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# Grand Generalized Weighted Morrey Spaces for RD-Spaces

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**Abstract** Let $(X, d, \mu)$ be an RD-space satisfying both the doubling condition in the sense of Coifman and Weiss and the reverse doubling condition. In this setting, the author obtains the definition of grand generalized weighted Morrey space on $(X, d, \mu)$, and also investigates some properties of these spaces. As an application, the boundedness of the Hardy-Littlewood maximal operator and the $\theta$-type Calderón-Zygmund operator on spaces $\mathcal{L}^{p),\varphi,\Phi}_{\omega}(X)$ is also obtained.

**Keywords** RD-space; Hardy-Littlewood maximal operator; $\theta$-type Calderón-Zygmund operator; grand generalized weighted Morrey space

**MR(2020) Subject Classification** 42B20; 42B35; 43A85

## 1. Introduction

As we all know, to investigate the local behaviour of solutions for the second order partial differential equations, Morrey in [1] has introduced the Morrey space, which generalizes the Lebesgue space and has important applications in harmonic analysis [2–4]. On the other hand, weighted inequalities play a key role in harmonic analysis, but their use is best justified by the variety of applications. In 2009, Komori and Shirai [5] introduced the weighted Morrey spaces on the classical Euclidean space equipped with Lebsgue measure, and also obtained the boundedness of Hardy-Littlewood maximal operator and Calderón-Zygmund operator on this space. In 2016, Nakamura [6] extended the Komori and Shirai's results to generalized weighted Morrey spaces. Moreover, many results from real analysis and harmonic analysis on Euclidean space have been proved valid underlying spaces replaced by metric measure spaces. One of the most important of these spaces is the space of homogeneous type in the sense of Coifman and Weiss [7,8]. Since then, many papers focus on the space of homogeneous type in the sense of Coifman and Weiss. For example, Nakai in [9] studied the Morrey spaces on the space of homogeneous type. Meskhi and Sawano provided several structural properties of grand variable exponent Lebesgue and Morrey spaces over spaces of homogeneous type [10]. In 2021, Kokilashvili and Meskhi in [11] obtained the boundedness of Hardy-Littlewood maximal operator and Riesz transforms in weighted grand Morrey spaces over space of homogeneous type. However, in 2008, Han et al. introduced a special

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space of homogeneous type, i.e., RD-space [12]. Recently, we mention that some papers focus on the different function spaces over RD-spaces and its applications; for example, see [13–16] and the references therein. For Morrey spaces in other setting of metric measure spaces, such as non-doubling measure and non-homogeneous metric measure spaces, we refer readers to see [17–20] and so on.

In this paper, the author first obtains the definition of grand generalized weighted Morrey space $\mathcal{L}_{\omega}^{p),\varphi,\Phi}(X)$ and establishes some properties about spaces $\mathcal{L}_{\omega}^{p),\varphi,\Phi}(X)$. Secondly, the author proves that Hardy-Littlewood maximal operator $M$ is bounded from space $\mathcal{L}_{\omega}^{p),\varphi,\Phi,\sigma}(X)$ into spaces $\mathcal{L}_{\omega}^{p),\varphi,\Phi,\sigma}(X)$, where $1<p<\infty$, $\varphi(\cdot)$ is a continuous positive function defined on $(0,p-1]$ with satisfying $\lim_{\varepsilon\to 0}\varphi(\varepsilon)=0$, $\Phi$ is an increasing function defined on $(0,\infty)$ and $0<\sigma<p-1$. Finally, the boundedness of $\theta$-type Calderón-Zygmund operator $T_\theta$ on spaces $L_{\omega}^{p),\varphi}(X)$ and on spaces $L_{\omega}^{p),\varphi,\Phi,\sigma}(X)$ is also investigated.

The definition of spaces of homogeneous type and RD-spaces is as follows.

**Definition 1.1** ([8,21]) *Let $(X,d)$ be a metric space equipped with a regular Borel measure $\mu$ such that all balls defined by $d$ have finite and positive measure. Then:*

*(i) The triple $(X,d,\mu)$ is called a space of homogeneous type if there exists a constant $C_1\in[1,\infty)$ such that, for any $x\in X$ and $r\in(0,\infty)$,*

$$\mu(B(x,2r))\le C_1\mu(B(x,r)). \tag{1.1}$$

*(ii) The triple $(X,d,\mu)$ is called an RD-space if there exists constants $\kappa\in(0,n]$ and $C_2\in[1,\infty)$ such that, for any $x\in X$, $r\in(0,\operatorname{diam}(X)/2)$ and $\lambda\in[1,\operatorname{diam}(X)/(2r))$,*

$$(C_2)^{-1}\lambda^{\kappa}\mu(B(x,r))\le\mu(B(x,\lambda r))\le C_2\lambda^{n}\mu(B(x,r)), \tag{1.2}$$

*where $\operatorname{diam}(X)=\sup_{x,y\in X}d(x,y)$ and $n$ measures the "dimension" of $X$.*

**Remark 1.2** (i) If we take $\lambda=2$ in (1.2), then RD-space is just the space of homogeneous type in the sense of Coifman and Weiss [7,8].

(ii) Han et al. in [12] have showed that, if measure $\mu$ satisfies (1.1), then $\mu$ satisfies (1.2) if and only if there exists $a_0,C_0\in(1,\infty)$ such that, for any $x\in X$ and $r\in(0,\operatorname{diam}(X)/a_0)$,

$$\mu(B(x,a_0r))\ge C_0\mu(B(x,r))$$

and equivalently, for any $x\in X$ and $r\in(0,\operatorname{diam}(X)/a_0)$, $B(x,a_0r)\setminus B(x,r)\ne\emptyset$.

