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Relative (b,c)-Inverses with Respect to a Ring Endomorphism

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Abstract We study the relative properties of (b,c)-inverses with respect to a ring endomorphism. A new class of generalized inverses named α-(b,c)-inverse is introduced and studied in a more general setting. We show by giving an example that (b,c)-inverses behave quite differently from α-(b,c)-inverses. The condition that an α-(b,c)-invertible element is precisely a (b,c)-invertible element is investigated. We also study the strongly clean decompositions for α-(b,c)-inverses. Some well-known results on (b,c)-inverses are extended and unified.

Keywords α-(b,c)-inverse; Cline's formula; Jacobson's lemma; strongly clean decomposition

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1. Introduction

Throughout this paper, R is a unitary associative ring and α is an endomorphism of R. The center and units of R are denoted by $C(R)$ and $U(R)$, respectively. Furthermore, we denote the set of all idempotent elements of R by $E(R)$. An involution $*$: $R \to R$ is an anti-isomorphism which satisfies $(a^*)^* = a$, $(ab)^* = b^*a^*$, $(a+b)^* = a^* + b^*$ for all $a, b \in R$. For any $a \in R$, we use $\mathrm{lann}(a) = \{x \in R : xa = 0\}$ and $\mathrm{rann}(a) = \{x \in R : ax = 0\}$ to denote the left and right annihilator of a, respectively. A ring R is abelian if every idempotent is central. According to [1], an endomorphism α of a ring R is called rigid if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$, and R is an α-rigid ring [2] if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism, and α-rigid rings are reduced rings. Recall from [3] that a ring R is right α-reversible if whenever $ab = 0$ for $a, b \in R$, then $b\alpha(a) = 0$.

An element $a \in R$ is called regular if there is $x \in R$ such that $axa = a$. Such an x is called an inner inverse of a and is denoted as a^- and the set of all inner invertible elements of R is denoted by R^-. An element $a \in R$ is group invertible if there is $y \in R$ such that $aya = a$, $yay = y$, $ay = ya$. The set of all group invertible elements is denoted by $R^{\#}$. It is well known that a is group invertible if and only if $a \in a^2R \cap Ra^2$. Given a ring R and $a, b, c, y \in R$, recall from [4] that y is the (b,c)-inverse of a if $yay = y$, $yR = bR$ and $Ry = Rc$, and is denoted by $a^{(b,c)}$. It was shown in [4, Theorem 2.2] that an element a is (b,c)-invertible if and only if $b \in Rcab$ and $c \in cabR$. The set of all (b,c)-invertible elements of R is denoted by $R^{(b,c)}$. More results

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on (b,c)-inverses can be found in [5–10]. According to [11], an element $a \in R$ is central Drazin invertible, if there is $x \in R$ such that $xa \in C(R)$, $xax = x$ and $a^{n+1}x = a^n$ for some integer $n \geq 0$.

In this paper, we further study the properties of (b,c)-inverses from a new perspective. More precisely, we study the relative properties of (b,c)-inverses with respect to a ring endomorphism. The new concept of α-(b,c)-inverses is introduced and investigated. In particular, it is easy to see that α-(b,c)-inverse is just the general (b,c)-inverse when $\alpha = 1_R$. However, we shall give an example to show that an α-(b,c)-invertible element need not be (b,c)-invertible, and a (b,c)-invertible element need not be α-(b,c)-invertible. Furthermore, the condition that an α-(b,c)-invertible element is precisely a (b,c)-invertible element is discussed. Various properties including Jacobson's lemma and Cline's formula of α-(b,c)-inverses are studied. Strongly clean decompositions for α-(b,c)-inverses are also considered.

This paper is organized as follows:

In Section 2, we define and investigate the α-(b,c)-inverse of an element in a unitary associative ring. An example is given to show that α-(b,c)-invertible elements are quite different from (b,c)-invertible elements (Example 2.2). If $a,b,c \in R$ and $\alpha(e) = e$ for any idempotent e, it is proved that a is (b,c)-invertible if and only if a is α-(b,c)-invertible with $b,c \in R^-$ (Proposition 2.3). In Section 3, we further study the properties of α-(b,c)-invertible elements, including Jacobson's lemma, strongly clean decompositions and Cline's formula (Corollary 3.12, Theorems 3.15 and 3.5). In particular, we obtain the strongly clean decomposition of Bott-Duffin (e,f)-inverse (Corollary 3.18).

2. α-(b,c)-inverses and their properties

In this section, we define and study a more general case of (b,c)-inverses that is closely related to an endomorphism of a ring, and is called α-(b,c)-inverse. However, we shall give an example to show that in general α-(b,c)-invertible elements are different with (b,c)-invertible elements.

We begin with the following definition.

Definition 2.1 *Let $a,b,c \in R$ and let α be an endomorphism of R. We say that a is α-(b,c)-invertible if there is $x \in R$ such that*

$$xax = x,\ xR = \alpha(b)R,\ Rx = R\alpha(c).$$

Any element x satisfying the above conditions is called the α-(b,c)-inverse of a, denoted as $a_\alpha^{(b,c)}$. The set of all α-(b,c)-invertible elements of R is denoted by $R_\alpha^{(b,c)}$.

In particular, if $\alpha = 1_R$, then it is clear that α-(b,c)-inverses coincide with the general (b,c)-inverses. Moreover, it is obvious that the α-(b,c)-inverse of an element is unique, and $a \in R$ is α-(b,c)-invertible if and only if $\alpha(b) \in R\alpha(c)a\alpha(b)$ and $\alpha(c) \in \alpha(c)a\alpha(b)R$.

The following example shows that α-(b,c)-invertible elements can be quite different from (b,c)-invertible elements.

Example 2.2 Let $\mathbb{Z}$ be the ring of integers. Consider the ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{Z} \right\}.$$

Let $\alpha : R \to R$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Take the elements

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

in R. Then it is clear that a is (b,c)-invertible. However, $\alpha(c) = c, \alpha(b) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. This shows that $\alpha(b) \in R\alpha(c)a\alpha(b)$ and $\alpha(c) \notin \alpha(c)a\alpha(b)R$. Therefore, a is not α-(b,c)-invertible.

On the other hand, let

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = c = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in R,$$

then $cab = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and a is α-(b,c)-invertible. However, $b \notin Rcab$ and $c \notin cabR$, that is, a is not (b,c)-invertible.

The next proposition shows the equivalence of α-(b,c)-invertibility and (b,c)-invertibility of an element.

Proposition 2.3 *Let $a, b, c \in R$ and $\alpha(e) = e$ for any $e \in E(R)$. Then $a \in R^{(b,c)}$ if and only if $a \in R_{\alpha}^{(b,c)}$ and $b, c \in R^{-}$.*

Proof If $a \in R^{(b,c)}$, then there exist $m, n \in R$ such that $b = mcab$, $c = cabn$. It is clear that $b, c \in R^{-}$. Since $bn = mc$, we get $abn, mca \in E(R)$. Since $c^{-}c \in E(R)$, it follows that

$$\alpha(b) = \alpha(mcab) = mca\alpha(b) = mc\alpha(c^{-})\alpha(c)a\alpha(b) \in R\alpha(c)a\alpha(b).$$

Similarly, we conclude that $\alpha(c) = \alpha(c)abn = \alpha(c)a\alpha(b)\alpha(b^{-})bn \in \alpha(c)a\alpha(b)R$.

