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# Verma Modules over Some Lie Algebras of $W$-Type

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**Abstract** In this paper, we describe the structure of Verma modules over the two kinds of Lie algebras $\mathfrak{g}(\lambda)$ of $W$-type. We determine the reducibility and the singular vectors of their Verma modules under some conditions.

**Keywords** $W$-algebra; Verma module; singular vector

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## 1. Introduction

The twisted Heisenberg-Virasoro algebra HV was first introduced in [1], it is the universal central extension of the Lie algebra of differential operators on a circle of order no more than one. Its structure and representation theory have been discussed by many authors. For example, the irreducibility of Verma modules over HV was discussed in [1,2], its derivations and automorphism group were computed in [3], the classification of irreducible Harish-Chandra modules over HV was discussed in [4].

For $a, b \in \mathbb{C}$, denote by $W(a, b)$ the complex Lie algebra with $\mathbb{C}$-basis $\{L_n, I_n, |n \in \mathbb{Z}\}$ and define the relations

$$
\begin{aligned}
[L_n, L_m] =&(m-n)L_{m+n},\\
[L_n, I_m] =&(a+m+bn)I_{m+n},\\
[I_n, I_m] =&0, \quad \text{where } m, n \in \mathbb{Z}.
\end{aligned}
$$

The $\mathrm{Vir}(a, b)$ is the universal central extension of $W(a, b)$ (see [5]). The algebra $\mathrm{Vir}(a, b)$ is very meaningful because it generalizes many important algebras, for example, the algebra $\mathrm{Vir}(0, 0)$ is the twisted Heisenberg-Virasoro algebra, the algebra $\mathrm{Vir}(0, -1)$ is the $W(2, 2)$ Lie algebra whose representations were discussed in [6]. Classification of non-weight $\mathrm{Vir}(0, b)$-modules over $\mathbb{C}[s, t]$ and the irreducibilities and isomorphic relations of these modules were constructed in [7].

Infinitesimal deformation of a Lie algebra is one way to give new Lie algebras. As a special $W$-algebra, the twisted Heisenberg-Virasoro algebra HV is a $\mathbb{Z}$-graded algebra. The infinitesimal deformations of the HV were given in [8], which were called deformed HV algebras. The deformed

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generalized Heisenberg-Virasoro algebra $\mathfrak{g}(\mathrm{G}, \lambda)$ was introduced in [9], where $\lambda$ is a deformation parameter, and $G$ is an additive subgroup of $\mathbb{C}$ such that $G$ is free of rank $\nu$ if $\lambda = -2$.

Verma module is a highest weight module, investigation of Verma module on infinite dimensional Lie algebras was initiated in many papers, such as the Verma module and its singular vector of the twisted Heisenberg-Virasoro algebra at level zero were determined in [2], the Verma module and its singular vector of the $W$-algebra $W(2,2)$ were determined in [6,10–12], the Verma modules over the generalized Heisenberg-Virasoro algebras were determined in [13], the Verma modules over the Virasoro algebra were determined in [14], the generalized Verma modules over some Block Lie algebra were studied in [15]. In [16], the author completely determined the irreducibility of the two type deformed generalized Heisenberg-Virasoro algebras, one is the deformed generalized Heisenberg-Virasoro algebra $\mathfrak{g}(G, \lambda)$ with the deformation parameter $\lambda \neq -1$, where $G$ is an additive subgroup of $\mathbb{C}$ such that $G$ is free of rank $\nu \geq 1$ if $\lambda = -2$, the other is the deformed Heisenberg-Virasoro algebra $\mathfrak{g}(\mathbb{Z}, \lambda)$. In particular, the author gave the necessary and sufficient condition of the Verma module over $\mathfrak{g}(G, \lambda)$ with $\lambda \neq 0, -1$.

In this paper, we want to make certain contributions to the reducibility of Verma modules over the two types of Lie algebras $\mathfrak{g}(\mathbb{Z}, \lambda)$ of $W$-type, denoted $\mathfrak{g}(\lambda)$ for short. One is $\mathfrak{g}(-1)$, this is the special $\mathrm{Vir}(a, b)$ with $a = 0$, $b = 1$, the other is $\mathfrak{g}(0)$. The rest of the paper is organized as follows. In Section 2, we introduce the $W$-algebras $\mathfrak{g}(\lambda)$, and their Verma modules. In Section 3, we determine the necessary and sufficient condition of the irreducibility for Verma module of $\mathfrak{g}(-1)$ and all its singular vectors. In Section 4, we determine the necessary and sufficient condition of the irreducibility condition for Verma module of $\mathfrak{g}(0)$.