Throughout the paper, $C$ represents a positive constant being independent of the main parameters involved, but it may be different from line to line. For any ball $B$, we use $c_B$ and $r_B$ to denote its center and radius, respectively. Given any ball $B$ and $\beta\in(0,\infty)$, $\beta B$ represents the ball which has the same center as ball $B$ and radius is $\beta$ times of $B$. For any $p\in[1,\infty)$, we denote by $p'$ its conjugate index, i.e., $\frac{1}{p}+\frac{1}{p'}=1$. Given a measurable set $E\subset X$, $\chi_E$ denotes its characteristic function.

## 2. Grand generalized weighted Morrey spaces

In this section, we establish two classes of definitions of grand generalized weighted Morrey spaces on RD-spaces, and then obtain the embedding, density and equivalence properties about the grand generalized weighted Morrey spaces on RD-spaces.

Here and in what follows, we always assume that $(X, d, \mu)$ is an RD-space with $\mu(X) < \infty$, and a function $\omega$ is called a weight if it is a locally integrable function on $(X, d, \mu)$ and take values in $(0, \infty)$ almost everywhere. Moreover, for any measurable set $E \subset X$ and a weighted $\omega$, we have $\omega(E) = \int_E \omega(x) \mathrm{d}\mu(x)$.

**Definition 2.1** (grand generalized weighted Morrey space) *Let $1 < p < \infty$, $\varphi : (0, p-1] \to (0, \infty)$ be a continuous positive function satisfying the condition $\lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0$, $\omega$ be a weight and $\Phi : (0, \infty) \to (0, \infty)$ be an increasing function. Then the grand generalized weighted Morrey space $\mathcal{L}_\omega^{p),\varphi,\Phi}(X)$ is defined by setting*

$$\mathcal{L}_\omega^{p),\varphi,\Phi}(X) := \{f \in L_{\mathrm{loc}}^p(\omega, X) : \|f\|_{\mathcal{L}_\omega^{p),\varphi,\Phi}(X)} < \infty\},$$

*where*

$$\begin{aligned} \|f\|_{\mathcal{L}_\omega^{p),\varphi,\Phi}(X)} &:= \sup_{0<\varepsilon<p-1} \sup_B \varphi(\varepsilon) \Big( \frac{1}{\Phi(\omega(B))} \int_B |f(x)|^{p-\varepsilon} \omega(x) \mathrm{d}\mu(x) \Big)^{\frac{1}{p-\varepsilon}} \\ &= \sup_{0<\varepsilon<p-1} \sup_B \frac{\varphi(\varepsilon)}{[\Phi(\omega(B))]^{\frac{1}{p-\varepsilon}}} \|f\|_{L_\omega^{p-\varepsilon}(B)} \end{aligned} \tag{2.1}$$

*and the supremum is taken over all balls $B \subset X$.*

*We also denote by $W\mathcal{L}_\omega^{p),\varphi,\Phi}(X)$ the grand generalized weighted weak Morrey space of all locally integrable functions satisfying*

$$\|f\|_{W\mathcal{L}_\omega^{p),\varphi,\Phi}(X)} := \sup_{0<\varepsilon<p-1} \sup_B \sup_{t>0} \varphi(\varepsilon) [\Phi(\omega(B))]^{-\frac{1}{p-\varepsilon}} t [\omega(\{x \in B : |f(x)| > t\})]^{\frac{1}{p-\varepsilon}}. \tag{2.2}$$

Moreover, for any $f \in \mathcal{L}_\omega^{p),\varphi,\Phi}(X)$, by applying (2.1) and (2.2), it is easy to verify that

$$\|f\|_{W\mathcal{L}_\omega^{p),\varphi,\Phi}(X)} \le \|f\|_{\mathcal{L}_\omega^{p),\varphi,\Phi}(X)}.$$

**Remark 2.2** (1) If we take $\varphi(\varepsilon) = \varepsilon^{\frac{\theta}{p-\varepsilon}}$ with $\theta > 0$ in (2.1) and (2.2), then $\mathcal{L}_\omega^{p),\theta,\Phi}(X) := \mathcal{L}_\omega^{p),\varphi,\Phi}(X)$ and $W\mathcal{L}_\omega^{p),\theta,\Phi}(X) := W\mathcal{L}_\omega^{p),\varphi,\Phi}(X)$, which are the grand generalized weighted Morrey space and the grand generalized weighted weak Morrey space in the original form.

(2) If we take $\omega \equiv 1$ in Definition 2.1, then spaces $\mathcal{L}_\omega^{p),\varphi,\Phi}(X)$ and spaces $W\mathcal{L}_\omega^{p),\varphi,\Phi}(X)$ are just the grand generalized Morrey space $\mathcal{L}^{p),\varphi,\Phi}(X)$ and the grand generalized weak Morrey space $W\mathcal{L}^{p),\varphi,\Phi}(X)$, respectively.