Conversely, if $a \in R_{\alpha}^{(b,c)}$, then there are $s, t \in R$ such that $\alpha(b) = s\alpha(c)a\alpha(b)$ and $\alpha(c) = \alpha(c)a\alpha(b)t$. This shows that

$$\alpha(b)\alpha(b^{-}) = s\alpha(c)a\alpha(b)\alpha(b^{-}), \quad \alpha(c^{-})\alpha(c) = \alpha(c^{-})\alpha(c)a\alpha(b)t.$$

Therefore, we have $bb^{-} = s\alpha(c)c^{-}cabb^{-}$ and $c^{-}c = c^{-}cabb^{-}\alpha(b)t$. Then $b = s\alpha(c)c^{-}cab \in Rcab$ and $c = cabb^{-}\alpha(b)t \in cabR$. □

Note that if R is an α-rigid ring, then $\alpha(e) = e$ for any $e \in E(R)$ by [3, Proposition 2.5]. Also if α is a monomorphism and R is a right α-reversible ring, then $\alpha(e) = e$ for any $e \in E(R)$ by [3, Theorem 2.13]. Thus the rings that satisfy the condition $\alpha(e) = e$ for any $e \in E(R)$ exist.

Proposition 2.4 *Let $a, b, c, x \in R$ such that $\alpha(e) = e$ for any $e \in E(R)$. If x is the (b,c)-inverse of a, then x is the α-(b,c)-inverse of a.*

Proof If x is the (b,c)-inverse of a, then $xab=b$, $cax=c$ and $b,c\in R^-$. It follows that $\alpha(b)=xa\alpha(b)$ and $\alpha(c)=\alpha(c)ax$ since $xa,\ ax\in E(R)$. Also since $x\in bR=\alpha(b)\alpha(b^-)bR\subseteq\alpha(b)R$ and $x\in Rc=Rc\alpha(c^-)\alpha(c)\subseteq R\alpha(c)$, we get $xR=\alpha(b)R$ and $Rx=R\alpha(c)$. Combining with $xax=x$, then x is the α-(b,c)-inverse of a. $\square$

In particular, if an endomorphism α of a ring R is an automorphism, then we have the following equivalence.

Theorem 2.5 *Let $a,b,c\in R$ and let α be an automorphism of R. Then $a\in R^{(b,c)}$ if and only if $\alpha(a)\in R_\alpha^{(b,c)}$.*

Proof If $\alpha(a)\in R_\alpha^{(b,c)}$, then there are $s,t\in R$ such that $\alpha(b)=s\alpha(c)\alpha(a)\alpha(b)$ and $\alpha(c)=\alpha(c)\alpha(a)\alpha(b)t$. Since α is an epimorphism, there are $g,h\in R$ such that $s=\alpha(g)$ and $t=\alpha(h)$. This implies that $\alpha(b)=\alpha(g)\alpha(c)\alpha(a)\alpha(b)=\alpha(gcab)$, $\alpha(c)=\alpha(c)\alpha(a)\alpha(b)\alpha(h)=\alpha(cabh)$. Since α is a monomorphism, we get $b=gcab\in Rcab$, $c=cabh\in cabR$. Therefore, we have $a\in R^{(b,c)}$. The converse is clear. $\square$

The next corollary shows a particular case of α-(b,c)-invertible elements.

Corollary 2.6 *Let $a\in R^-, k\in\mathbb{N}$ and $\alpha(e)=e$ for any idempotent e. If $e\in C(R)$, then a is α-(a^k,a^k)-invertible if and only if a is central Drazin invertible.*

The proof of the following auxiliary lemma is similar to that of [12, Corollary 2.4].

Lemma 2.7 *Let $a,b,c,x\in R$. If x is the α-(b,c)-inverse of a, then we have the following assertions:*

(1) If $\alpha(b),\alpha(c)\in comm(a)$, then $x\in comm(a)$.

(2) If $\alpha(b),\alpha(c)\in comm^2(a)$, then $x\in comm(\alpha(b),\alpha(c))$ and $\alpha(b)\in comm(\alpha(c))$.

The following example shows that the endomorphism α in Lemma 2.7 actually exists.

Example 2.8 Let R and $\alpha:R\to R$ be the ring and the ring endomorphism in Example 2.2. Take $a=b=\begin{pmatrix}1&0\\0&0\end{pmatrix}$. Then $\alpha(b)=b$. It is clear that $\alpha(b)\in comm(a)$. Moreover, let any $k\in R$ such that

$$k=\begin{pmatrix}p&q\\0&s\end{pmatrix}\in comm(a)$$

for some $p,q,s\in\mathbb{Z}$. Then it can be easily checked that k has the form of $k=\begin{pmatrix}p&0\\0&s\end{pmatrix}$. Therefore, $\alpha(b)\in comm(k)$, that is, $\alpha(b)\in comm^2(a)$.

For any two elements $a,d\in R$, the next proposition shows the equivalence of α-(b,c)-invertibility of a and $a+d$ under some suitable conditions.

Proposition 2.9 *Let $a,b,c,d\in R$ with $d\in C(R)$ and $d^2=0$. Then $a\in R_\alpha^{(b,c)}$ if and only if $a+d\in R_\alpha^{(b,c)}$.*

Proof If $a\in R_\alpha^{(b,c)}$, then there exist $m,n\in R$ such that $\alpha(b)=m\alpha(c)a\alpha(b)$ and $\alpha(c)=$

$\alpha(c)a\alpha(b)n$. Since $d \in C(R)$ and $d^2 = 0$, it follows that

$$\begin{aligned}\alpha(b) &= m\alpha(c)(a+d)\alpha(b) - m\alpha(c)d\alpha(b)\\ &= m\alpha(c)(a+d)\alpha(b) - m\alpha(c)d\alpha(b) - m\alpha(c)dm\alpha(c)d\alpha(b)\\ &= [m - m\alpha(c)dm]\alpha(c)(a+d)\alpha(b) \in R\alpha(c)(a+d)\alpha(b).\end{aligned}$$

Similarly, we can get $\alpha(c) = \alpha(c)(a+d)\alpha(b)[n - nd\alpha(b)n] \in \alpha(c)(a+d)\alpha(b)R$.

