$ such that $\mathscr{A}\mathscr{C}\mathscr{B} = a_0\varepsilon_{12}I + a_1\varepsilon_{12}V + \cdots + a_{n-1}\varepsilon_{12}V^{n-1}$. Similarly, for any $\mathscr{D} \in U$, it can be written as

$$U = (s_0\varepsilon_{11} + t_0\varepsilon_{22} + r_0\varepsilon_{12})I + (s_1\varepsilon_{11} + t_1\varepsilon_{22} + r_1\varepsilon_{12})V + \cdots + (s_{n-1}\varepsilon_{11} + t_{n-1}\varepsilon_{22} + r_{n-1}\varepsilon_{12})V^{n-1}$$

for some $s_0, t_0, r_0, \ldots, s_{n-1}, t_{n-1}, r_{n-1} \in S$. It follows that $\mathscr{D}\mathscr{A}\mathscr{C}\mathscr{B} = (r_0a_0)\varepsilon_{12}I + (r_0a_1 + r_1a_0)\varepsilon_{12}V + \cdots + (r_0a_{n-1} + r_1a_{n-2} + \cdots + r_{n-1}a_0)\varepsilon_{12}V^{n-1}$ by the virtue of matrix units. In the case S being a commutative reduced ring, then $\mathscr{D}' = r_0\varepsilon_{22}I + r_1\varepsilon_{22}V + \cdots + r_{n-1}\varepsilon_{22}V^{n-1}$ satisfies $\mathscr{D}\mathscr{A}\mathscr{C}\mathscr{B} = \mathscr{A}\mathscr{C}\mathscr{B}\mathscr{D}'$ by a direct computation. This proves that $\mathscr{A}\mathscr{C}\mathscr{B} \in Q_l(U)$, and so U is an LQCS ring. Similarly, it can be proved that U is an RQCS ring in this case. In another case, we need to apply [27, Lemma 1.7] which states that if S is a strongly regular ring and $r_0, a_0, r_1, a_1, \ldots, a_{n-1}, r_{n-1} \in S$, then the following system of linear equations

$$r_0a_0 = a_0x_0$$

$$r_0a_1 + r_1a_0 = a_1x_0 + a_0x_1$$

$$\vdots$$

$$r_0a_{n-1} + r_1a_{n-2} + \cdots + r_{n-1}a_0 = a_{n-1}x_0 + a_{n-2}x_1 \cdots + a_0x_{n-1}$$

is solvable in S. Let $x_0 = s_0, x_1 = s_1, \ldots, x_{n-1} = s_{n-1}$ be a solution and $\mathscr{D}' = s_0\varepsilon_{22}I + s_1\varepsilon_{22}V + \cdots + s_{n-1}\varepsilon_{22}V^{n-1}$. There is no difficulty to check that $\mathscr{D}\mathscr{A}\mathscr{C}\mathscr{B} = \mathscr{A}\mathscr{C}\mathscr{B}\mathscr{D}'$. Therefore, U is an LQCS ring. Analogously, it can be proved that U is an RQCS ring. □

In what follows, a 1×1 matrix over a ring R is denoted by (b) for some $b \in R$.

Lemma 3.9 *(1) Let R be a right duo ring. For any $b \in R$ and $\beta = (c_1, c_2, \ldots, c_n)^{\mathrm{T}} \in R^n$, there exists $\beta' \in R^n$ such that $\beta(b) = bI_n\beta'$.*

(2) Let R be a reduced ring, $b_1 \in R$ and $\alpha_1 = (d_1, d_2, \ldots, d_n)^{\mathrm{T}} \in R^n$. If $\alpha_1(b_1)^2 = 0$, then we have $\alpha_1(b_1) = 0$.