(3) If we take $\Phi(t) = t^{\frac{p-\varepsilon}{q}-1}$ with $t > 0$ and $1 < p \le q < \infty$ in (2.1) and (2.2), then grand generalized weighted Morrey space $\mathcal{L}_\omega^{p),\varphi,\Phi}(X)$ and grand generalized weighted weak Morrey space $W\mathcal{L}_\omega^{p),\varphi,\Phi}(X)$ are just the grand weighted Morrey space $M_\omega^{p),q,\varphi}(X)$ and the grand weighted weak Morrey space $WM_\omega^{p),q,\varphi}(X)$, respectively, see [11].

(4) If we take $\Phi(t) \equiv 1$ with $t > 0$ in (2.1) and (2.2), then grand generalized weighted Morrey space $\mathcal{L}_\omega^{p),\varphi,\Phi}(X)$ and grand generalized weighted weak Morrey space $W\mathcal{L}_\omega^{p),\varphi,\Phi}(X)$ are

just the grand weighted Lebesgue space $L^{p),\varphi}_\omega(X)$ and the grand weighted weak Lebesgue space $WL^{p),\varphi}_\omega(X)$.

We now recall the following notion of Muckenhoupt class $A_p(\mu)$.

**Definition 2.3** ([22]) *Let $1<p<\infty$. A weight $\omega$ belongs to the Muckenhoupt $A_p(\mu)$ if*

$$\|\omega\|_{A_p(\mu)} := \sup_B \Big(\frac{1}{\mu(B)}\int_B \omega(x)\mathrm{d}\mu(x)\Big)\Big\{\frac{1}{\mu(B)}\int_B [\omega(x)]^{1-p'}\mathrm{d}\mu(x)\Big\}^{p-1} < \infty, \tag{2.3}$$

*where the supremum is taken over all balls $B\subset X$.*

*Further, a weight $\omega$ is called an $A_1(\mu)$ weight if there exists a positive constant $C$ such that, for any ball $B\subset X$,*

$$\frac{1}{\mu(B)}\int_B \omega(x)\mathrm{d}\mu(x) \le C \operatorname*{ess\,inf}_{y\in B}\omega(y). \tag{2.4}$$

*As in the classical setting, let $A_\infty(\mu):=\bigcup_{p=1}^\infty A_p(\mu)$.*

The embedding and density properties on the spaces $\mathcal{L}^{p),\varphi,\Phi}_\omega(X)$ are as follows.

**Lemma 2.4** (embedding property) *Let $1<r<p<\infty$. Assume that the mapping $t\mapsto \Phi(t)/t$ is almost decreasing, namely, there exists a constant $c>0$ such that*

$$\frac{\Phi(s)}{s} \le c\frac{\Phi(t)}{t} \quad \text{for all } s\ge t>0. \tag{2.5}$$

*Then*

$$\mathcal{L}^{p,\Phi}_\omega(X) \hookrightarrow \mathcal{L}^{p),\theta,\Phi}_\omega(X) \hookrightarrow \mathcal{L}^{r),\theta,\Phi}_\omega(X).$$

*Moreover, if $\mu(X)<\infty$, then*

$$\mathcal{L}^{p),\varphi,\Phi}_\omega(X) \hookrightarrow \mathcal{L}^{r),\varphi,\Phi}_\omega(X). \tag{2.6}$$

**Proof** For any $f\in\mathcal{L}^{p,\Phi}_\omega(X)$, by applying (2.1) and the Hölder inequality with exponent $\frac{p}{p-\varepsilon}$, we have

$$\begin{aligned}
\|f\|_{\mathcal{L}^{p),\theta,\Phi}_\omega(X)} &= \sup_{0<\varepsilon<p-1}\sup_B\Big[\frac{\varepsilon^\theta}{\Phi(\omega(B))}\Big]^{\frac{1}{p-\varepsilon}}\Big(\int_B |f(x)|^{p-\varepsilon}\omega(x)\mathrm{d}\mu(x)\Big)^{\frac{1}{p-\varepsilon}}\\
&= \sup_{0<\varepsilon<p-1}\sup_B\Big[\frac{\varepsilon^\theta}{\Phi(\omega(B))}\Big]^{\frac{1}{p-\varepsilon}}\Big\{\int_B |f(x)|^{p-\varepsilon}[\omega(x)]^{\frac{p-\varepsilon}{p}+\frac{\varepsilon}{p}}\mathrm{d}\mu(x)\Big\}^{\frac{1}{p-\varepsilon}}\\
&\le \sup_{0<\varepsilon<p-1}\sup_B\Big[\frac{\varepsilon^\theta}{\Phi(\omega(B))}\Big]^{\frac{1}{p-\varepsilon}}\Big\{\int_B |f(x)|^{p}\omega(x)\mathrm{d}\mu(x)\Big\}^{\frac{1}{p}}\Big(\int_B\omega(x)\mathrm{d}\mu(x)\Big)^{\frac{1}{p(p-\varepsilon)}}\\
&= \sup_{0<\varepsilon<p-1}\sup_B \varepsilon^{\frac{\theta}{p-\varepsilon}}\Big\{\frac{1}{\Phi(\omega(B))}\int_B |f(x)|^{p}\omega(x)\mathrm{d}\mu(x)\Big\}^{\frac{1}{p}}\Big[\frac{\Phi(\omega(B))}{\omega(B)}\Big]^{\frac{1}{p}-\frac{1}{p-\varepsilon}}.
\end{aligned}$$