Conversely, if $a + d \in R_\alpha^{(b,c)}$, then there is $s \in R$ such that $\alpha(b) = s\alpha(c)(a+d)\alpha(b)$. This implies that

$$\alpha(b) = s\alpha(c)a\alpha(b) + s\alpha(c)ds\alpha(c)(a+d)\alpha(b) = [s + s\alpha(c)ds]\alpha(c)a\alpha(b) \in R\alpha(c)a\alpha(b).$$

Also we can prove $\alpha(c) \in \alpha(c)a\alpha(b)R$ in a similar way, as desired. □

Theorem 2.10 *Let $a,b,c,d \in R$ with $\alpha(b),\alpha(c) \in comm(a,d)$. If $a,d \in R_\alpha^{(b,c)}$, then $(da)^k \in R_\alpha^{(b,c)}$ for $k \in \mathbb{N}$.*

Proof If $a \in R_\alpha^{(b,c)}$, then there exist $m,n \in R$ such that $\alpha(b) = m\alpha(c)a\alpha(b)$ and $\alpha(c) = \alpha(c)a\alpha(b)n$. Since $\alpha(b),\alpha(c) \in comm(a)$, we have $a\alpha(b)n = m\alpha(c)a$ by Lemma 2.7. If $d \in R_\alpha^{(b,c)}$, then there exist $s,t \in R$ such that $\alpha(b) = s\alpha(c)d\alpha(b)$ and $\alpha(c) = \alpha(c)d\alpha(b)t$. Since $\alpha(b),\alpha(c) \in comm(d)$, we get $d\alpha(b)t = s\alpha(c)d$. We conclude that

$$\begin{aligned}\alpha(b) &= m\alpha(c)d\alpha(b)ta\alpha(b) = m\alpha(c)s\alpha(c)a\alpha(b)nda\alpha(b) = \alpha(b)ns\alpha(c)m\alpha(c)ada\alpha(b)\\ &= \alpha(b)ns\alpha(c)m\alpha(c)d\alpha(b)tada\alpha(b) = \alpha(b)ns\alpha(c)m\alpha(c)s\alpha(c)dada\alpha(b)\\ &= \cdots = [m\alpha(c)s\alpha(c)]^{k-1}m\alpha(c)s\alpha(c)(da)^k\alpha(b) \in R\alpha(c)(da)^k\alpha(b),\\ \alpha(c) &= \alpha(c)dm\alpha(c)a\alpha(b)t = \alpha(c)das\alpha(c)d\alpha(b)n\alpha(b)t = \alpha(c)dad\alpha(b)t\alpha(b)n\alpha(b)t\\ &= \alpha(c)dadm\alpha(c)a\alpha(b)t\alpha(b)n\alpha(b)t = \alpha(c)dada\alpha(b)n\alpha(b)t\alpha(b)n\alpha(b)t\\ &= \cdots = \alpha(c)(da)^k\alpha(b)n\alpha(b)t[\alpha(b)n\alpha(b)t]^{k-1} \in \alpha(c)(da)^k\alpha(b)R.\end{aligned}$$

Therefore, $(da)^k \in R_\alpha^{(b,c)}$ for $k \in \mathbb{N}$. □

Corollary 2.11 *Let $a,b,c \in R$ with $\alpha(b),\alpha(c) \in comm(a)$. If $a \in R_\alpha^{(b,c)}$, then $a^k \in R_\alpha^{(b,c)}$ for $k \in \mathbb{N}$. In this case, $(a^k)_\alpha^{(b,c)} = (a_\alpha^{(b,c)})^k$.*

Proof If $a \in R_\alpha^{(b,c)}$, then $a^k \in R_\alpha^{(b,c)}$ by Theorem 2.10. Let $x = a_\alpha^{(b,c)}$. Then we have

$$\alpha(b) = xaxa\alpha(b) = x^2a^2\alpha(b) = \cdots = x^ka^k\alpha(b),$$

$$\alpha(c) = \alpha(c)axax = \alpha(c)a^2x^2 = \cdots = \alpha(c)a^kx^k$$

by Lemma 2.7. Since $x \in \alpha(b)R$ and $x \in R\alpha(c)$, we have $x^k \in \alpha(b)R$ and $x^k \in R\alpha(c)$. Hence, $a^k \in R_\alpha^{(b,c)}$ and $(a^k)_\alpha^{(b,c)} = (a_\alpha^{(b,c)})^k$. □

3. Further results on α-(b,c)-invertible elements

In this section, we continue to study some topics related to α-(b,c)-invertible elements. We also explore the Jacobson's lemma, Cline's formula and strongly clean decompositions for α-

(b,c)-invertible elements.

Theorem 3.1 *Let $a,b,c,d\in R$ such that $a,d\in R_\alpha^{(b,c)}$ and $ad^2=dad$. If $\alpha(b),\alpha(c)\in comm(a,d)$, then $1+a_\alpha^{(b,c)}d\in R_\alpha^{(b,c)}$ if and only if $a+d\in R_\alpha^{(b,c)}$.*

Proof Since $d\in R_\alpha^{(b,c)}$, there are $s,t\in R$ such that $\alpha(b)=s\alpha(c)d\alpha(b)$ and $\alpha(c)=\alpha(c)d\alpha(b)t$. Since $\alpha(b),\alpha(c)\in comm(a)$, we have $aa_\alpha^{(b,c)}=a_\alpha^{(b,c)}a$ by Lemma 2.7. If $1+a_\alpha^{(b,c)}d\in R_\alpha^{(b,c)}$, then there exist $m,n\in R$ such that $\alpha(b)=m\alpha(c)(1+a_\alpha^{(b,c)}d)\alpha(b)$ and $\alpha(c)=\alpha(c)(1+a_\alpha^{(b,c)}d)\alpha(b)n$. It follows that

$$\begin{aligned}\alpha(b)&=m\alpha(c)aa_\alpha^{(b,c)}(1+a_\alpha^{(b,c)}d)\alpha(b)=m\alpha(c)a_\alpha^{(b,c)}(a+d)\alpha(b)\\&=m\alpha(c)k\alpha(c)(a+d)\alpha(b)\in R\alpha(c)(a+d)\alpha(b),\end{aligned}$$

since $a_\alpha^{(b,c)}=k\alpha(c)=\alpha(b)l$ for $k,l\in R$. Also we can conclude that

$$\begin{aligned}\alpha(c)&=\alpha(c)(1+a_\alpha^{(b,c)}d)a_\alpha^{(b,c)}a\alpha(b)n\\&=\alpha(c)a_\alpha^{(b,c)}a\alpha(b)n+\alpha(c)a_\alpha^{(b,c)}da\alpha(b)l\alpha(b)n\\&=\alpha(c)a_\alpha^{(b,c)}a\alpha(b)n+\alpha(c)a_\alpha^{(b,c)}dads\alpha(c)\alpha(b)l\alpha(b)n\\&=\alpha(c)a_\alpha^{(b,c)}a\alpha(b)n+\alpha(c)a_\alpha^{(b,c)}ad^2s\alpha(c)\alpha(b)l\alpha(b)n\\&=\alpha(c)a\alpha(b)l\alpha(b)n+\alpha(c)d\alpha(b)l\alpha(b)n\\&=\alpha(c)(a+d)\alpha(b)l\alpha(b)n\in\alpha(c)(a+d)\alpha(b)R,\end{aligned}$$

since $\alpha(b),\alpha(c)\in comm(d)$. Therefore, $a+d\in R_\alpha^{(b,c)}$.