## 2. $W$-algebras $\mathfrak{g}(\lambda)$ and their Verma modules

In this section, we recall the W-algebras $\mathfrak{g}(\lambda)$ and their Verma modules. The W-algebras $\mathfrak{g}(\lambda)$ are a kind of infinite-dimensional Lie algebras related with the parameter $\lambda$ with the $\mathbb{C}$-basis

$$\{L_n, I_n, C_1, C_2 | n \in \mathbb{Z}\}$$

and the Lie brackets given by

$$\begin{aligned}
[L_n, L_m] =& (m-n)L_{m+n} + \delta_{m+n,0}\frac{1}{12}(n^3-n)C_1,\\
[L_n, I_m] =& (m-\lambda n)I_{m+n} + \delta_{m+n,0}\delta_{\lambda,1}\frac{1}{12}(n^3-n)C_2 + \delta_{m+n,0}\delta_{\lambda,-1}nC_2,\\
[I_n, I_m] =& [C_i, \mathfrak{g}] = 0, \quad \text{where } m, n \in \mathbb{Z},\ i = 1, 2.
\end{aligned}$$

It is clear that the W-algebras $\mathfrak{g}(\lambda)$ are a kind of $\mathbb{Z}$-graded Lie algebras and have triangular decomposition

$$\mathfrak{g}(\lambda) = \mathfrak{g}(\lambda)_{(-)} \oplus \mathfrak{g}(\lambda)_{(0)} \oplus \mathfrak{g}(\lambda)_{(+)},$$

where

$$\mathfrak{g}(\lambda)_{(0)} = \mathrm{Span}_{\mathbb{C}}\{L_0, I_0, C_1, C_2\},$$

$$\mathfrak{g}(\lambda)_{(\pm)} = \mathrm{Span}_{\mathbb{C}}\{L_n, I_n | n \in \pm\mathbb{N}\}.$$

Let $c_1, c_2, h, h_I \in \mathbb{C}$. Denote by $I(c_1, c_2, h, h_I)$ the left ideal of the universal enveloping algebra $U(\mathfrak{g}(\lambda))$ generated by the elements

$$\{L_i, I_j | i, j > 0\} \cup \{L_0 - h, I_0 - h_I, C_1 - c_1, C_2 - c_2\}.$$

The Verma module with highest weight $(c_1, c_2, h, h_I)$ over $\mathfrak{g}(\lambda)$ is defined as

$$M(c_1, c_2, h, h_I) = U(\mathfrak{g})/I(c_1, c_2, h, h_I),$$

which is a highest weight module with a basis consisting of all vectors of the form

$$I_{-m_1} \cdots I_{-m_s} L_{-n_1} \cdots L_{-n_r} v,$$

where $r, s \geq 0$, $n_1 \geq \cdots \geq n_r > 0$, $m_1 \geq \cdots \geq m_s > 0$, $v = 1 + I(c_1, c_2, h, h_I)$. For simplicity denote $M = M(c_1, c_2, h, h_I)$. Clearly, $M$ is graded by the $L_0$-eigenvalues:

$$M = \oplus_{n \geq 0} M_n,$$

where

$$M_n = \{w \in M | L_0 w = (n + h)w\}$$

is spanned by vectors of the form $I_{-m_1} \cdots I_{-m_s} L_{-n_1} \cdots L_{-n_r} v$ such that $m_1 + \cdots + m_s + n_1 + \cdots + n_r = n$.

A nonzero homogeneous vector $\xi$ in a highest weight $\mathfrak{g}(\lambda)$-module is called singular if $\mathfrak{g}(\lambda)_{(+)}\xi = 0$. $M$ has a unique maximal submodule $J(c_1, c_2, h, h_I)$ so that

$$\bar{M}(c_1, c_2, h, h_I) = M/J(c_1, c_2, h, h_I)$$

is an irreducible highest weight module.

Let $P = \{(m_1, ..., m_s) | m_1 \geq \cdots \geq m_s > 0, s \in \mathbb{N}\}$. For $a = (a_1, ..., a_k), b = (b_1, ..., b_l) \in P$, denote by $|a|$ the length of $a$. We may define a total order $\succ$ on $P$ as follows. If $k = |a| > |b| = l$, set $b_{l+1} = \cdots = b_k = 0$, then

$$a \succ b \text{ if and only if there exists } 1 \leq i \leq k \text{ such that } a_i > b_i \text{ and } a_j = b_j \text{ for } j < i.$$

The algebra $\mathfrak{g}(\lambda)$ has an anti-involution $\sigma : \mathfrak{g}(\lambda) \to \mathfrak{g}(\lambda)$ defined by

$$\sigma(L_n) = L_{-n},\ \sigma(I_n) = I_{-n},\ \sigma(C_i) = C_i, \text{ for } i = 1, 2.$$

