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Existence Result to a Class of Parabolic Equations with Nonstandard Growth Condition and Zero Order Term

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Abstract In the setting of variable exponent, an existence result to a class of parabolic equations with zero order term is proved. The proof of existence relies essentially on selecting some suitable test functions based upon the integrability of the source term and the zero order term simultaneously. By virtue of a priori estimates and some limit analyses, the weak limit of the nonlinear principal term is identified via the Young measures method.

Keywords nonstandard growth condition; zero order term; weak solutions

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1. Introduction

Assume that $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $\partial\Omega$ is the boundary of Ω, $T > 0$ is finite, $Q_T = \Omega \times (0, T)$ is a cylinder, and $\Gamma_T = \partial\Omega \times (0, T)$ stands for the lateral boundary. Consider a class of parabolic equations:

$$\begin{cases} \partial_t u - \operatorname{div} A(x, t, \nabla u) + g(x, t, u) = f(x, t), & (x, t) \in Q_T, \\ u(x, t) = 0, & (x, t) \in \Gamma_T, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

The hypotheses on Eq. (1.1) are given as follows:

(H1) Let $p(x)$ be a continuous function in $\overline{\Omega}$, $p^+ := \max_{\overline{\Omega}} p(x)$, $p^- := \min_{\overline{\Omega}} p(x)$ and $1 < p^- \le p(x) \le p^+ < +\infty$. Then $p(x)$ is known as the variable exponent. Assume that $p(x)$ also satisfies the log-Hölder continuity condition in [1], i.e.,

$$\forall\, x_1, x_2 \in \overline{\Omega},\ |x_1 - x_2| < 1,\ |p(x_1) - p(x_2)| < \omega(|x_1 - x_2|),$$

where $\omega : (0, \infty) \to \mathbb{R}$ is a nondecreasing function with $\limsup_{\alpha\to 0^+} \omega(\alpha) \ln(\frac{1}{\alpha}) < +\infty$.

(H2) $A(x, t, \eta) : Q_T \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function and assume that $A(x, t, \eta)$ satisfies the structure conditions

$$[A(x, t, \eta) - A(x, t, \eta')] \cdot (\eta - \eta') > 0; \tag{1.2}$$

$$A(x, t, \eta) \cdot \eta \ge \alpha|\eta|^{p(x)}; \tag{1.3}$$

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$$|A(x,t,\eta)| \le \beta|\eta|^{p(x)-1}, \tag{1.4}$$

where $\eta, \eta' \in \mathbb{R}^N$ with $\eta \neq \eta'$; and the constants $\alpha, \beta > 0$.

(H3) $g(x,t,s)$ is a Carathéodory function and satisfies

$$g(x,t,s)s \ge 0; \tag{1.5}$$

$$\sup_{|s|\le k} |g(x,t,s)| := h_k(x,t) \in L^1(Q_T), \forall k > 0. \tag{1.6}$$

(H4) $f \in L^m(Q_T)$, $m \ge [p^-(1+\frac{2}{N})]'$; here $'$ means the Hölder conjugate exponent. Assume that the initial value $u_0 \in L^2(\Omega)$.

The mathematical models with variable exponent $p(x)$, like Problem (1.1), are related to electro-rheological fluids, which can be seen as a class of non-Newtonian fluids or smart fluids (see monograph [2] and the references therein). Compared with the constant case p, the variable exponent $p(x)$ is able to describe the diffusion phenomenon more refined in the divergence term.

In the classical constant exponent setting, [3] investigated a steady problem with zero order term $g(x,u)$ and a divergence term $-\mathrm{div}\Phi(u)$, where $\Phi(u)$ is only a continuous field. Since there is no growth condition on $\Phi(u)$, the authors had to consider the existence of renormalized solutions although the source term $f(x) \in W^{-1,p'}(\Omega)$. With respect to the elliptic or parabolic equations with finite Radon measures, we refer to Boccardo and Gallouët's pioneer work [4], in which some classical results have been proved. Here we only state them in a simple form: if $p > 2 - \frac{1}{N}$, $f \in L^1(\Omega)$, then the steady equation

$$\begin{cases} -\Delta_p u = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases}$$

has a weak solution $u \in W_0^{1,q}(\Omega)$ with $1 \le q < \frac{N(p-1)}{N-1}$. For the time-dependent problem

$$\begin{cases} \partial_t u - \Delta_p u = f(x,t), & (x,t) \in Q_T, \\ u(x,t) = 0, & (x,t) \in \Gamma_T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

if $p > 2 - \frac{1}{N+1}$, $f \in L^1(Q_T)$, then the problem has a weak solution $u \in L^q(0,T;W_0^{1,q}(\Omega))$ with $1 \le q < p - \frac{N}{N+1}$. For the variable exponents equations, it should be emphasized that the functional setting for the parabolic equations with $p(x)$ structure was established in [1]. In the L^1 data framework, the authors of [1] also proved the existence of renormalized solutions without the zero order term. For the well-posedness of entropy solutions in the elliptic case, see [5].