Conversely, if $a+d\in R_\alpha^{(b,c)}$, then there exist $s',t'\in R$ such that $\alpha(b)=s'\alpha(c)(a+d)\alpha(b)$ and $\alpha(c)=\alpha(c)(a+d)\alpha(b)t'$. This implies that

$$\begin{aligned}\alpha(b)&=s'\alpha(c)a\alpha(b)+s'\alpha(c)aa_\alpha^{(b,c)}d\alpha(b)=s'\alpha(c)a(1+a_\alpha^{(b,c)}d)\alpha(b)\\&=s'a\alpha(c)(1+a_\alpha^{(b,c)}d)\alpha(b)\in R\alpha(c)(1+a_\alpha^{(b,c)}d)\alpha(b).\end{aligned}$$

In addition, we also have

$$\begin{aligned}\alpha(c)&=\alpha(c)a\alpha(b)t'+\alpha(c)a_\alpha^{(b,c)}ad\alpha(b)t'=\alpha(c)a\alpha(b)t'+\alpha(c)a_\alpha^{(b,c)}ads\alpha(c)d\alpha(b)t'\\&=\alpha(c)a\alpha(b)t'+\alpha(c)a_\alpha^{(b,c)}ad^2s\alpha(c)\alpha(b)t'=\alpha(c)a\alpha(b)t'+\alpha(c)a_\alpha^{(b,c)}dads\alpha(c)\alpha(b)t'\\&=\alpha(c)a\alpha(b)t'+\alpha(c)a_\alpha^{(b,c)}da\alpha(b)t'=\alpha(c)(1+a_\alpha^{(b,c)}d)a\alpha(b)t'\\&=\alpha(c)(1+a_\alpha^{(b,c)}d)\alpha(b)at'\in\alpha(c)(1+a_\alpha^{(b,c)}d)\alpha(b)R.\end{aligned}$$

Therefore, $1+a_\alpha^{(b,c)}d\in R_\alpha^{(b,c)}$ and we are done. □

Corollary 3.2 *Let $a,b,c\in R$ such that $a\in R^{(b,c)}$. If $b,c\in comm(a)$, then $1+a^{(b,c)}\in R^{(b,c)}$ if and only if $1+a\in R^{(b,c)}$.*

Proof Since $a\in R^{(b,c)}$, there exist $m,n\in R$ such that $b=mcab$ and $c=cabn$. Also since $b,c\in comm(a)$, we have $a^{(b,c)}=a^{(b,c)}a$ by [12, Corollary 2.4]. If $1+a^{(b,c)}\in R^{(b,c)}$, then there are $g,h\in R$ such that $b=gc(1+a^{(b,c)})b$ and $c=c(1+a^{(b,c)})bh$. It yields that

$$b=gcaa^{(b,c)}(1+a^{(b,c)})bh=gc(aa^{(b,c)}+a^{(b,c)})b$$

$$= gca^{(b,c)}(1+a)b = gcpc(1+a)b,$$

since $a^{(b,c)} = pc = bq$ for $p, q \in R$. Therefore, we conclude that

$$c = c(1+a^{(b,c)})a^{(b,c)}abh = caa^{(b,c)}bh + ca^{(b,c)}bh = cabqbh + cbqbh$$
$$= c(1+a)bqbh \in c(1+a)bR.$$

Therefore, $1+a \in R^{(b,c)}$.

Conversely, if $1+a \in R^{(b,c)}$, then there are $m', n' \in R$ such that $b = m'c(1+a)b$ and $c = c(1+a)bn'$. This shows that

$$b = m'caa^{(b,c)}b + m'cab = m'aca^{(b,c)}b + m'acb$$
$$= m'ac(1+a^{(b,c)})b \in Rc(1+a^{(b,c)})b.$$

We also have

$$c = ca^{(b,c)}abn' + cabn' = ca^{(b,c)}ban' + cban'$$
$$= c(1+a^{(b,c)})ban' \in c(1+a^{(b,c)})bR.$$

This implies that $1+a^{(b,c)} \in R^{(b,c)}$ and we are done. □

It was proved in [13] that if ab is Drazin invertible, then so is ba, and $(ba)^D = b[(ab)^D]^2a$. This equality is called Cline's formula. Next, we discuss the Cline's formula for α-(b,c)-invertible elements.

Proposition 3.3 *Let $a, b, c, d, g, h \in R$ such that $adh = hgh$ and $hga = ada$. If $\alpha(b), \alpha(c) \in comm(ad, hg)$, then $ad \in R_\alpha^{(b,c)}$ if and only if $hg \in R_\alpha^{(b,c)}$. In this case, $(ad)_\alpha^{(b,c)} = (hg)_\alpha^{(b,c)}$.*

Proof If $ad \in R_\alpha^{(b,c)}$, then there exist $m, n \in R$ such that $\alpha(b) = m\alpha(c)ad\alpha(b)$ and $\alpha(c) = \alpha(c)ad\alpha(b)n$. Since $\alpha(b), \alpha(c) \in comm(ad, hg)$, it follows that

$$\alpha(b) = m\alpha(c)adadm\alpha(c)\alpha(b) = m\alpha(c)hgadm\alpha(c)\alpha(b),$$
$$\alpha(c) = \alpha(c)adad\alpha(b)n\alpha(b)n = \alpha(c)hgad\alpha(b)n\alpha(b)n$$

by Lemma 2.7. Therefore, we have

$$\alpha(b) = m\alpha(c)hg\alpha(b) \in R\alpha(c)hg\alpha(b),$$

$$\alpha(c) = \alpha(c)hg\alpha(b)n \in \alpha(c)hg\alpha(b)R.$$

Let $x = (ad)_\alpha^{(b,c)}$. This implies that

$$x = xadadx^2 = xhgadx^2 = xhgx,$$

$$xhg\alpha(b) = xhgxad\alpha(b) = xad\alpha(b) = \alpha(b),$$

$$\alpha(c)hgx = \alpha(c)adxhgx = \alpha(c)adx = \alpha(c).$$

Combining with $x \in \alpha(b)R$ and $x \in R\alpha(c)$, we get $(ad)_\alpha^{(b,c)} = (hg)_\alpha^{(b,c)}$. Conversely, if $hg \in R_\alpha^{(b,c)}$, then we can show $ad \in R_\alpha^{(b,c)}$ similarly. □

Corollary 3.4 *Let R be an abelian ring and let $a,b,c,d,g,h\in R$ such that $adh=hgh$ and $hga=ada$. Then $ad\in R_\alpha^{(b,c)}$ if and only if $hg\in R_\alpha^{(b,c)}$. In this case, $(ad)_\alpha^{(b,c)}=(hg)_\alpha^{(b,c)}$.*

More generally, we can extend Proposition 3.3 to the following new version of Cline's formula for α-(b,c)-invertible elements.