Then we get a symmetric bilinear form $(\cdot|\cdot)$ on $M$ defined by

$$(xv|yv)v = \pi(\widetilde{\sigma}(x)y)v,$$

for $x, y \in U(\mathfrak{g}(\lambda))$, where $\pi : U(\mathfrak{g}(\lambda)) \to U(\mathfrak{g}(\lambda)_{(0)})$ denotes the projection and the anti-involution $\widetilde{\sigma} : U(\mathfrak{g}(\lambda)) \to U(\mathfrak{g}(\lambda))$ is given as follows:

$$\widetilde{\sigma}(x_1 \cdots x_n) = \sigma(x_n) \cdots \sigma(x_1) \text{ for any } x_1, ..., x_n \in \mathfrak{g}(\lambda).$$

Clearly, we get $(v|v) = 1$ and

$$(L_m\mu|\nu) = (\mu|L_{-m}\nu), (I_m\mu|\nu) = (\mu|I_{-m}\nu),$$

where $m\in\mathbb{Z},\ \mu,\nu\in M$. Moreover, the distinct graded components of $M$ are orthogonal, that is

$$(M_m|M_n)=0 \text{ for } m\neq n.$$

We know the radical of the symmetric bilinear form is the maximal $\mathfrak{g}(\lambda)$-submodule of $M$. So it is enough to consider the restriction of the bilinear form on each $M_n$ when we determine the irreducibility of $M$.

Let $S_n$ be the set of the basis of $M_n$ consisting of vectors of the form

$$I_{-m_1}\cdots I_{-m_s}L_{-n_1}\cdots L_{-n_r}v.$$

We introduce the total order on $S_n$ as follows:

$$I_{-m_1}\cdots I_{-m_s}L_{-n_1}\cdots L_{-n_r}v\succ I_{-k_1}\cdots I_{-k_p}L_{-l_1}\cdots L_{-l_q}v$$

if one of the following conditions stands,

(1) $\sum m_i<\sum k_i$;

(2) $\sum m_i=\sum k_i,(m_1,\dots,m_s)\succ(k_1,\dots,k_p)$;

(3) $\sum m_i=\sum k_i,(m_1,\dots,m_s)=(k_1,\dots,k_p),(n_1,\dots,n_r)\prec(l_1,\dots,l_q)$.

Clearly, if $S_n=\{\mu_1,\dots,\mu_d\}$ with $\mu_i\succ\mu_j$ if $i<j$, we know that $d=\dim M_n$. For example, when $n=2$, we have $L_{-1}^2v\succ L_{-2}v\succ I_{-1}L_{-1}v\succ I_{-2}v\succ I_{-1}^2v$ and

$$S_2=\{L_{-1}^2v,L_{-2}v,I_{-1}L_{-1}v,I_{-2}v,I_{-1}^2v\}.$$

Denote by $A_n=(A_{ij})$ the $d\times d$ matrix with $A_{ij}=(\mu_i|\mu_{d+1-j})$, next we compute the determinant $\det A_n$ of $A_n$.

## 3. Verma module over the $W$-algebra $\mathfrak{g}(-1)$

Let $\lambda=-1$. We know that the $W$-algebra $\mathfrak{g}(-1)$ becomes $\mathrm{Vir}(0,1)$ Lie algebra, its Lie brackets are

$$\begin{aligned}[][L_n,L_m]=&(m-n)L_{m+n}+\delta_{m+n,0}\frac{1}{12}(n^3-n)C_1,\\ [L_n,I_m]=&(m+n)I_{m+n}+\delta_{m+n,0}nC_2,\\ [I_n,I_m]=&[C_i,\mathrm{Vir}(0,1)]=0,\quad \text{where } m,n\in\mathbb{Z},\ i=1,2.\end{aligned}$$

In this section, we discuss the reducible property of Verma module and the corresponding singular vectors.

**Lemma 3.1** *If* $(n_1,\dots,n_r)\succ(m_1,\dots,m_s)$, $r,s>0$, $n_1\geq\cdots\geq n_r>0$, $m_1\geq\cdots\geq m_s>0$, *then*

$$(L_{-n_1}\cdots L_{-n_r}v|I_{-m_1}\cdots I_{-m_s}v)=(I_{-m_1}\cdots I_{-m_s}v|L_{-n_1}\cdots L_{-n_r}v)=0.$$

**Proof** For any integer $m\geq m_1$, we have

$$\begin{aligned}L_mI_{-m_1}\cdots I_{-m_s}v=&I_{-m_1}L_mI_{-m_2}\cdots I_{-m_s}v+[L_m,I_{-m_1}]I_{-m_2}\cdots I_{-m_s}v\\ =&I_{-m_1}L_mI_{-m_2}\cdots I_{-m_s}v+(m-m_1)I_{m-m_1}I_{-m_2}\cdots I_{-m_s}v+\\ &\delta_{m-m_1,0}mC_2I_{-m_2}\cdots I_{-m_s}v.\end{aligned}$$