Case 1. If $r_B<1$, then from (2.5) and the monotonicity on function $t^{\frac{1}{p}-\frac{1}{p-\varepsilon}}$ with $t>0$ and $0<\varepsilon<p-1$, it follows that

$$\begin{aligned}
\|f\|_{\mathcal{L}^{p),\theta,\Phi}_\omega(X)} \le& C_p \sup_{0<\varepsilon<p-1}\sup_B\Big\{\frac{1}{\Phi(\omega(B))}\int_B |f(x)|^{p}\omega(x)\mathrm{d}\mu(x)\Big\}^{\frac{1}{p}}\Big[\frac{1}{c}\times\frac{\Phi(\omega(B(c_B,1)))}{\omega(B(c_B,1))}\Big]^{\frac{1}{p}-\frac{1}{p-\varepsilon}}\\
\le& C_p\|f\|_{\mathcal{L}^{p,\Phi}_\omega(X)}.
\end{aligned}$$

Case 2. If $r_B \geq 1$, then by applying (2.5), we can deduce that

$$\begin{aligned}
&\sup_{0<\varepsilon<p-1} \sup_B \varepsilon^{\frac{\theta}{p-\varepsilon}} \Big\{ \frac{1}{\Phi(\omega(B))} \int_B |f(x)|^p \omega(x) \mathrm{d}\mu(x) \Big\}^{\frac{1}{p}} [\frac{\Phi(\omega(B))}{\omega(B)}]^{\frac{1}{p}-\frac{1}{p-\varepsilon}} \\
&\leq C_p \|f\|_{\mathcal{L}^{p,\Phi}_\omega(X)} \sup_{0<\varepsilon<p-1} \sup_B [\frac{\Phi(\omega(B))}{\omega(B)} \times \frac{\omega(B(c_B,1))}{\Phi(\omega(c_B,1))}]^{\frac{1}{p}-\frac{1}{p-\varepsilon}} [\frac{\Phi(\omega(c_B,1))}{\omega(B(c_B,1))}]^{\frac{1}{p}-\frac{1}{p-\varepsilon}} \\
&\leq C_p \|f\|_{\mathcal{L}^{p,\Phi}_\omega(X)}.
\end{aligned}$$

Combining the Cases 1 and 2, we obtain that the embedding $\mathcal{L}^{p,\Phi}_\omega(X) \hookrightarrow \mathcal{L}^{p),\theta,\Phi}_\omega(X)$.

Now we show the embedding $\mathcal{L}^{p),\theta,\Phi}_\omega(X) \hookrightarrow \mathcal{L}^{r),\theta,\Phi}_\omega(X)$. For any $f \in \mathcal{L}^{p),\theta,\Phi}_\omega(X)$, from (2.1), Remark 2.2 (1) and the Hölder inequality with exponent $\frac{p-\varepsilon}{r-\varepsilon}$, it then follows that

$$\begin{aligned}
&[\frac{\varepsilon^\theta}{\Phi(\omega(B))}]^{\frac{1}{r-\varepsilon}} \|f\|_{L^{r-\varepsilon}_\omega(B)} \\
&= [\frac{\varepsilon^\theta}{\Phi(\omega(B))}]^{\frac{1}{r-\varepsilon}} \Big\{ \int_B |f(x)|^{r-\varepsilon} [\omega(x)]^{\frac{r-\varepsilon}{p-\varepsilon}+\frac{p-r}{p-\varepsilon}} \mathrm{d}\mu(x) \Big\}^{\frac{1}{r-\varepsilon}} \\
&\leq [\frac{\varepsilon^\theta}{\Phi(\omega(B))}]^{\frac{1}{r-\varepsilon}} \Big( \int_B |f(x)|^{p-\varepsilon} \omega(x) \mathrm{d}\mu(x) \Big)^{\frac{1}{p-\varepsilon}} \times \Big( \int_B \omega(x) \mathrm{d}\mu(x) \Big)^{\frac{p-r}{(p-\varepsilon)(r-\varepsilon)}} \\
&\leq \varepsilon^{\theta(\frac{1}{r-\varepsilon}-\frac{1}{p-\varepsilon})} [\frac{\Phi(\omega(B)}{\omega(B)}]^{\frac{1}{p-\varepsilon}-\frac{1}{r-\varepsilon}} \Big( \frac{\varepsilon^\theta}{\Phi(\omega(B))} \int_B |f(x)|^{p-\varepsilon} \omega(x) \mathrm{d}\mu(x) \Big)^{\frac{1}{p-\varepsilon}} \\
&\leq C_p \|f\|_{\mathcal{L}^{p),\theta,\Phi}_\omega(X)}.
\end{aligned}$$

Taking the supremum for $0 < \varepsilon < r-1$ at the left of the above inequality, we have

$$\|f\|_{\mathcal{L}^{r),\theta,\Phi}_\omega(X)} \leq C_p \|f\|_{\mathcal{L}^{p),\theta,\Phi}_\omega(X)}.$$

Thus, we complete the proof of Lemma 2.4. □

**Example 2.5** Let $(X, d, \mu) := (\mathbb{R}, |\cdot|, \mathrm{d}x)$, where $|\cdot|$ denotes the Euclidean distance and $\mathrm{d}x$ represents the Lebesgue measure on $\mathbb{R}$, namely, let $I = (0, 1] \subset \mathbb{R}$, $p \in (1, \infty)$, $\omega(x) \equiv 1$, $\varphi(\varepsilon) = \varepsilon^{\frac{1}{p-\varepsilon}}$, $f(x) = x^{-\frac{1}{p}}\chi_I(x)$ and $\Phi(t) \equiv t$ with $t > 0$. Then it is easy to show that $f \in \mathcal{L}^{p),\varphi,\Phi}(I)$ and $f \notin \mathcal{L}^{p,\Phi}(I)$.