Theorem 3.5 *Let $a,b,c,d,g,h\in R$ such that $\alpha(b),\alpha(c)\in comm(ad,g,h)$. If $h\in R_\alpha^{(b,c)}$ with $adh=hgh$ and $hga=ada$, then $ad\in R_\alpha^{(b,c)}$ if and only if $gh\in R_\alpha^{(b,c)}$. In this case, we have $(gh)_\alpha^{(b,c)}=g((ad)_\alpha^{(b,c)})^2h$, $(ad)_\alpha^{(b,c)}=h((gh)_\alpha^{(b,c)})^2g$.*

Proof If $gh\in R_\alpha^{(b,c)}$, then there exist $s,t\in R$ such that $\alpha(b)=s\alpha(c)gh\alpha(b)=s\alpha(c)s\alpha(c)ghgh\alpha(b)$ and $\alpha(c)=\alpha(c)gh\alpha(b)t=\alpha(c)ghgh\alpha(b)t\alpha(b)t$ since $\alpha(b),\alpha(c)\in comm(gh)$. Hence $\alpha(b)=(s\alpha(c))^2gadh\alpha(b)$ and $\alpha(c)=\alpha(c)gadh(\alpha(b)t)^2$. Since $h\in R_\alpha^{(b,c)}$, there is $n'\in R$ such that $\alpha(c)=\alpha(c)h\alpha(b)n'$. Then we have

$$(s\alpha(c))^2gadh\alpha(b)n'=\alpha(b)n',$$

$$\alpha(c)=\alpha(c)h\alpha(b)n'gadh(\alpha(b)t)^2=\alpha(c)\alpha(b)n'hgadh(\alpha(b)t)^2,$$

since $\alpha(b),\alpha(c)\in comm(h)$. This implies that

$$\alpha(b)=hs\alpha(c)s\alpha(c)gad\alpha(b)=hs\alpha(c)sg\alpha(c)ad\alpha(b)\in R\alpha(c)ad\alpha(b)$$

and $\alpha(c)h=h\alpha(c)=h\alpha(c)\alpha(b)n'hgadh(\alpha(b)t)^2$. Thus, we have

$$\begin{aligned}\alpha(c)&=\alpha(c)hgadh(\alpha(b)t)^2\alpha(b)n'=\alpha(c)adhgh(\alpha(b)t)^2\alpha(b)n'\\&=\alpha(c)adh\alpha(b)ts\alpha(c)gh\alpha(b)n'=\alpha(c)ad\alpha(b)hts\alpha(c)g\in\alpha(c)ad\alpha(b)R.\end{aligned}$$

Moreover, it is clear that $(gh)_\alpha^{(b,c)}=s\alpha(c)=\alpha(b)t$. Let $y=h((gh)_\alpha^{(b,c)})^2g$. Then $yad\alpha(b)=\alpha(b)$ and $\alpha(c)ady=\alpha(c)$. Moreover, we have

$$yady=h((gh)_\alpha^{(b,c)})^2ghgh((gh)_\alpha^{(b,c)})^2g=h((gh)_\alpha^{(b,c)})^2g=y,\ y=\alpha(b)ht(gh)_\alpha^{(b,c)}g\in\alpha(b)R,$$

$$y=h((gh)_\alpha^{(b,c)})^2g=h(gh)_\alpha^{(b,c)}s\alpha(c)g=h(gh)_\alpha^{(b,c)}sg\alpha(c)\in R\alpha(c),$$

that is, $(ad)_\alpha^{(b,c)}=h((gh)_\alpha^{(b,c)})^2g$. The converse can be proved similarly. □

Specifically, we have the following Cline's formula for α-(b,c)-invertible elements.

Corollary 3.6 *Let $a,b,c,d\in R$ such that $\alpha(b),\alpha(c)\in comm(a,d)$. If $a\in R_\alpha^{(b,c)}$, then $ad\in R_\alpha^{(b,c)}$ if and only if $da\in R_\alpha^{(b,c)}$. In this case, we have $(da)_\alpha^{(b,c)}=d((ad)_\alpha^{(b,c)})^2a$, $(ad)_\alpha^{(b,c)}=a((da)_\alpha^{(b,c)})^2d$.*

Corollary 3.7 *Let $a,b,c,d\in R$ such that $b,c\in comm(a,d)$. If $a\in R^{(b,c)}$, then $ad\in R^{(b,c)}$ if and only if $da\in R^{(b,c)}$.*

A ring R is called semicommutative if $ab=0$ implies $aRb=0$ for $a,b\in R$. It can be easily checked that every semicommutative ring is abelian.

Proposition 3.8 *Let R be a semicommutative ring and let $a,b,c,d,g,h\in R$ such that $adh=hgh$ and $hga=ada$. Then $ad\in R_\alpha^{(b,c)}$ if and only if $gh\in R_\alpha^{(b,c)}$. In this case, we have $(gh)_\alpha^{(b,c)}=g((ad)_\alpha^{(b,c)})^2h$, $(ad)_\alpha^{(b,c)}=h((gh)_\alpha^{(b,c)})^3gad$.*

Proof If $ad \in R_{\alpha}^{(b,c)}$, then there exist $m, n \in R$ such that $\alpha(b) = m\alpha(c)ad\alpha(b) = m\alpha(c)hg\alpha(b)$ and $\alpha(c) = \alpha(c)ad\alpha(b)n = \alpha(c)hg\alpha(b)n$ by Corollary 3.4. Since $hga = ada$ and $ad\alpha(b)n, m\alpha(c)hg$, $hg\alpha(b)n \in E(R)$, we conclude that

$$\begin{aligned}\alpha(b) &= m\alpha(c)ad\alpha(b)nad\alpha(b) = \alpha(b)m\alpha(c)adad\alpha(b)n\\ &= \alpha(b)m\alpha(c)hgad\alpha(b)n = \alpha(b)m\alpha(c)had\alpha(b)ng,\end{aligned}$$

$$hgm\alpha(c) = hgm\alpha(c)hg\alpha(b)n = m\alpha(c)hghg\alpha(b)n = m\alpha(c)hg.$$

Therefore, we have

$$\begin{aligned}\alpha(b)h &= \alpha(b)m\alpha(c)ad\alpha(b)nhad\alpha(b)ngh = \alpha(b)adad\alpha(b)n\alpha(b)nm\alpha(c)hadm\alpha(c)gh\\ &= \alpha(b)hgad(\alpha(b)n)^3hadm\alpha(c)gh = \alpha(b)hg(\alpha(b)n)^2hadm\alpha(c)gh,\end{aligned}$$

that is, $\alpha(b)(h - hg(\alpha(b)n)^2hadm\alpha(c)gh) = 0$. Since R is semicommutative, we get $\alpha(b)nh = \alpha(b)nhg(\alpha(b)n)^2hadm\alpha(c)gh$. Hence $\alpha(b)nh(1 - g(\alpha(b)n)^2hadm\alpha(c)gh) = 0$. Then we have $\alpha(b)nhg = \alpha(b)nhg^2(\alpha(b)n)^2hadm\alpha(c)gh$. This shows that

$$\alpha(b) = \alpha(b)nhg\alpha(b) = \alpha(b)nhg^2(\alpha(b)n)^2hadm\alpha(c)gh\alpha(b) \in R\alpha(c)gh\alpha(b).$$

Next, we have

$$\begin{aligned}\alpha(c) &= ad\alpha(b)n\alpha(c) = m\alpha(c)adad\alpha(b)n\alpha(c)\\ &= m\alpha(c)hgad\alpha(b)n\alpha(c) = m\alpha(c)had\alpha(b)ng\alpha(c).\end{aligned}$$

This implies that $(1 - m\alpha(c)had\alpha(b)ng)\alpha(c) = 0$. Since R is a semicommutative ring, we have $m\alpha(c) = m\alpha(c)had\alpha(b)ngm\alpha(c)$. Therefore, we have

$$\begin{aligned}m\alpha(c)hg &= m\alpha(c)hgad\alpha(b)ngm\alpha(c)h = gm\alpha(c)hgad\alpha(b)nm\alpha(c)h\\ &= ghgm\alpha(c)ad\alpha(b)nm\alpha(c)h = ghm\alpha(c)adgm\alpha(c)m\alpha(c)h,\end{aligned}$$

since $m\alpha(c)h(1 - ad\alpha(b)ngm\alpha(c)h) = 0$. Then we have

$$\begin{aligned}\alpha(c) &= \alpha(c)hg\alpha(b)n = \alpha(c)hgm\alpha(c) = \alpha(c)m\alpha(c)hg\\ &= \alpha(c)gh\alpha(b)nadgm\alpha(c)m\alpha(c)h \in \alpha(c)gh\alpha(b)R.\end{aligned}$$