Continuing to compute, we get

$$L_m I_{-m_1} \cdots I_{-m_s} v = mC_2 \frac{\partial(I_{-m_1} \cdots I_{-m_s})}{\partial I_{-m}} v.$$

We know there exists $1 \le t \le \min\{r, s\}$ such that $n_t > m_t$ and $n_i = m_i$ for $i < t$, so we obtain that $L_{n_r} \cdots L_{n_1} I_{-m_1} \cdots I_{-m_s} v = 0$, that is

$$(v|L_{n_r} \cdots L_{n_1} I_{-m_1} \cdots I_{-m_s} v) = (L_{-n_1} \cdots L_{-n_r} v|I_{-m_1} \cdots I_{-m_s} v) = 0.$$

Because of the symmetry, we also have

$$(L_{n_r} \cdots L_{n_1} I_{-m_1} \cdots I_{-m_s} v|v) = (I_{-m_1} \cdots I_{-m_s} v|L_{-n_1} \cdots L_{-n_r} v) = 0. \quad \square$$

**Lemma 3.2** *The determinant* $\det A_n$ *is a product of a nonzero integer and some*

$$f(m) = mc_2, m \in \mathbb{Z}+.$$

**Proof** Set $1 \le b < a \le d$, for $\mu_a, \mu_b \in S_n$, we have $\mu_a \prec \mu_b$. Write

$$\mu_a = I_{-n_1} \cdots I_{-n_r} L_{-m_1} \cdots L_{-m_s} v, \quad \mu_b = I_{-k_1} \cdots I_{-k_p} L_{-l_1} \cdots L_{-l_q} v.$$

Then we obtain

$$\mu_{d+1-a} = I_{-m_1} \cdots I_{-m_s} L_{-n_1} \cdots L_{-n_r} v,$$
$$\mu_{d+1-b} = I_{-l_1} \cdots I_{-l_q} L_{-k_1} \cdots L_{-k_p} v.$$

Next we consider the three cases of $\succ$ on $S_n$. If case (1) stands, so we get

$$\sum_{i=1}^{s} m_i < \sum_{j=1}^{q} l_j.$$

Then we have

$$(I_{-n_1} \cdots I_{-n_r} v|L_{-k_1} \cdots L_{-k_p} v) = 0.$$

It follows from Lemma 3.1 that $I_{n_r} \cdots I_{n_1} L_{-k_1} \cdots L_{-k_p} v = 0$. Hence

$$L_{m_s} \cdots L_{m_1} I_{-l_1} \cdots I_{-l_q} I_{n_r} \cdots I_{n_1} L_{-k_1} \cdots L_{-k_p} v = 0.$$

So

$$\begin{aligned} A_{ab} = (\mu_a|\mu_{d+1-b}) =& (I_{-n_1} \cdots I_{-n_r} L_{-m_1} \cdots L_{-m_s} v|I_{-l_1} \cdots I_{-l_q} L_{-k_1} \cdots L_{-k_p} v) \\ =& (v|L_{m_s} \cdots L_{m_1} I_{n_r} \cdots I_{n_1} I_{-l_1} \cdots I_{-l_q} L_{-k_1} \cdots L_{-k_p} v) \\ =& (v|L_{m_s} \cdots L_{m_1} I_{-l_1} \cdots I_{-l_q} I_{n_r} \cdots I_{n_1} L_{-k_1} \cdots L_{-k_p} v) \\ =& (L_{-m_1} \cdots L_{-m_s} v|I_{-l_1} \cdots I_{-l_q} v)(I_{-n_1} \cdots I_{-n_r} v|L_{-k_1} \cdots L_{-k_p} v) \\ =& 0. \end{aligned}$$

If case (2) stands, that is $\sum_{i=1}^{r} n_i = \sum_{j=1}^{p} k_j$ and $(n_1, ..., n_r) \prec (k_1, ..., k_p)$, by Lemma 3.1, we have

$$(L_{-k_1} \cdots L_{-k_p} v|I_{-n_1} \cdots I_{-n_r} v) = (I_{-n_1} \cdots I_{-n_r} v|L_{-k_1} \cdots L_{-k_p} v) = 0.$$