Different from the parabolic problem in [4], in our Problem (1.1), the integrability of f can ensure the weak energy solutions, other than the solutions like $u \in L^q(0,T;W_0^{1,q}(\Omega))$ in the equations with constant exponent. They are from different perspectives. As stated in [6], identifying the weak limit of the nonlinear term is vital to the nonlinear problem. Compared with the steady problem in [3], one should select new and appropriate test functions to the parabolic equations. To be more specific, as pointed out in [7, Remark 3.3], the time derivative term prevents us from directly using the test function $T_i[u^\epsilon - T_j(u)]$ as in [3]. We need the Landes time regularization for help. It is interesting that the integrability of f, assumed in (H4), is neither high nor low. In [8], in order to establish the strong convergence of the gradient sequence $\{\nabla u^\epsilon\}_\epsilon$, the trick is to utilize $u^\epsilon - u^\eta$ as the test function in the difference of the approximate equation for ϵ and η. However, in our Problem (1.1), the zero order term $g(x,t,u)$ makes this trick in vain, since the integrability of f is not high enough to ensure the uniform L^∞ bound of the approximate solution u^ϵ. When $g(x,t,u) \equiv 0$ and the source term f^ϵ is bounded in L^1 (non-reflexive Banach space) or weakly converges to L^1, the almost everywhere convergence of the gradient sequence $\{\nabla u^\epsilon\}_\epsilon$ is attributed to the pioneering paper [9]. Nevertheless, in our problem, since the zero order term exists and f has a relatively higher integrability, the Young measures method can be more efficiently used to prove the almost everywhere convergence of the gradient sequence $\{\nabla u^\epsilon\}_\epsilon$.

We recall the solutions space in [1]: $\mathbb{V} := \{v \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)), |\nabla v| \in L^{p(x)}(Q_T)\}$. The norm in $\mathbb{V}$ is defined as $\|v\|_{\mathbb{V}} = \|v\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} + |\nabla v|_{L^{p(x)}(Q_T)}$ or an equivalent norm $\|v\|_{\mathbb{V}} = |\nabla v|_{L^{p(x)}(Q_T)}$. The dual of $\mathbb{V}$ is $\mathbb{V}^*$. For more details on functional spaces for the parabolic equations with $p(x)$ structure, we refer to the fundamental works in [1].

Theorem 1.1 *Let the Assumptions (H1)–(H4), be satisfied. Then there exists a weak solution u to Problem (1.1). Here, we say that u is a weak solution to Eq. (1.1), provided that $u \in L^\infty(0,T;L^2(\Omega)) \cap \mathbb{V}$,*

$$g(x,t,u) \in L^1(Q_T), \quad g(x,t,u)u \in L^1(Q_T); \tag{1.7}$$

and the equality

$$\begin{aligned} &-\int_\Omega u_0\phi(x,0)\mathrm{d}x - \int_{Q_T} u\partial_t\phi \mathrm{d}x\mathrm{d}t + \int_{Q_T} A(x,t,u,\nabla u)\cdot\nabla\phi \mathrm{d}x\mathrm{d}t + \\ &\int_{Q_T} g(x,t,u)\phi \mathrm{d}x\mathrm{d}t = \int_{Q_T} f\phi \mathrm{d}x\mathrm{d}t \end{aligned} \tag{1.8}$$

holds for every $\phi \in \mathcal{D}(\Omega \times [0,T))$.

2. Proof of the existence result

This section is devoted to proving the existence result. For clarity, the proof is divided into several steps.

Step 1. Approximate problem and some estimates.

We define an approximate equation corresponding to Problem (1.1):

$$\begin{cases} \partial_t u^\epsilon - \mathrm{div} A(x,t,\nabla u^\epsilon) + g^\epsilon(x,t,u^\epsilon) = f^\epsilon(x,t), & (x,t) \in Q_T, \\ u^\epsilon(x,t) = 0, & (x,t) \in \Gamma_T, \\ u^\epsilon(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where $g^\epsilon(x,t,s) = \frac{g(x,t,s)}{1+\epsilon|g(x,t,s)|}$; $f^\epsilon(x,t) = \frac{f(x,t)}{1+\epsilon|f(x,t)|}$. For fixed $0 < \epsilon < 1$, the existence of weak solutions $u^\epsilon \in \mathbb{V}$ to Problem (2.1) is ensured by the pseudo-monotone operator theory in [10] or the Rothe method in [11].