It is clear that $m\alpha(c) = \alpha(b)n$ is the α-(b,c)-inverse of ad. Let $x = (ad)_{\alpha}^{(b,c)}$. Then $x = m\alpha(c) = \alpha(b)n$. Thus, $x = (hg)_{\alpha}^{(b,c)}$ by Corollary 3.4. Since $xad = adx \in E(R)$, we conclude that

$$\alpha(b) = \alpha(b)nhg^2(\alpha(b)n)^2hadm\alpha(c)gh\alpha(b) = m\alpha(c)hg^2\alpha(b)n\alpha(b)nhgh\alpha(b).$$

Since $m\alpha(c)hg\alpha(b) = \alpha(b)$, we have $(m\alpha(c)hg - 1)\alpha(b) = 0$. Thus we get $m\alpha(c)hg^2\alpha(b) = g\alpha(b)$. Therefore, $\alpha(b) = g\alpha(b)n\alpha(b)nhgh\alpha(b) = gx^2hgh\alpha(b)$. Similarly, we obtain

$$\alpha(c) = \alpha(c)gh\alpha(b)nadgm\alpha(c)m\alpha(c)h = \alpha(c)ghgm\alpha(c)m\alpha(c)h = \alpha(c)ghgx^2h.$$

Let $y = gx^2h$. Then $\alpha(b) = ygh\alpha(b)$ and $\alpha(c) = \alpha(c)ghy$. And we have

$$yghy = gx^2hghgx^2h = gx^2h = y, \quad gx^2h = gx^2hadm\alpha(c) \in R\alpha(c),$$

$$gx^2h = gm\alpha(c)ad\alpha(b)nxh = \alpha(b)nadgx^2h \in \alpha(b)R.$$

This implies that $gx^2hR = \alpha(b)R$ and $Rgx^2h = R\alpha(c)$. Therefore, we have $(gh)_\alpha^{(b,c)} = y = gx^2h = g((ad)_\alpha^{(b,c)})^2h$. The converse can be proved similarly. □

Proposition 3.9 *Let $a,b,c,d,g \in R$ such that $a(ga)^2 = adaga$ and $agada = (ad)^2a$. If R is a semicommutative ring, then $ad \in R_\alpha^{(b,c)}$ if and only if $ga \in R_\alpha^{(b,c)}$.*

Proof If $ad \in R_\alpha^{(b,c)}$, then there exist $m,n \in R$ such that $\alpha(b) = m\alpha(c)ad\alpha(b)$ and $\alpha(c) = \alpha(c)ad\alpha(b)n$. Because R is semicommutative (hence abelian), we conclude that

$$\begin{aligned}\alpha(b)n &= \alpha(b)nm\alpha(c)adadad\alpha(b)n\alpha(b)n = \alpha(b)nm\alpha(c)agadad\alpha(b)n\alpha(b)n\\ &= \alpha(b)nm\alpha(c)aad\alpha(b)nad\alpha(b)ng = \alpha(b)nad\alpha(b)nm\alpha(c)aad\alpha(b)ng.\end{aligned}$$

Then $\alpha(b)na = \alpha(b)nad\alpha(b)nm\alpha(c)aad\alpha(b)nga$. Since R is semicommutative and $\alpha(b)na(1-d\alpha(b)nm\alpha(c)aad\alpha(b)nga) = 0$, we have

$$\begin{aligned}\alpha(b) &= \alpha(b)nad\alpha(b) = \alpha(b)nad^2\alpha(b)nm\alpha(c)a^2d\alpha(b)nga\alpha(b)\\ &= d\alpha(b)nm\alpha(c)a^2dm\alpha(c)ga\alpha(b) \in R\alpha(c)ga\alpha(b).\end{aligned}$$

Since $\alpha(c) = m\alpha(c)adadad\alpha(b)n\alpha(b)n\alpha(c) = m\alpha(c)agad\alpha(b)nad\alpha(b)n\alpha(c)$, we get

$$(1 - m\alpha(c)agad\alpha(b)n)\alpha(c) = 0.$$

Then $m\alpha(c)a = m\alpha(c)agad\alpha(b)nm\alpha(c)a$ and thus

$$m\alpha(c)ad = m\alpha(c)adgad\alpha(b)nm\alpha(c)a = gam\alpha(c)ad^2\alpha(b)nm\alpha(c)a.$$

Also it is clear that $m\alpha(c)ad = adm\alpha(c)$ since $ad\alpha(b)n, m\alpha(c)ad \in E(R)$. Then we have $\alpha(c) = \alpha(c)ga\alpha(b)nad^2\alpha(b)nm\alpha(c)a \in \alpha(c)ga\alpha(b)R$, which implies that $ga \in R_\alpha^{(b,c)}$. The converse can be proved in a similar way. □

Corollary 3.10 *Let R be a semicommutative ring and $a,b,c,d \in R$. Then $ad \in R^{(b,c)}$ if and only if $da \in R^{(b,c)}$.*

It is well-known that $1+ab$ is invertible if and only if $1+ba$ is invertible, it is called the Jacobson's Lemma [14, Exercise 1.6]. The next proposition shows the similar result for α-(b,c)-invertible elements.

Proposition 3.11 *Let $a,b,c,d,g,h \in R$ such that $\alpha(b),\alpha(c) \in comm(ad,hg)$, $adh = hgh$ and $hga = ada$. Then $(1-ad) \in R_\alpha^{(b,c)}$ if and only if $(1-hg) \in R_\alpha^{(b,c)}$.*

Proof If $(1-ad) \in R_\alpha^{(b,c)}$, then there exist $m,n \in R$ such that $\alpha(b) = m\alpha(c)(1-ad)\alpha(b)$ and $\alpha(c) = \alpha(c)(1-ad)\alpha(b)n$. Since $adh = hgh$ and $hga = ada$, we get $(hg-ad)(1-hg) = hg-ad$ and $(1-hg)(hg-ad) = (1-ad)(hg-ad)$. Therefore, we have

$$\begin{aligned}\alpha(b) &= m\alpha(c)(1-hg)\alpha(b) + m\alpha(c)(hg-ad)\alpha(b)\\ &= m\alpha(c)(1-hg)\alpha(b) + m(hg-ad)\alpha(c)(1-hg)\alpha(b),\\ &= [m + m(hg-ad)]\alpha(c)(1-hg)\alpha(b) \in R\alpha(c)(1-hg)\alpha(b)\\ \alpha(c) &= \alpha(c)(1-hg)\alpha(b)n + \alpha(c)(hg-ad)(1-ad)\alpha(b)n\end{aligned}$$

$$= \alpha(c)(1-hg)\alpha(b)n + \alpha(c)m\alpha(c)(1-ad)(hg-ad)\alpha(b)n(1-ad)$$
$$= \alpha(c)(1-hg)\alpha(b)n + \alpha(c)m\alpha(c)(1-hg)(hg-ad)\alpha(b)n(1-ad),$$

since $\alpha(b), \alpha(c) \in comm(ad, hg)$. This shows that

$$\begin{aligned}\alpha(c)(1-ad)\alpha(b) &= (1-ad)\alpha(c)\alpha(b)\\ &= (1-ad)\alpha(c)(1-hg)\alpha(b)n\alpha(b) + \alpha(c)(1-hg)(hg-ad)\alpha(b)\\ &= (1-ad)\alpha(c)(1-hg)\alpha(b)n\alpha(b) + \alpha(c)(1-hg)\alpha(b)(hg-ad).\end{aligned}$$