Thus

$$A_{ab} = (\mu_a|\mu_{d+1-b}) = (I_{-n_1} \cdots I_{-n_r} L_{-m_1} \cdots L_{-m_s} v|I_{-l_1} \cdots I_{-l_q} L_{-k_1} \cdots L_{-k_p} v)$$

$$
\begin{aligned}
&=(v|L_{m_s}\cdots L_{m_1}I_{n_r}\cdots I_{n_1}I_{-l_1}\cdots I_{-l_q}L_{-k_1}\cdots L_{-k_p}v)\\
&=(v|L_{m_s}\cdots L_{m_1}I_{-l_1}\cdots I_{-l_q}I_{n_r}\cdots I_{n_1}L_{-k_1}\cdots L_{-k_p}v)\\
&=(L_{-m_1}\cdots L_{-m_s}v|I_{-l_1}\cdots I_{-l_q}v)(I_{-n_1}\cdots I_{-n_r}v|L_{-k_1}\cdots L_{-k_p}v)\\
&=0.
\end{aligned}
$$

If case (3) stands, that is $\sum_{i=1}^{r} n_i = \sum_{j=1}^{p} k_j$, $(n_1, ..., n_r) = (k_1, ..., k_p)$, $(m_1, ..., m_s) \succ (l_1, ..., l_q)$, by Lemma 3.1, we have

$$(L_{-m_1}\cdots L_{-m_s}v|I_{-l_1}\cdots I_{-l_q}v) = (I_{-l_1}\cdots I_{-l_q}v|L_{-m_1}\cdots L_{-m_s}v) = 0.$$

Thus

$$
\begin{aligned}
A_{ab} = (\mu_a|\mu_{d+1-b}) &=(I_{-n_1}\cdots I_{-n_r}L_{-m_1}\cdots L_{-m_s}v|I_{-l_1}\cdots I_{-l_q}L_{-k_1}\cdots L_{-k_p}v)\\
&=(v|L_{m_s}\cdots L_{m_1}I_{n_r}\cdots I_{n_1}I_{-l_1}\cdots I_{-l_q}L_{-k_1}\cdots L_{-k_p}v)\\
&=(v|L_{m_s}\cdots L_{m_1}I_{-l_1}\cdots I_{-l_q}I_{n_r}\cdots I_{n_1}L_{-k_1}\cdots L_{-k_p}v)\\
&=(L_{-m_1}\cdots L_{-m_s}v|I_{-l_1}\cdots I_{-l_q}v)(I_{-n_1}\cdots I_{-n_r}v|L_{-k_1}\cdots L_{-k_p}v)\\
&=0.
\end{aligned}
$$

From the above three cases, we see that if $1 \le b < a \le d$, we have $A_{ab} = 0$, so the matrix $A_n$ is upper triangular. Thus the determinant $\det A_n$ is the product of diagonal elements. By Lemma 3.1, we have

$$
\begin{aligned}
A_{aa} = (\mu_a|\mu_{d+1-a}) &=(I_{-n_1}\cdots I_{-n_r}L_{-m_1}\cdots L_{-m_s}v|I_{-m_1}\cdots I_{-m_s}L_{-n_1}\cdots L_{-n_r}v)\\
&=(v|L_{m_s}\cdots L_{m_1}I_{n_r}\cdots I_{n_1}I_{-m_1}\cdots I_{-m_s}L_{-n_1}\cdots L_{-n_r}v)\\
&=(v|L_{m_s}\cdots L_{m_1}I_{-m_1}\cdots I_{-m_s}I_{n_r}\cdots I_{n_1}L_{-n_1}\cdots L_{-n_r}v)\\
&=(L_{-m_1}\cdots L_{-m_s}v|I_{-m_1}\cdots I_{-m_s}v)(I_{-n_1}\cdots I_{-n_r}v|L_{-n_1}\cdots L_{-n_r}v)\\
&=K_a\prod_{i=1}^{s} f(m_i)^{x_i}\prod_{j=1}^{r} f(n_j)^{y_j},
\end{aligned}
$$

where $K_a$ is some nonzero integers, $x_i$, $y_j$ are the times of $n_i$, $m_i$ appearing in $(n_1, ..., n_r)$, $(m_1, ..., m_s)$. This completes the proof of the lemma. □

Next is our main result.

**Theorem 3.3** *The Verma module $M$ over $\mathfrak{g}(-1)$ is irreducible if and only if $c_2 \neq 0$.*

**Proof** If $c_2 \neq 0$, then $f(m) \neq 0$ for any $m \in \mathbb{Z}_+$. So the bilinear form on $M$ is non-degenerate. The radical as the max submodule of $M$ is zero, which implies that the $\mathfrak{g}(-1)$-module $M$ is irreducible.

Suppose $c_2 = 0$, the bilinear form on $M$ is degenerate, so the radical of the bilinear form is nonzero and is a proper $\mathfrak{g}(-1)$-submodule of $M$, which contradicts the irreducibility of $M$. □

Next, we suppose $c_2 = 0$, then $J(c_1, 0, h, h_I) \neq 0$.