Proof (1) By hypothesis, $Rb \subseteq bR$ holds. So for each c_i there exists $c_i' \in R$ such that $c_i b = bc_i'$. Let $\beta' = (c_1', c_2', \ldots, c_n')^{\mathrm{T}}$. We have $\beta(b) = (c_1 b, c_2 b, \ldots, c_n b)^{\mathrm{T}} = (bc_1', bc_2', \ldots, bc_n')^{\mathrm{T}} = bI_n\beta'$.

(2) From $\alpha_1(b_1)^2 = 0$, we have $d_i b_1 b_1 = 0$ for each i. This implies $d_i b_1 d_i b_1 = 0$ by the semicommutativity of R. This means $d_i b_1 = 0$ by the reduceness of R, entailing $\alpha_1(b_1) = 0$. $\square$

Noticing that any reduced ring R is reversible, $ab = 0$ if and only if $ba = 0$ for $a, b \in R$. In the sequel we will use this fact freely without mention. For any $A \in T_n^k(R)$, we write $A = (a_{ij}) \in T_n(R)$ such that $a_{11} = a_{22} = \cdots = a_{nn} = a_1$, $a_{12} = a_{23} = \cdots = a_{n-1,n} = a_2, \ldots$ and $a_{1k} = a_{2,k+1} = \cdots = a_{n-k+1,n} = a_k$. Moreover, for matrices $A = (a_{il})_{m\times s}$, $B = (b_{lj})_{s\times n}$ over R, we write $[AB]_{i,j} = 0$ to mean that $a_{il}b_{lj} = 0$ for $l = 1, 2, \ldots, s$.

Theorem 3.10 *Let R be a ring and k a positive integer. If R is a reduced right (left) duo ring, then $T_{2k+2}^k(R)$ is an LQCS (RQCS) ring.*

Proof Assume that $A, B \in T_{2k+2}^k(R)$ with $AB = 0$. We need to show $ACB \in Q_l(T_{2k+2}^k(R))$ for any $C \in T_{2k+2}^k(R)$. Represent $A = \left(\begin{smallmatrix} A_1 & \alpha_1 \\ 0 & a_1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} B_1 & \beta_1 \\ 0 & b_1 \end{smallmatrix}\right)$ as partitioned matrices, where $A_1, B_1 \in T_{2k+1}^k(R)$, $\alpha_1, \beta_1 \in R^{2k+1}$ and $a_1, b_1 \in R$. We may identify a_1, b_1 with $(a_1), (b_1)$ for simplification. Now $AB = 0$ gives $A_1B_1 = 0$, $a_1b_1 = 0$ and $A_1\beta_1 + \alpha_1 b_1 = 0$.

The last equality implies $A_1\beta_1 b_1 + \alpha_1 b_1^2 = 0$. Since R is a right duo ring, there exists $\beta_1' \in R^{2k+1}$ such that $\beta_1 b_1 = b_1 I_{2k+1}\beta_1'$ by Lemma 3.9 (1). Meanwhile $A_1B_1 = 0$ implies $[A_1B_1]_{i,j} = 0$ by [28, Lemma 1]. In particular, we have $A_1 b_1 I_{2k+1} = 0$ and hence $A_1\beta_1 b_1 = A_1 b_1 I_{2k+1}\beta_1' = 0$ by Lemma 3.9 (1). From $A_1\beta_1 b_1 + \alpha_1 b_1^2 = 0$, it yields that $\alpha_1 b_1^2 = 0$. This implies that $\alpha_1 b_1 = 0$ by Lemma 3.9 (2). So $A_1\beta_1 + \alpha_1 b_1 = 0$ gives $A_1\beta_1 = 0$. Write A_1 as a row partitioned matrix,

$$A_1 = \begin{pmatrix} \alpha_{2k+1} \\ \alpha_{2k} \\ \vdots \\ \alpha_1 \end{pmatrix}.$$

It is easy to see $\alpha_i = (0, \ldots, 0, a_1, \ldots, a_i)$ for $i = 1, 2, \ldots, k$ and there is no difficulty to check $\alpha_{k+i} = (0, \ldots, 0, a_1, a_2, \ldots, a_k, a_{k+2-i,2k+2-i}, \ldots, a_{k+2-i,2k+1})$ where the occurrence of 0 is $k + 1 - i$. Moreover $\beta_1 = (b_{1,2k+2}, b_{2,2k+2}, \ldots, b_{k+2,2k+2}, b_k, b_{k-1}, \ldots, b_3, b_2)^{\mathrm{T}}$ which lies in the last column of the matrix B. Remember that we have assumed $AB = 0$ and so $A_1B_1 = 0$.