Since $(hg-ad)(hg-ad) = 0$, it follows that

$$m\alpha(c)(1-ad)\alpha(b)(hg-ad) = m(1-ad)\alpha(c)(1-hg)\alpha(b)n\alpha(b)(hg-ad).$$

This implies that $\alpha(b)(hg-ad)n = (1-ad)\alpha(b)n(1-hg)\alpha(b)n\alpha(b)(hg-ad)n$, that is, $\alpha(c)(hg-ad)\alpha(b)n = \alpha(c)(1-hg)\alpha(b)n(hg-ad)\alpha(b)n$. Therefore, we conclude that

$$\begin{aligned}\alpha(c) &= \alpha(c)(1-hg)\alpha(b)n + \alpha(c)(1-hg)\alpha(b)n(hg-ad)\alpha(b)n\\ &= \alpha(c)(1-hg)\alpha(b)[n + n(hg-ad)\alpha(b)n] \in \alpha(c)(1-hg)\alpha(b)R,\end{aligned}$$

proving $(1-hg) \in R_\alpha^{(b,c)}$. The converse can be proved similarly. □

Corollary 3.12 *Let $a, b, c, d \in R$ such that $\alpha(b), \alpha(c) \in comm(ad, da)$, $ad^2 = dad$ and $da^2 = ada$. Then $(1-ad) \in R_\alpha^{(b,c)}$ if and only if $(1-da) \in R_\alpha^{(b,c)}$.*

If R is an abelian ring, then we have the similar result as follows.

Proposition 3.13 *Let R be an abelian ring such that $adh = hgh$ and $hga = ada$ for $a, b, c, d, g, h \in R$. Then $(1-ad) \in R_\alpha^{(b,c)}$ if and only if $(1-hg) \in R_\alpha^{(b,c)}$.*

Proof If $(1-ad) \in R_\alpha^{(b,c)}$, then there exist $m, n \in R$ such that $\alpha(b) = m\alpha(c)(1-ad)\alpha(b)$, $\alpha(c) = \alpha(c)(1-ad)\alpha(b)n$. Since R is an abelian ring, we have $m\alpha(c)(1-ad) = (1-ad)m\alpha(c) \in C(R)$. This shows that

$$\begin{aligned}\alpha(b)n &= \alpha(b)nm\alpha(c)(1-hg) + \alpha(b)nm\alpha(c)(hg-ad)\\ &= \alpha(b)nm\alpha(c)(1-hg) + \alpha(b)nm\alpha(c)(hg-ad)(1-hg)\\ &= \alpha(b)nm\alpha(c)(1-hg) + \alpha(b)n\alpha(b)n(hg-ad)(1-ad)m\alpha(c)(1-hg)\end{aligned}$$

since $adh = hgh$ and $hga = ada$. It follows that $\alpha(b)n(1-ad) = m\alpha(c)(1-hg) + m\alpha(c)(hg-ad)m\alpha(c)(1-hg)$. Therefore, we have

$$\begin{aligned}\alpha(b) &= m\alpha(c)(1-hg)\alpha(b) + m\alpha(c)(hg-ad)m\alpha(c)(1-hg)\alpha(b)\\ &= [m + m\alpha(c)(hg-ad)m]\alpha(c)(1-hg)\alpha(b) \in R\alpha(c)(1-hg)\alpha(b).\end{aligned}$$

Next, since $(hg-ad)(1-ad) = hg-ad$, we also have

$$\begin{aligned}m\alpha(c) &= (1-hg+hg-ad)\alpha(b)nm\alpha(c)\\ &= (1-hg)\alpha(b)nm\alpha(c) + m\alpha(c)(1-ad)(hg-ad)m\alpha(c)\\ &= (1-hg)\alpha(b)nm\alpha(c) + m\alpha(c)(1-hg)(hg-ad)(1-ad)m\alpha(c)\end{aligned}$$

$$= (1-hg)\alpha(b)nm\alpha(c) + m\alpha(c)(1-hg)(hg-ad).$$

Then $m\alpha(c)(1-ad) = (1-hg)m\alpha(c)+m\alpha(c)(1-hg)(hg-ad)$. Since $(hg-ad)(hg-ad) = 0$, we deduce that $m\alpha(c)(1-ad)(hg-ad) = (1-hg)m\alpha(c)(hg-ad)$. This implies that $m\alpha(c)(1-ad) = (1-hg)\alpha(b)n + (1-hg)\alpha(b)n(hg-ad)$. Hence, we have

$$\begin{aligned}\alpha(c) &= \alpha(c)(1-ad)\alpha(b)n = \alpha(c)m\alpha(c)(1-ad)\\ &= \alpha(c)(1-hg)\alpha(b)[n+n(hg-ad)] \in \alpha(c)(1-hg)\alpha(b)R.\end{aligned}$$

The converse can be proved similarly. □

It was shown in [9, Theorem 2.19] that (b,c)-inverse has the analogous version for Jacobson's lemma. Similarly, we have the following result for α-(b,c)-inverses.

Proposition 3.14 *Let $a,b,c,m \in R$. If $y \in R$ is the α-(b,c)-inverse of a, then the following statements are equivalent:*

(1) $m \in R_\alpha^{(b,c)}$;

(2) $1+(m-a)y$ *is invertible;*

(3) $1+y(m-a)$ *is invertible.*

Proof The proof is similar to that of [9, Theorem 2.19]. □

The following theorem shows the strongly clean decompositions for α-(b,c)-invertible elements.

Theorem 3.15 *Let $a,b,c \in R$ with $\alpha(b),\alpha(c) \in comm^2(a)$. Then the following statements are equivalent:*

(1) $a \in R_\alpha^{(b,c)}$ and $a \in R_\alpha^{(c,b)}$;

(2) $\alpha(b) = u+e$, $\alpha(c) = v+e$ *are strongly clean decompositions,* $e\alpha(b) = \alpha(b)e = e\alpha(c) = \alpha(c)e = 0$, $\alpha(b) \in R\alpha(c)a$, $\alpha(c) \in a\alpha(b)R$ *and* $\alpha(c) \in R\alpha(b)a$, $\alpha(b) \in a\alpha(c)R$, *where* $u,v \in U(R)$, $e \in E(R)$;

(3) $\alpha(b) = u+e$, $\alpha(c) = v+e$, $ue = eu$, $ve = ev$, $\alpha(b)R \cap eR = \alpha(c)R \cap eR = \{0\}$ *and* $\alpha(b) \in R\alpha(c)a$, $\alpha(c) \in a\alpha(b)R$, $\alpha(c) \in R\alpha(b)a$, $\alpha(b) \in a\alpha(c)R$, *where* $u,v \in U(R)$ *and* $e \in E(R)$.