**Lemma 3.4** *The singular vectors of the Verma module $M(c_1, 0, h, h_I)$ must be in $U(I_-)v$ where $I_- = \bigoplus_{n\in\mathbb{N}} \mathbb{C}I_{-n}$.*

**Proof** Suppose $s \in M(c_1, 0, h, h_I)$ is a singular vector and homogeneous, then we can write $s = Sv$ for some $S \in U(\mathrm{Vir}(0,1)_-)$. We can obtain

$$I_0 s = I_0 S v = S I_0 v + [I_0, S]v = h_I s + [I_0, S]v.$$

It is easy to see that if $S \notin U(I_-)$, $[I_0, S] \neq kS$ for any $k \in \mathbb{C}$. So $s \in U(I_-)v$. □

**Lemma 3.5** *If $c_2 = 0$, then the Verma module $M(c_1, 0, h, h_I)$ possesses a singular vector $\mu^{'} \in M(c_1, 0, h, h_I)_p$ for some $p \in \mathbb{N}$, and up to a scalar factor, it is unique and can be written as*

$$\mu' = I_{-1}^p v.$$

**Proof** It is easy to show that $I_m I_{-1}^p v = 0$ for $m \geq 1$, $I_0 I_{-1}^p v = h_I I_{-1}^p v$ and

$$L_m I_{-1}^p v = p I_{-1}^{p-1}[L_m, I_{-1}]v = p(m-1) I_{-1}^{p-1} I_{m-1} v = 0 \text{ for } m \geq 1.$$

So $I_{-1}^p v$ is singular vector. By Lemma 3.4 and the definition of $M(c_1, 0, h, h_I)_p$, if

$$(aI_{-p} + bI_{-(p-1)}I_{-1} + cI_{-(p-2)}I_{-2} + dI_{-(p-2)}I_{-1}^2 + \cdots + eI_{-2}I_{-1}^{p-2})v,$$

where $a, b, c, ..., e \in \mathbb{C}$, is also singular vector, for $i \geq 1$, we have

$$\begin{aligned} &L_i(aI_{-p} + bI_{-(p-1)}I_{-1} + cI_{-(p-2)}I_{-2} + dI_{-(p-2)}I_{-1}^2 + \cdots + eI_{-2}I_{-1}^{p-2})v \\ &= (a(i-p)I_{i-p} + b(i+1-p)I_{i+1-p}I_{-1} + b(i-1)I_{-(p-1)}I_{i-1} + \cdots + e(i-2)I_{i-2}I_{-1}^{p-2})v. \end{aligned}$$

Then, choosing different $i$, we obtain that these coefficients $a = b = c = \cdots = e = 0$. So the singular vector $\mu' \in M(c_1, 0, h, h_I)_p$ is $I_{-1}^p v$ for some $p \in \mathbb{N}$. □

**Theorem 3.6** *Let $c_2 = 0$. Up to a scalar vector, all the singular vectors of $M(c_1, 0, h, h_I)$ are $(\mu')^n v$ for $n \geq 1$.*

**Proof** By Lemma 3.5, we have

$$I_m (I_{-1}^p)^n v = 0, \text{ for } m \geq 1,$$

$$I_0 (I_{-1}^p)^n v = h_I (I_{-1}^p)^n v$$

and

$$L_m (I_{-1}^p)^n v = np I_{-1}^{np-1}[L_m, I_{-1}]v = np(m-1) I_{-1}^{np-1} I_{m-1} v = 0, \text{ for } m \geq 1.$$

So $(I_{-1}^p)^n v$ is singular vector. If there are some other singular vectors $\nu$, then by Lemma 3.4, we have $\nu \in U(I_-)v$. Choosing the leading term of $\nu$ to be $I_{-q}$, and using $L_{q-1}$ to act on $\nu$, we get $L_{q-1}\nu \neq 0$. So all the singular vectors of $M(c_1, 0, h, h_I)$ are $(\mu')^n v$ for $n \geq 1$. □

## 4. Verma module over the $W$-algebra $\mathfrak{g}(0)$

When $\lambda = 0$, the basis of the Lie algebra $\mathfrak{g}(0)$ is

$$\{L_n, I_n, C_1 | n \in \mathbb{Z}\}$$

with the Lie brackets given by

$$\begin{aligned}[L_n, L_m] =&(m-n)L_{m+n} + \delta_{m+n,0}\frac{1}{12}(n^3-n)C_1,\\ [L_n, I_m] =&mI_{m+n},\\ [I_n, I_m] =&[C_1, \mathfrak{g}(0)] = 0, \quad \text{where } m, n \in \mathbb{Z}.\end{aligned}$$

In this section, we discuss its Verma module and the corresponding singular vectors.