Claim. $A_1\beta_1 = 0$ implies $[\alpha_m\beta_1]_{i,j} = 0$ for all $m = 1, 2, \ldots, 2k$.

Case 1. In the case $1 \le m \le k-1$, all $\alpha_m\beta_1 = 0$ if and only if the following equalities

$$a_1b_2 = 0$$

$$a_1b_3 + a_2b_2 = 0$$

$$a_1b_4 + a_2b_3 + a_3b_2 = 0$$

$$\vdots$$

$$a_1b_k + a_2b_{k-1} + \cdots + a_{k-2}b_3 + a_{k-1}b_2 = 0$$

hold. On the other hand, it is easily checked that $A_1B_1 = 0$ implies the following equalities

$$a_1b_1 = 0$$

$$a_1b_2 + a_2b_1 = 0$$

$$a_1b_3 + a_2b_2 + a_3b_1 = 0$$

$$\vdots$$

$$a_1b_k + a_2b_{k-1} + \cdots + a_{k-2}b_3 + a_{k-1}b_2 + a_kb_1 = 0.$$

As previously mentioned, $A_1B_1 = 0$ implies $[A_1B_1]_{i,j} = 0$ by [28, Lemma 1]. In particular, $\alpha_m\beta_1 = 0$ implies $[\alpha_m\beta_1]_{i,j} = 0$ for $1 \le m \le k-1$, proving the validity of Claim in Case 1.

Case 2. In the case $k \le m \le 2k-1$, we proceed from $m = k$. Assume $\alpha_k\beta_1 = 0$, i.e.,

$$a_1b_{k+2,2k+2} + a_2b_k + \cdots + a_{k-1}b_3 + a_kb_2 = 0. \tag{3.1}$$

Applying the conclusion of Case 1, we have $b_ka_1 = b_{k-1}a_1 = \cdots = b_4a_1 = b_3a_1 = b_2a_1 = 0$. Multiplying $a_1b_{k+2,2k+2}$ on the right sides of (3.1) yields $(a_1b_{k+2,2k+2})^2 = 0$. This implies $a_1b_{k+2,2k+2} = 0$ by the reduceness of R. Thus (3.1) can be simplified to the following equality

$$a_2b_k + a_3b_{k-1} + \cdots + a_{k-1}b_3 + a_kb_2 = 0. \tag{3.2}$$

Similarly, multiplying a_2b_k on the right sides of (3.2), we can obtain $a_2b_k = 0$. Continuing this process, finally we get $a_3b_{k-1} = \cdots = a_{k-1}b_3 = a_kb_2 = 0$. Now it can be concluded that

$$a_1b_{k+2,2k+2} = a_2b_k = \cdots = a_{k-1}b_3 = a_kb_2 = 0.$$

It follows from the previous argument that $\alpha_m\beta_1 = 0$ implies $[\alpha_m\beta_1]_{i,j} = 0$ for $1 \le m \le k$.