**Lemma 2.6** (density property) *Let $1 < p < \infty$, $\omega$ be a weight, $(X, d, \mu)$ be an RD-space, and $\mu(X) < \infty$. Then, for all $f \in \overline{\mathcal{L}}^{p),\varphi,\Phi}_\omega(X)$, the following equation $\lim_{\varepsilon\to 0} \|f\|_{\mathcal{L}^{p-\varepsilon,\Phi}_\omega(X)} = 0$ holds, where $\overline{\mathcal{L}}^{p),\varphi,\Phi}_\omega(X)$ is the closure of space $\mathcal{L}^{p,\Phi}_\omega(X)$ in space $\mathcal{L}^{p),\varphi,\Phi}_\omega(X)$.*

**Proof** Let $f \in \overline{\mathcal{L}}^{p),\varphi,\Phi}_\omega(X)$ and $\epsilon > 0$. Assume that the mapping $t \mapsto \Phi(t)/t$ is almost decreasing which satisfies (2.5). Then there exists a function $f_{n_0} \in \mathcal{L}^{p,\Phi}_\omega(X)$ such that

$$\|f - f_{n_0}\|_{\mathcal{L}^{p),\varphi,\Phi}_\omega(X)} < \frac{\epsilon}{2}. \tag{2.7}$$

For $f_{n_0}$ and $\epsilon$, by applying Definition 2.1, the Hölder inequality with exponent $\frac{p}{p-\varepsilon}$ and (2.5), we obtain

$$\sup_B \frac{\varphi(\varepsilon)}{[\Phi(\omega(B))]^{\frac{1}{p-\varepsilon}}} \|f_{n_0}\|_{L^{p-\varepsilon}_\omega(B)}$$

$$
\begin{aligned}
&= \sup_B \frac{\varphi(\varepsilon)}{[\Phi(\omega(B))]^{\frac{1}{p-\varepsilon}}} \Big\{ \int_B |f_{n_0}(x)|^{p-\varepsilon} [\omega(x)]^{\frac{p-\varepsilon}{p}+\frac{\varepsilon}{p}} \mathrm{d}\mu(x) \Big\}^{\frac{1}{p-\varepsilon}} \\
&\le \sup_B \frac{\varphi(\varepsilon)}{[\Phi(\omega(B))]^{\frac{1}{p-\varepsilon}}} \Big( \int_B |f_{n_0}(x)|^{p} \omega(x) \mathrm{d}\mu(x) \Big)^{\frac{1}{p}} [\omega(B)]^{\frac{1}{p-\varepsilon}-\frac{1}{p}} \\
&= \sup_B \frac{\varphi(\varepsilon)}{[\Phi(\omega(B))]^{\frac{1}{p-\varepsilon}-\frac{1}{p}}} \Big( \frac{1}{\Phi(\omega(B))} \int_B |f_{n_0}(x)|^{p} \omega(x) \mathrm{d}\mu(x) \Big)^{\frac{1}{p}} [\omega(B)]^{\frac{1}{p-\varepsilon}-\frac{1}{p}} \\
&\le M\varphi(\varepsilon) \|f_{n_0}\|_{L^{p,\Phi}_\omega(X)} < \frac{\epsilon}{2},
\end{aligned}
$$

when $\varepsilon$ is sufficiently small, because $\lim_{\varepsilon\to 0}\varphi(\varepsilon)=0$ and $M=\sup_B[\frac{\Phi(\omega(B))}{\omega(B)}]^{\frac{1}{p}-\frac{1}{p-\varepsilon}}$.

Further, taking such function $f_{n_0}$, for any small $\epsilon$, by the Minkowski inequality, we have

$$
\varphi(\varepsilon)\|f\|_{\mathcal{L}^{p-\varepsilon,\Phi}_\omega(X)} \le \varphi(\varepsilon)\|f-f_{n_0}\|_{\mathcal{L}^{p-\varepsilon,\Phi}_\omega(X)} + \varphi(\varepsilon)\|f_{n_0}\|_{\mathcal{L}^{p-\varepsilon,\Phi}_\omega(X)} < \frac{\epsilon}{2}+\frac{\epsilon}{2} < \epsilon,
$$

moreover, $\varphi$ defined on $(0,p-1]$ is a continuous positive function. Thus, we complete the proof of Lemma 2.6. □

We now introduce another definition of grand generalized weighted Morrey space depending on more parameters, $\sigma\in(0,p-1)$, and is defined by

$$
\|f\|_{\mathcal{L}^{p),\varphi,\Phi,\sigma}_\omega(X)} := \sup_{0<\varepsilon<\sigma}\sup_B \varphi(\varepsilon)\Big(\frac{1}{\Phi(\omega(B))}\int_B |f(x)|^{p-\varepsilon}\omega(x)\mathrm{d}\mu(x)\Big)^{\frac{1}{p-\varepsilon}}, \quad 1<p<\infty. \tag{2.8}
$$

The following lemma is valid.