**Lemma 4.1** *If* $(n_1, ..., n_r) \succ (m_1, ..., m_s), r, s > 0, n_1 \geq \cdots \geq n_r > 0, m_1 \geq \cdots \geq m_s > 0$, *then*

$$(L_{-n_1} \cdots L_{-n_r} v | I_{-m_1} \cdots I_{-m_s} v) = (I_{-m_1} \cdots I_{-m_s} v | L_{-n_1} \cdots L_{-n_r} v) = 0.$$

**Proof** For any integer $m \geq m_1$, we have

$$\begin{aligned}L_m I_{-m_1} \cdots I_{-m_s} v =& I_{-m_1} L_m I_{-m_2} \cdots I_{-m_s} v + [L_m, I_{-m_1}] I_{-m_2} \cdots I_{-m_s} v\\ =& I_{-m_1} L_m I_{-m_2} \cdots I_{-m_s} v + (-m_1) I_{m-m_1} I_{-m_2} \cdots I_{-m_s} v.\end{aligned}$$

Continuing to compute, we get

$$L_m I_{-m_1} \cdots I_{-m_s} v = -mh_I \frac{\partial(I_{-m_1} \cdots I_{-m_s})}{\partial I_{-m}} v.$$

We know there exists $1 \leq t \leq \min\{r, s\}$ such that $n_t > m_t$ and $n_i = m_i$ for $i < t$, so we obtain that $L_{n_r} \cdots L_{n_1} I_{-m_1} \cdots I_{-m_s} v = 0$, that is

$$(v | L_{n_r} \cdots L_{n_1} I_{-m_1} \cdots I_{-m_s} v) = (L_{-n_1} \cdots L_{-n_r} v | I_{-m_1} \cdots I_{-m_s} v) = 0.$$

Because of the symmetry, we also have

$$(L_{n_r} \cdots L_{n_1} I_{-m_1} \cdots I_{-m_s} v | v) = (I_{-m_1} \cdots I_{-m_s} v | L_{-n_1} \cdots L_{-n_r} v) = 0. \quad \square$$

**Lemma 4.2** *The determinant* $\det A_n$ *is a product of a nonzero integer and some*

$$f(m) = -mh_I, m \in \mathbb{Z}_+.$$

**Proof** Set $1 \leq b < a \leq d$, for $\mu_a, \mu_b \in S_n$, we have $\mu_a \prec \mu_b$. Write

$$\mu_a = I_{-n_1} \cdots I_{-n_r} L_{-m_1} \cdots L_{-m_s} v, \quad \mu_b = I_{-k_1} \cdots I_{-k_p} L_{-l_1} \cdots L_{-l_q} v.$$

Then we obtain

$$\mu_{d+1-a} = I_{-m_1} \cdots I_{-m_s} L_{-n_1} \cdots L_{-n_r} v,$$

$$\mu_{d+1-b} = I_{-l_1} \cdots I_{-l_q} L_{-k_1} \cdots L_{-k_p} v.$$

Next we consider the three cases of $\succ$ on $S_n$. If case (1) stands, so we get $\sum_{i=1}^{s} m_i < \sum_{j=1}^{q} l_j$. Then we have

$$(I_{-n_1} \cdots I_{-n_r} v | L_{-k_1} \cdots L_{-k_p} v) = 0.$$

It follows from Lemma 4.1 that $I_{n_r} \cdots I_{n_1} L_{-k_1} \cdots L_{-k_p} v = 0$. Hence

$$L_{m_s} \cdots L_{m_1} I_{-l_1} \cdots I_{-l_q} I_{n_r} \cdots I_{n_1} L_{-k_1} \cdots L_{-k_p} v = 0.$$

Thus we have

$$\begin{aligned}A_{ab}=(\mu_a|\mu_{d+1-b})=&(I_{-n_1}\cdots I_{-n_r}L_{-m_1}\cdots L_{-m_s}v|I_{-l_1}\cdots I_{-l_q}L_{-k_1}\cdots L_{-k_p}v)\\=&(v|L_{m_s}\cdots L_{m_1}I_{n_r}\cdots I_{n_1}I_{-l_1}\cdots I_{-l_q}L_{-k_1}\cdots L_{-k_p}v)\\=&(v|L_{m_s}\cdots L_{m_1}I_{-l_1}\cdots I_{-l_q}I_{n_r}\cdots I_{n_1}L_{-k_1}\cdots L_{-k_p}v)\\=&(L_{-m_1}\cdots L_{-m_s}v|I_{-l_1}\cdots I_{-l_q}v)(I_{-n_1}\cdots I_{-n_r}v|L_{-k_1}\cdots L_{-k_p}v)\\=&0.\end{aligned}$$