In the case $m = k+1$, then $\alpha_{k+1}\beta_1 = 0$ is equivalent to the following equality

$$a_1b_{k+1,2k+2} + a_2b_{k+2,2k+2} + a_3b_k + \cdots + a_{k-1}b_4 + a_kb_3 + a_{k+1,2k+1}b_2 = 0. \tag{3.3}$$

Multiplying $a_1b_{k+1,2k+2}$ on the right sides of (3.3), we have $(a_1b_{k+1,2k+2})^2 = 0$ with help of $[\alpha_m\beta_1]_{i,j} = 0$ for $1 \le m \le k$, and hence $a_1b_{k+1,2k+2} = 0$ by the virtue of R. This implies that

$$a_2b_{k+2,2k+2} + a_3b_k + \cdots + a_{k-1}b_4 + a_kb_3 + a_{k+1,2k+1}b_2 = 0. \tag{3.4}$$

Similarly, multiplying $a_2b_{k+2,2k+2}$ on the right sides of (3.4) yields $(a_2b_{k+2,2k+2})^2 = 0$, and so $a_2b_{k+2,2k+2} = 0$ by the reduceness of R. Thus (3.4) can be simplified into the next equality

$$a_3b_k + a_4b_{k-1} + \cdots + a_{k-1}b_4 + a_kb_3 + a_{k+1,2k+1}b_2 = 0. \tag{3.5}$$

Applying the same technique to (3.5), we can get $a_3b_k = 0$. Continuing this process, finally we have $[\alpha_{k+1}\beta_1]_{i,j} = 0$. It follows that $\alpha_m\beta_1 = 0$ implies $[\alpha_m\beta_1]_{i,j} = 0$ when $m = 1, 2, \ldots, k+1$. Inductively, assume that Claim is valid in the case $m = k+i$ for $i < k-1$. We prove its validity for $m = k+i+1$. Noticing that $\alpha_{k+i} = (0, \ldots, 0, a_1, a_2, \ldots, a_k, a_{k+2-i,2k+2-i}, \ldots, a_{k+2-i,2k+1})$ in which the occurrence of 0 is $k+1-i$, there are $k+i$ nonzero components in α_{k+i} formally.

By inductive hypothesis, we have $[\alpha_m\beta_1]_{i,j}=0$ for all $1\le m\le k+i$. In particular, $\alpha_{k+i}\beta_1=0$ implies $[\alpha_{k+i}\beta_1]_{l,j}=0$, equivalently, each term on the right side of the following equality

$$\begin{aligned}&a_1b_{k+2-i,2k+2}+a_2b_{k+3-i,2k+2}+\cdots+a_{i+1}b_{k+2,2k+2}+a_{i+2}b_k+\cdots+a_kb_{i+2}+\\&\quad a_{k+2-i,2k+2-i}b_{i+1}+a_{k+2-i,2k+3-i}b_i+\cdots+a_{k+2-i,2k+1}b_2=0\end{aligned}\tag{3.6}$$

is zero. Substituting i for $i+1$ in the equality (3.6), we obtain the expression $\alpha_{k+i+1}\beta_1=0$,

$$\begin{aligned}&a_1b_{k+1-i,2k+2}+a_2b_{k+2-i,2k+2}+\cdots+a_{i+2}b_{k+2,2k+2}+a_{i+3}b_k+\cdots+a_kb_{i+3}+\\&\quad a_{k+1-i,2k+1-i}b_{i+2}+a_{k+1-i,2k+2-i}b_{i+1}+\cdots+a_{k+1-i,2k+1}b_2=0.\end{aligned}\tag{3.7}$$

Multiplying $a_1b_{k+1-i,2k+2}$ on the right sides of (3.7), we have $(a_1b_{k+1-i,2k+2})^2=0$ by the conclusion $[\alpha_m\beta_1]_{i,j}=0$ for all $1\le m\le k+i$, and so $a_1b_{k+1-i,2k+2}=0$. Thus (3.7) becomes

$$\begin{aligned}&a_2b_{k+2-i,2k+2}+a_3b_{k+3-i,2k+2}+\cdots+a_{i+2}b_{k+2,2k+2}+a_{i+3}b_k+\cdots+a_kb_{i+3}+\\&\quad a_{k+1-i,2k+1-i}b_{i+2}+a_{k+1-i,2k+2-i}b_{i+1}+\cdots+a_{k+1-i,2k+1}b_2=0.\end{aligned}\tag{3.8}$$

Similarly, multiplying $a_2b_{k+2-i,2k+2}$ on the right sides of (3.8), we may get $a_2b_{k+2-i,2k+2}=0$. Continuing this process, there is no doubt that we can get $[\alpha_{k+i+1}\beta_1]_{s,t}=0$ in the final.