**Lemma 2.7** *Assume that the mapping $t\mapsto\Phi(t)/t$ is almost decreasing and satisfying (2.5). Then the norm $\|f\|_{\mathcal{L}^{p),\varphi,\Phi,\sigma}_\omega(X)}$ is equivalent to $\|f\|_{\mathcal{L}^{p),\varphi,\Phi}_\omega(X)}$; that is, there exists a constant $c>1$ independent of $f$ such that*

$$
c^{-1}\|f\|_{\mathcal{L}^{p),\varphi,\Phi,\sigma}_\omega(X)} \le \|f\|_{\mathcal{L}^{p),\varphi,\Phi}_\omega(X)} \le c\|f\|_{\mathcal{L}^{p),\varphi,\Phi,\sigma}_\omega(X)}.
$$

**Proof** From Definition 2.1 and (2.8), it is easy to see that $\|f\|_{\mathcal{L}^{p),\varphi,\Phi,\sigma}_\omega(X)} \le \|f\|_{\mathcal{L}^{p),\varphi,\Phi}_\omega(X)}$ holds. Hence, we only need to show that there exists some constant $C_{\varphi,\sigma,p}>0$ such that

$$
\|f\|_{\mathcal{L}^{p),\varphi,\Phi}_\omega(X)} \le C_{\varphi,\sigma,p}\|f\|_{\mathcal{L}^{p),\varphi,\Phi,\sigma}_\omega(X)}.
$$

Since

$$
\begin{aligned}
&\sup_{0<\varepsilon<p-1}\sup_B \varphi(\varepsilon)\Big[\frac{1}{\Phi(\omega(B))}\int_B |f(x)|^{p-\varepsilon}\omega(x)\mathrm{d}\mu(x)\Big]^{\frac{1}{p-\varepsilon}} \\
&\le \sup_{0<\varepsilon<\sigma}\sup_B \varphi(\varepsilon)\Big[\frac{1}{\Phi(\omega(B))}\int_B |f(x)|^{p-\varepsilon}\omega(x)\mathrm{d}\mu(x)\Big]^{\frac{1}{p-\varepsilon}} + \\
&\quad \sup_{\sigma\le\varepsilon<p-1}\sup_B \varphi(\varepsilon)\Big[\frac{1}{\Phi(\omega(B))}\int_B |f(x)|^{p-\varepsilon}\omega(x)\mathrm{d}\mu(x)\Big]^{\frac{1}{p-\varepsilon}},
\end{aligned}
$$

we only need to estimate the second part of the right hand. For $\sigma\le\varepsilon<p-1$, by applying the Hölder inequality with $\frac{p-\sigma}{p-\varepsilon}$, we have

$$
\frac{\varphi(\varepsilon)}{[\Phi(\omega(B))]^{\frac{1}{p-\varepsilon}}}\Big(\int_B |f(x)|^{p-\varepsilon}\omega(x)\mathrm{d}\mu(x)\Big)^{\frac{1}{p-\varepsilon}}
$$

$$= \frac{\varphi(\varepsilon)}{[\Phi(\omega(B))]^{\frac{1}{p-\varepsilon}}} \Big\{ \int_B |f(x)|^{p-\varepsilon} [\omega(x)]^{\frac{p-\varepsilon}{p-\sigma}+\frac{\varepsilon-\sigma}{p-\sigma}} \mathrm{d}\mu(x) \Big\}^{\frac{1}{p-\varepsilon}}$$
$$\leq \frac{\varphi(\varepsilon)}{[\Phi(\omega(B))]^{\frac{1}{p-\varepsilon}}} \Big( \int_B |f(x)|^{p-\sigma} \omega(x) \mathrm{d}\mu(x) \Big)^{\frac{1}{p-\sigma}} [\omega(B)]^{\frac{\varepsilon-\sigma}{(p-\varepsilon)(p-\sigma)}}$$
$$\leq \frac{\varphi(\varepsilon)}{[\Phi(\omega(B))]^{\frac{1}{p-\varepsilon}}} \frac{\varphi(\sigma)}{[\Phi(\omega(B))]^{-\frac{1}{p-\sigma}}} [\omega(B)]^{\frac{\varepsilon-\sigma}{(p-\varepsilon)(p-\sigma)}} \times$$
$$\varphi(\sigma) \Big( \frac{1}{\Phi(\omega(B))} \int_B |f(x)|^{p-\sigma} \omega(x) \mathrm{d}\mu(x) \Big)^{\frac{1}{p-\sigma}}$$
$$\leq \|f\|_{\mathcal{L}_\omega^{p),\varphi,\Phi,\sigma}(X)} \varphi(p-1) [\varphi(\sigma)]^{-1} [\frac{\Phi(\omega(B))}{\omega(B)}]^{\frac{1}{p-\sigma}-\frac{1}{p-\varepsilon}}$$
$$\leq C_{\varphi,\sigma,p} \|f\|_{\mathcal{L}_\omega^{p),\varphi,\Phi,\sigma}(X)}.$$

Thus, we obtain the desired result. □

## 3. Boundedness of Hardy-Littlewood maximal operators

In this section, we investigate the boundedness of Hardy-Littlewood maximal operator $M$ on spaces $\mathcal{L}_\omega^{p),\varphi,\Phi,\sigma}(X)$. Respectively, the Hardy-Littlewood maximal operator $M$ is defined by setting

$$Mf(x) := \sup_B \frac{1}{\mu(B)} \int_B |f(y)| \mathrm{d}\mu(y), \quad \text{for any } x \in X. \tag{3.1}$$

The main results of this section are stated as follows.