Proof (1)⇒ (2). If $a \in R_\alpha^{(b,c)}$, then there exist $m,n \in R$ such that $\alpha(b) = m\alpha(c)a\alpha(b)$ and $\alpha(c) = \alpha(c)a\alpha(b)n$. If $a \in R_\alpha^{(c,b)}$, then there exist $s,t \in R$ such that $\alpha(c) = s\alpha(b)a\alpha(c)$ and $\alpha(b) = \alpha(b)a\alpha(c)t$. By Lemma 2.7, we have

$$\alpha(b)\alpha(c) = \alpha(c)\alpha(b), \quad m\alpha(c)a = am\alpha(c),$$

$$m\alpha(c)\alpha(b) = \alpha(b)m\alpha(c), \quad m\alpha(c)\alpha(c) = \alpha(c)m\alpha(c),$$

since $\alpha(b),\alpha(c) \in comm^2(a)$. This shows that $\alpha(b) \in R\alpha(c)a, \alpha(c) \in a\alpha(b)R$ and $\alpha(c) \in R\alpha(b)a, \alpha(b) \in a\alpha(c)R$. Let $p \in rann(\alpha(b))$. Then we have $m\alpha(c)ap = ms\alpha(b)a\alpha(c)ap = msa\alpha(c)a\alpha(b)p = 0$, that is, $\mathrm{rann}(\alpha(b)) \subseteq \mathrm{rann}(m\alpha(c)a)$. It follows that $m\alpha(c)ana\alpha(b) = m\alpha(c)a$ since $\alpha(b)na\alpha(b) = \alpha(b)$. Also since $\alpha(b)nam\alpha(c)a = \alpha(b)na$, we get $m\alpha(c)anam\alpha(c)a =$

$m\alpha(c)ana$. Let $u=\alpha(b)-1+m\alpha(c)a$. Then we have

$$\begin{aligned}&(\alpha(b)-1+m\alpha(c)a)(m\alpha(c)ana-1+m\alpha(c)a)\\&=(m\alpha(c)ana-1+m\alpha(c)a)(\alpha(b)-1+m\alpha(c)a)=1,\end{aligned}$$

which shows $u\in U(R)$. Take $e=1-m\alpha(c)a$. Then $\alpha(b)=u+e$ and $e\alpha(b)=\alpha(b)e=e\alpha(c)=\alpha(c)e=0$. Similarly, let $k\in lann(\alpha(c))$. We deduce that

$$km\alpha(c)a=k\alpha(b)na=k\alpha(b)a\alpha(c)tna=k\alpha(c)\alpha(b)atna=0,$$

that is, $lann(\alpha(c))\subseteq lann(m\alpha(c)a)$. Because $\alpha(c)am\alpha(c)=\alpha(c)$ and $am\alpha(c)am\alpha(c)=am\alpha(c)$, we also have $\alpha(c)amm\alpha(c)a=m\alpha(c)a$, and $am\alpha(c)amm\alpha(c)a=amm\alpha(c)a$. This shows that

$$\begin{aligned}&(\alpha(c)-1+m\alpha(c)a)(amam\alpha(c)-1+m\alpha(c)a)\\&=(amam\alpha(c)-1+m\alpha(c)a)(\alpha(c)-1+m\alpha(c)a)=1.\end{aligned}$$

Let $v=\alpha(c)-1+m\alpha(c)a$. Then $v\in U(R)$ and $\alpha(c)=v+e$.

(2)$\Rightarrow$ (1). Since $\alpha(b)=u+e$ and $\alpha(c)=v+e$, we get $\alpha(b)\alpha(b)=(u+e)\alpha(b)=u\alpha(b)$ and $\alpha(c)\alpha(c)=\alpha(c)(v+e)=\alpha(c)v$. Because $u,v\in U(R),\alpha(b)\in R\alpha(c)a$ and $\alpha(c)\in a\alpha(b)R$, we have $\alpha(b)=u^{-1}\alpha(b)\alpha(b)\in R\alpha(c)a\alpha(b)$, $\alpha(c)=\alpha(c)\alpha(c)v^{-1}\in\alpha(c)a\alpha(b)R$. Similarly, we can get $\alpha(b)\in\alpha(b)a\alpha(c)R$ and $\alpha(c)\in R\alpha(b)a\alpha(c)$, that is, $a\in R_\alpha^{(b,c)}$ and $a\in R_\alpha^{(c,b)}$.

(2)$\Leftrightarrow$ (3). It is obvious. □

Theorem 3.16 *Let R be an abelian ring and $a\in R_\alpha^{(b,c)}$. Then $\alpha(b)=u+e$, $\alpha(c)=v+e$ are strongly clean decompositions, where $u,v\in U(R)$ and $e\in E(R)$.*

Proof (1)$\Rightarrow$ (2). If $a\in R_\alpha^{(b,c)}$, then there exist $m,n\in R$ such that $\alpha(b)=m\alpha(c)a\alpha(b)$ and $\alpha(c)=\alpha(c)a\alpha(b)n$. It follows that $na\alpha(b),\alpha(c)am\in E(R)$ since $\alpha(b)n=m\alpha(c)$. Also since R is an abelian ring, we get

$$\alpha(b)na=\alpha(b)na\alpha(b)na=na\alpha(b),$$

$$m\alpha(c)a=\alpha(c)amm\alpha(c)a=\alpha(c)am=am\alpha(c).$$

This yields that

$$\begin{aligned}&(\alpha(b)-1+m\alpha(c)a)(m\alpha(c)ana-1+m\alpha(c)a)\\&=(m\alpha(c)ana-1+m\alpha(c)a)(\alpha(b)-1+m\alpha(c)a)=1.\end{aligned}$$

Let $u=\alpha(b)-1+m\alpha(c)a$. Then $u\in U(R)$. Take $e=1-m\alpha(c)a$. Then $\alpha(b)=u+e$. Similarly, let $v=\alpha(c)-1+m\alpha(c)a$. Then we have

$$\begin{aligned}&(\alpha(c)-1+m\alpha(c)a)(amam\alpha(c)-1+m\alpha(c)a)\\&=(amam\alpha(c)-1+m\alpha(c)a)(\alpha(c)-1+m\alpha(c)a)=1.\end{aligned}$$

Thus, $v\in U(R)$ and $\alpha(c)=v+e$. □

Corollary 3.17 *If R is an abelian ring and $a\in R^{(b,c)}$, then $b=u+e$, $c=v+e$ are strongly clean decompositions, where $u,v\in U(R)$, $e\in E(R)$ and $a,b,c\in R$.*

According to [4], $a \in R$ is Bott-Duffin (e, f)-invertible if there is $y \in R$ such that $y = ey = yf$, $yae = e$ and $fay = f$, where $e, f \in E(R)$.

Corollary 3.18 *If R is an abelian ring and $a \in R$ is Bott-Duffin (e, f)-invertible, then $e = u + w$ and $f = v + w$ are strongly clean decompositions, where $u, v \in U(R)$ and $e, f, w \in E(R)$.*

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