Case 3. In the case $m=2k$, we proceed by using the conclusions of Cases 1 and 2.

In this case $\alpha_{2k-1}\beta_1=a_1b_{3,2k+2}+a_2b_{4,2k+2}+\cdots+a_kb_{k+2,2k+2}+a_{3,k+2}b_k+\cdots+a_{3,2k+1}b_2=0$, $\alpha_{2k}\beta_1=a_1b_{2,2k+2}+a_2b_{3,2k+2}+\cdots+a_kb_{k+1,2k+2}+a_{2,k+2}b_{k+2,2k+2}+a_{2,k+3}b_k+\cdots+a_{2,2k+1}b_2$. Note that $[\alpha_m\beta_1]_{i,j}=0$ for all $m\le 2k-1$ by the conclusions of the previous Cases. Multiplying $a_1b_{2,2k+2}$ on the right sides of $\alpha_{2k}\beta_1=0$, we have $(a_1b_{2,2k2})^2=0$, and so $a_1b_{2,2k2}=0$. Thus

$$\begin{aligned}&a_2b_{3,2k+2}+a_3b_{4,2k+2}\cdots+a_kb_{k+1,2k+2}+a_{2,k+2}b_{k+2,2k+2}+\\&\quad a_{2,k+3}b_k+\cdots+a_{2,2k+1}b_2=0\end{aligned}\tag{3.9}$$

from $\alpha_{2k}\beta_1=0$. Similarly, multiplying $a_2b_{3,2k+2}$ on the right sides of (3.9), it follows that $a_2b_{3,2k+2}=0$. Continuing this process, finally $[\alpha_{2k}\beta_1]_{i,j}=0$, proving the validity of Claim.

Claim implies that there exists $r\in R$ such that $A_1\beta_1=(r,0,\ldots,0)^{\mathrm{T}}$.

Now we prove that $AB=0$ implies $ACB\in Q_l(T^k_{2k+2}(R))$ for any $C\in T^k_{2k+2}(R)$. Write $C=\left(\begin{smallmatrix}C_1&\gamma_1\\0&c_1\end{smallmatrix}\right)$, where $C_1\in T^k_{2k+1}(R)$, $\gamma_1,\in R^{2k+1}$ and $c_1\in R$. Then it is easily checked

$$ACB=\begin{pmatrix}A_1C_1B_1 & A_1C_1\beta_1+A_1\gamma_1b_1+\alpha_1c_1b_1\\ 0 & a_1c_1b_1\end{pmatrix}.$$

We have showed that $AB=0$ implies $A_1B_1=0$, $a_1b_1=0$, and $\alpha_1b=0$. Since R is a reduced ring, $T^k_{2k+1}(R)$ is a semicommutative ring by [28, Theorem 1]. It follows that $A_1C_1B_1=0$ and $a_1c_1b_1=0$. From the right duo property of R, we have $Rb_1\subseteq b_1R$. This gives $c_1b_1=b_1c_1'$ for some $c_1'\in R$. Thus $\alpha_1b_1=0$ implies $\alpha_1c_1b_1=0$ by taking into account the components of α_1. Meanwhile there exists $\gamma'\in R^{2k+1}$ such that $\gamma b_1=b_1I_{2k+1}\gamma'$ with help of Lemma 3.9 (1) and $A_1B_1=0$ implies $A_1b_1I_{2k+1}=0$ by [28, Lemma 1]. This gives $A_1\gamma_1b_1=A_1b_1I_{2k+1}\gamma'=0$. By the multiplication of block matrix, it is easy to obtain the following expression of

$$A_1C_1\beta_1 = \begin{pmatrix} \alpha_{2k+1}C_1\beta_1 \\ \alpha_{2k}C_1\beta_1 \\ \vdots \\ \alpha_1C_1\beta_1 \end{pmatrix}.$$