**Theorem 3.1** *Let $1 < p < \infty$, $\mu(X) < \infty$, $\omega \in A_p(X)$ and $\varphi : (0, p-1] \to (0, \infty)$ be a continuous positive function satisfying the condition $\lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0$. Then $M$ defined as in (3.1) is bounded on spaces $L_\omega^{p),\varphi}(X)$.*

**Theorem 3.2** *Let $1 < p < \infty$, $\mu(X) < \infty$, $\omega \in A_p(X)$, $\varphi : (0, p-1] \to (0, \infty)$ be a continuous positive function satisfying the condition $\lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0$, $\Phi : (0, \infty) \to (0, \infty)$ be an increasing function and $\sigma \in (0, p-1)$. Then the operator $M$ defined as in (3.1) is bounded on spaces $\mathcal{L}_\omega^{p),\varphi,\Phi,\sigma}(X)$.*

Once Theorem 3.2 holds, by applying Lemma 2.7, it is easy to obtain the following corollary.

**Corollary 3.3** *Let $1 < p < \infty$, $\mu(X) < \infty$, $\omega \in A_p(X)$, $\varphi : (0, p-1] \to (0, \infty)$ be a continuous positive function satisfying the condition $\lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0$, $\Phi : (0, \infty) \to (0, \infty)$ be an increasing function and $\sigma \in (0, p-1)$. Then the operator $M$ defined as in (3.1) is bounded on spaces $\mathcal{L}_\omega^{p),\varphi,\Phi}(X)$.*

To prove the above main theorems, we need to recall the following lemma in [23].

**Lemma 3.4** *Let $p \in [1, \infty)$ and $\omega \in A_p(X)$. Then $M$ defined as in (3.1) is bounded on spaces $L^p(\omega)$ for $p \in (1, \infty)$ and bounded from spaces $L^1(\omega)$ to spaces $L^{1,\infty}(\omega)$.*

**Proof of Theorem 3.1** By the openness property of the Muckenhoupt class $A_p(X)$ (see [23]), there exists some number $\delta$ such that $0 < \delta < p-1$ and $\omega \in A_p(\mu)$. By applying Lemma 3.4, we have

$$\|Mf\|_{L^{p-\delta}_\omega(X)} \le C_1\|f\|_{L^{p-\delta}_\omega(X)}, \quad f\in L^{p-\delta}_\omega(X),$$

$$\|Mf\|_{L^{p}_\omega(X)} \le C_2\|f\|_{L^{p}_\omega(X)}, \quad f\in L^{p}_\omega(X),$$

where the positive constants $C_1$ and $C_2$ are independent of $f$.

By virtue of Calderón-Zygmund interpolation theorem for sublinear operators, we conclude that there exists some constant $C>0$ independent of $\varepsilon$ and $f$ such that

$$\|Mf\|_{L^{p-\varepsilon}_\omega(X)} \le C_2\|f\|_{L^{p-\varepsilon}_\omega(X)}, \quad f\in L^{p-\varepsilon}_\omega(X),\ 0<\varepsilon<\delta. \tag{3.2}$$

Write

$$\begin{aligned}\|Mf\|_{L^{p),\varphi}_\omega(X)} &= \sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\Big(\int_X |Mf(x)|^{p-\varepsilon}\omega(x)\mathrm{d}\mu(x)\Big)^{\frac{1}{p-\varepsilon}}\\ &\le \sup_{0<\varepsilon<\delta}\varphi(\varepsilon)\Big(\int_X |Mf(x)|^{p-\varepsilon}\omega(x)\mathrm{d}\mu(x)\Big)^{\frac{1}{p-\varepsilon}}+\\ &\quad \sup_{\delta\le\varepsilon<p-1}\varphi(\varepsilon)\Big(\int_X |Mf(x)|^{p-\varepsilon}\omega(x)\mathrm{d}\mu(x)\Big)^{\frac{1}{p-\varepsilon}}\\ &=:\mathrm{D}_1+\mathrm{D}_2.\end{aligned}$$

By (3.2), we have

$$\mathrm{D}_1\le \sup_{0<\varepsilon<\delta}\varphi(\varepsilon)\|Mf\|_{L^{p-\varepsilon}_\omega(X)}\le C\sup_{0<\varepsilon<\delta}\varphi(\varepsilon)\|f\|_{L^{p-\varepsilon}_\omega(X)}\le C\|f\|_{L^{p),\varphi}_\omega(X)}.$$

From the Hölder inequality with exponent $\frac{p-\delta}{p-\varepsilon}$ and Lemma 3.4, it then follows that

$$\begin{aligned}\mathrm{D}_2 &\le \sup_{\delta\le\varepsilon<p-1}\varphi(\varepsilon)\Big(\int_X |Mf(x)|^{p-\delta}\omega(x)\mathrm{d}\mu(x)\Big)^{\frac{1}{p-\delta}}[\omega(X)]^{\frac{\varepsilon-\delta}{(p-\varepsilon)(p-\delta)}}\\ &\le C\sup_{\delta\le\varepsilon<p-1}\varphi(\varepsilon)[\varphi(\delta)]^{-1}\varphi(\delta)\Big(\int_X |f(x)|^{p-\delta}\omega(x)\mathrm{d}\mu(x)\Big)^{\frac{1}{p-\delta}}[\omega(X)]^{\frac{\varepsilon-\delta}{(p-\varepsilon)(p-\delta)}}\\ &\le C\|f\|_{L^{p),\varphi}_\omega(X)}\varphi(p-1)[\varphi(\delta)]^{-1}[\omega(X)]^{p-1-\delta}\le C_p\|f\|_{L^{p),\varphi}_\omega(X)}.\end{aligned}$$