If case (2) stands, that is $\sum_{i=1}^{r}n_i=\sum_{j=1}^{p}k_j$ and $(n_1,...,n_r)\prec(k_1,...,k_p)$, by Lemma 4.1, we have

$$(L_{-k_1}\cdots L_{-k_p}v|I_{-n_1}\cdots I_{-n_r}v)=(I_{-n_1}\cdots I_{-n_r}v|L_{-k_1}\cdots L_{-k_p}v)=0.$$

Thus

$$\begin{aligned}A_{ab}=(\mu_a|\mu_{d+1-b})=&(I_{-n_1}\cdots I_{-n_r}L_{-m_1}\cdots L_{-m_s}v|I_{-l_1}\cdots I_{-l_q}L_{-k_1}\cdots L_{-k_p}v)\\=&(v|L_{m_s}\cdots L_{m_1}I_{n_r}\cdots I_{n_1}I_{-l_1}\cdots I_{-l_q}L_{-k_1}\cdots L_{-k_p}v)\\=&(v|L_{m_s}\cdots L_{m_1}I_{-l_1}\cdots I_{-l_q}I_{n_r}\cdots I_{n_1}L_{-k_1}\cdots L_{-k_p}v)\\=&(L_{-m_1}\cdots L_{-m_s}v|I_{-l_1}\cdots I_{-l_q}v)(I_{-n_1}\cdots I_{-n_r}v|L_{-k_1}\cdots L_{-k_p}v)\\=&0.\end{aligned}$$

If case (3) stands, that is $\sum_{i=1}^{r}n_i=\sum_{j=1}^{p}k_j$, $(n_1,...,n_r)=(k_1,...,k_p)$, $(m_1,...,m_s)\succ(l_1,...,l_q)$, by Lemma 4.1, we have

$$(L_{-m_1}\cdots L_{-m_s}v|I_{-l_1}\cdots I_{-l_q}v)=(I_{-l_1}\cdots I_{-l_q}v|L_{-m_1}\cdots L_{-m_s}v)=0.$$

Thus

$$\begin{aligned}A_{ab}=(\mu_a|\mu_{d+1-b})=&(I_{-n_1}\cdots I_{-n_r}L_{-m_1}\cdots L_{-m_s}v|I_{-l_1}\cdots I_{-l_q}L_{-k_1}\cdots L_{-k_p}v)\\=&(v|L_{m_s}\cdots L_{m_1}I_{n_r}\cdots I_{n_1}I_{-l_1}\cdots I_{-l_q}L_{-k_1}\cdots L_{-k_p}v)\\=&(v|L_{m_s}\cdots L_{m_1}I_{-l_1}\cdots I_{-l_q}I_{n_r}\cdots I_{n_1}L_{-k_1}\cdots L_{-k_p}v)\\=&(L_{-m_1}\cdots L_{-m_s}v|I_{-l_1}\cdots I_{-l_q}v)(I_{-n_1}\cdots I_{-n_r}v|L_{-k_1}\cdots L_{-k_p}v)\\=&0.\end{aligned}$$

From the above three cases, we see that if $1\le b<a\le d$, we have $A_{ab}=0$, so the matrix $A_n$ is upper triangular. Thus the determinant $detA_n$ is the product of diagonal elements. By Lemma 4.1, we have

$$\begin{aligned}A_{aa}=(\mu_a|\mu_{d+1-a})=&(I_{-n_1}\cdots I_{-n_r}L_{-m_1}\cdots L_{-m_s}v|I_{-m_1}\cdots I_{-m_s}L_{-n_1}\cdots L_{-n_r}v)\\=&(v|L_{m_s}\cdots L_{m_1}I_{n_r}\cdots I_{n_1}I_{-m_1}\cdots I_{-m_s}L_{-n_1}\cdots L_{-n_r}v)\\=&(v|L_{m_s}\cdots L_{m_1}I_{-m_1}\cdots I_{-m_s}I_{n_r}\cdots I_{n_1}L_{-n_1}\cdots L_{-n_r}v)\\=&(L_{-m_1}\cdots L_{-m_s}v|I_{-m_1}\cdots I_{-m_s}v)(I_{-n_1}\cdots I_{-n_r}v|L_{-n_1}\cdots L_{-n_r}v)\\=&K_a\prod_{i=1}^{s}f(m_i)^{x_i}\prod_{j=1}^{r}f(n_j)^{y_j},\end{aligned}$$

where $K_a$ is some nonzero integers, $x_i, y_j$ are the times of $n_i, m_i$ appearing in $(n_1, \ldots, n_r)$, $(m_1, \ldots, m_s)$. This completes the lemma. □

Next is our main result.

**Theorem 4.3** *The Verma module $M$ over the $W$-algebra $\mathfrak{g}(0)$ is irreducible if and only if $h_I \neq 0$.*

It follows from a similar proof as the one of Theorem 3.3.

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