We wish to prove $\alpha_{2k}C_1\beta_1 = \alpha_{2k-1}C_1\beta_1 = \cdots = \alpha_1C_1\beta_1 = 0$. Firstly, we show $\alpha_{2k}C_1\beta_1 = 0$. Write C_1 as a row partitioned matrix. There exist $1 \times (2k+1)$ matrices $\xi_{2k+1}, \ldots, \xi_1$ such that

$$C_1 = \begin{pmatrix} \xi_{2k+1} \\ \xi_{2k} \\ \vdots \\ \xi_1 \end{pmatrix} \text{ and so } C_1\beta_1 = \begin{pmatrix} \xi_{2k+1}\beta_1 \\ \xi_{2k}\beta_1 \\ \vdots \\ \xi_1\beta_1 \end{pmatrix},$$

where $\xi_i = (0, \ldots, 0, c_1, \ldots, c_i)$, $\xi_{k+i} = (0, \ldots, 0, c_1, c_2, \ldots, c_k, c_{k+2-i,2k+2-i}, \ldots, c_{k+2-i,2k+1})$ for $i = 1, 2, \ldots, k$ and the occurrence of 0 in the component of ξ_{k+i} is $k+1-i$. It yields that

$$\alpha_{2k}C_1\beta_1 = a_1\xi_{2k}\beta_1 + a_2\xi_{2k-1}\beta_1 + \cdots + a_k\xi_{k+1}\beta_1 + a_{2,k+2}\xi_{k+2}\beta_1 + \cdots + a_{2,2k+1}\xi_1\beta_1.$$

Now we show that each term of $\alpha_{2k}C_1\beta_1$ is zero. By a simple computation, we have

$$\xi_{2k}\beta_1 = c_1b_{2,2k+2} + c_2b_{3,2k+2} + \cdots + c_kb_{k+1,2k+2} + c_{2,k+2}b_{k+2,2k+2} + c_{2,k+3}b_k + \cdots + c_{2,2k}b_3 + c_{2,2k+1}b_2.$$

On the other hand, with help of the conclusions of Claim, it yields the following equalities

$$a_1b_{2,2k+2} = a_1b_{3,2k+2} = \cdots = a_1b_{k+1,2k+2} = a_1b_{k+2,2k+2} = a_1b_k = \cdots = a_1b_2 = 0.$$

We conclude that $a_1c_1b_{2,2k+2} = a_1c_2b_{3,2k+2} = \cdots = a_1c_kb_{k+1,2k+2} = \cdots = a_1c_{2,2k+1}b_2 = 0$, since R is a semicommutative ring. This implies that the first term of $\alpha_{2k}C_1\beta_1$ is zero, i.e., $a_1\xi_{2k}\beta_1 = 0$ from the previous argument. Similarly, it can be proved that

$$a_2\xi_{2k-1}\beta_1 = \cdots = a_k\xi_{k+1}\beta_1 = \cdots = a_{2,2k+1}\xi_1\beta_1 = 0$$

and so $\alpha_{2k}C_1\beta_1 = 0$. Continuing this process, we have $\alpha_{2k-1}C_1\beta_1 = \cdots = \alpha_1C_1\beta_1 = 0$. We conclude $A_1C_1\beta_1 = (a, 0, \ldots, 0)^{\mathrm{T}}$ for some $a \in R$, i.e., $ACB = aE_{1,2k+2}$. It is easily checked $ACB \in Q_l(T^k_{2k+2}(R))$ by the right duo property of R. This completes the proof of Theorem 3.10. □

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