Combining with the estimate of $\mathrm{D}_1$, we complete the proof of Theorem 3.1. □

**Proof of Theorem 3.2** By applying (2.8) and (3.2), we have

$$\begin{aligned}&\sup_{0<\varepsilon<\sigma}\sup_B\varphi(\varepsilon)\Big(\frac{1}{\Phi(\omega(B))}\int_B |Mf(x)|^{p-\varepsilon}\omega(x)\mathrm{d}\mu(x)\Big)^{\frac{1}{p-\varepsilon}}\\ &\le \sup_{0<\varepsilon<\sigma}\sup_B\frac{\varphi(\varepsilon)}{[\Phi(\omega(B))]^{\frac{1}{p-\varepsilon}}}\|Mf\|_{L^{p-\varepsilon}_\omega(X)}\\ &\le C\sup_{0<\varepsilon<\sigma}\sup_B\frac{\varphi(\varepsilon)}{[\Phi(\omega(B))]^{\frac{1}{p-\varepsilon}}}\Big(\int_B |f(x)|^{p-\varepsilon}\omega(x)\mathrm{d}\mu(x)\Big)^{\frac{1}{p-\varepsilon}}\\ &\le C\|f\|_{\mathcal{L}^{p),\varphi,\Phi,\sigma}_\omega(X)}.\end{aligned}$$

Hence, the proof of Theorem 3.2 is completed. □

## 4. Boundedness of $\theta$-tyep Calderón-Zygmund operators

Before stating the main theorems of this section, we first recall some necessary definitions and notation and give the lemma needed in the proof.

For any $x, y \in X$ and $\delta \in (0, \infty)$, set

$$V_\delta(x) := \mu(B(x, \delta)) \text{ and } V(x, y) := \mu(x, d(x, y)).$$

It follows from (1.1) that $V(x, y) \sim V(y, x)$.

We now recall the definition of $\theta$-type Calderón-Zygmund integral operator as follows

**Definition 4.1** ([24]) *Let $\theta$ be a non-negative, non-decreasing function defined on $[0, +\infty)$ and satisfy the following condition*

$$\int_0^1 \frac{\theta(t)}{t} \mathrm{d}t < \infty. \tag{4.1}$$

*And a measurable function $K_\theta(\cdot, \cdot)$ on $X \times X \setminus \{(x, x) : x \in X\}$ is called a $\theta$-type kernel if there exists some constant $C > 0$ such that*

*(i) For all $x, y \in X$ with $x \neq y$,*

$$|K_\theta(x, y)| \leq \frac{C}{V(x, y)}. \tag{4.2}$$

*(ii) For all $x, x', y \in X$ satisfying $d(x, x') \leq 2d(x, y)$,*

$$|K_\theta(x, y) - K_\theta(x', y)| + |K_\theta(y, x) - K_\theta(y, x')| \leq C\theta(\frac{d(x, x')}{d(x, y)})\frac{1}{V(x, y)}. \tag{4.3}$$

Let $L_b^\infty(\mu)$ be the space of all $L^\infty(\mu)$ functions with bounded support. A linear operator $T_\theta$ is called a $\theta$-type Calderón-Zygmund singular integral operator with kernel $K_\theta$ satisfying (4.2) and (4.3) if, for all $f \in L_b^\infty(\mu)$ and $x \in (X \setminus \mathrm{supp}(f))$,

$$T_\theta(f)(x) := \int_X K_\theta(x, y) f(y) \mathrm{d}\mu(y). \tag{4.4}$$

The main theorems of this section are stated as follows:

**Theorem 4.2** *Let $1 < p < \infty$, $\mu(X) < \infty$, $\omega \in A_p(X)$ and $\varphi : (0, p-1] \to (0, \infty)$ be a continuous positive function satisfying the condition $\lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0$. Suppose that the $\theta$-type Calderón-Zygmund operator $T_\theta$ defined by (4.4) is bounded on $L^2(\mu)$. Then $T_\theta$ is bounded on spaces $L_\omega^{p),\varphi}(X)$.*

**Theorem 4.3** *Let $1 < p < \infty$, $\mu(X) < \infty$, $\omega \in A_p(X)$, $\varphi : (0, p-1] \to (0, \infty)$ be a continuous positive function satisfying the condition $\lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0$, $\Phi : (0, \infty) \to (0, \infty)$ be an increasing function and $\sigma \in (0, p-1)$. Suppose that the $\theta$-type Calderón-Zygmund operator $T_\theta$ defined as in (4.4) is bounded on $L^2(\mu)$. Then $T_\theta$ is bounded on spaces $\mathcal{L}_\omega^{p),\varphi,\Phi,\sigma}(X)$.*

To prove the above theorems, we need to recall the following lemma in [24].

**Lemma 4.4** *Let $1 < p < \infty$ and $\omega \in A_p(X)$. Suppose that the $\theta$-type Calderón-Zygmund operator $T_\theta$ defined as in (4.4) is bounded on $L^2(\mu)$. Then $T_\theta$ is bounded on spaces $L_\omega^p(X)$ for*

$p \in (1, \infty)$, *and also bounded from spaces* $L^1_\omega(X)$ *to spaces* $L^{1,\infty}(\omega)$.

**Remark 4.5** By using Lemma 4.4 and similar arguments used in the proofs of Theorems 3.1 and 3.2, it is easy to show that Theorems 4.2 and 4.3 hold, respectively.

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