Let us take u^ϵ as a test function in (2.1); by (1.3) we get

$$\begin{aligned}&\frac{1}{2}\operatorname*{ess\,sup}_{\tau\in(0,T)}\int_\Omega |u^\epsilon|^2(\tau)\mathrm{d}x+\alpha\int_{Q_T}|\nabla u^\epsilon|^{p(\cdot)}\mathrm{d}x\mathrm{d}t+\int_{Q_T}g^\epsilon(x,t,u^\epsilon)u^\epsilon\mathrm{d}x\mathrm{d}t\\&\le\frac{3}{2}\|u_0\|_{2,\Omega}^2+3\int_{Q_T}f^\epsilon u^\epsilon\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{2.2}$$

Employing the Hölder inequality, parabolic embedding in [12] (the embedding constant is denoted by γ_{N,p^-}), Young inequality with ε, we have

$$\begin{aligned}&\int_{Q_T}f^\epsilon u^\epsilon\mathrm{d}x\mathrm{d}t\le\|f^\epsilon\|_{[p^-(1+\frac{2}{N})]',Q_T}\|u^\epsilon\|_{p^-(1+\frac{2}{N}),Q_T}\\&\le\|f^\epsilon\|_{[p^-(1+\frac{2}{N})]',Q_T}\gamma_{N,p^-}\Big[\operatorname*{ess\,sup}_{\tau\in(0,T)}\int_\Omega|u^\epsilon|^2(\tau)\mathrm{d}x+\int_{Q_T}|\nabla u^\epsilon|^{p^-}\mathrm{d}x\mathrm{d}t\Big]^{\frac{1+\frac{p^-}{N}}{p^-(1+\frac{2}{N})}}\\&\le\varepsilon\Big[\operatorname*{ess\,sup}_{\tau\in(0,T)}\int_\Omega|u^\epsilon|^2(\tau)\mathrm{d}x+\int_{Q_T}|\nabla u^\epsilon|^{p(\cdot)}\mathrm{d}x\mathrm{d}t+|Q_T|\Big]+\\&\varepsilon^{-\frac{1+\frac{p^-}{N}}{p^-(1+\frac{1}{N})-1}}[\gamma_{N,p^-}\|f\|_{[p^-(1+\frac{2}{N})]',Q_T}]^{\frac{p^-(1+\frac{2}{N})}{p^-(1+\frac{1}{N})-1}},\end{aligned}\tag{2.3}$$

where $|Q_T|$ is the Lebesgue measure of Q_T.

Denote $Q_t=\Omega\times(0,t)$ and define

$$\mathbb{E}(u^\epsilon;0,t):=\sup_{\tau\in(0,t)}\int_\Omega|u^\epsilon|^2(\tau)\mathrm{d}x+\int_{Q_t}|\nabla u^\epsilon|^{p(\cdot)}\mathrm{d}x\mathrm{d}t.$$

Choosing $\varepsilon=\frac{1}{6}\min\{\frac{1}{2},\alpha\}$ in (2.3) and from (1.5), (2.2), we actually see that

$$\frac{1}{2}\min\{\frac{1}{2},\alpha\}\mathbb{E}(u^\epsilon;0,T)\le C;\tag{2.4}$$

$$\int_{Q_T}g^\epsilon(x,t,u^\epsilon)u^\epsilon\mathrm{d}x\mathrm{d}t\le C\tag{2.5}$$

with

$$\begin{aligned}C=&\frac{3}{2}\|u_0\|_{2,\Omega}^2+3(\frac{1}{6}\min\{\frac{1}{2},\alpha\})^{-\frac{1+\frac{p^-}{N}}{p^-(1+\frac{1}{N})-1}}[\gamma_{N,p^-}\|f\|_{[p^-(1+\frac{2}{N})]',Q_T}]^{\frac{p^-(1+\frac{2}{N})}{p^-(1+\frac{1}{N})-1}}+\\&\frac{1}{6}\min\{\frac{1}{2},\alpha\}|Q_T|,\end{aligned}$$

which is independent of ϵ.

The estimates (2.4), (2.5) and (1.5) show that $\{u^\epsilon\}_\epsilon$ is bounded in $L^\infty(0,T;L^2(\Omega))\cap\mathbb{V}$; and $\{g^\epsilon(x,t,u^\epsilon)u^\epsilon\}_\epsilon$ is bounded in $L^1(Q_T)$. Thus, there exist a function $u\in L^\infty(0,T;L^2(\Omega))\cap\mathbb{V}$ and a subsequence of $\{u^\epsilon\}_\epsilon$, not relabeled again, such that $u^\epsilon\rightharpoonup u$ weakly* in $L^\infty(0,T;L^2(\Omega))$; $u^\epsilon\rightharpoonup u$ weakly in $\mathbb{V}$; and

$$\nabla u^\epsilon\rightharpoonup\nabla u\ \text{ weakly in }\ (L^{p(\cdot)}(Q_T))^N.\tag{2.6}$$

Inequalities (1.4) and (2.4) lead to that $|A(x,t,\nabla u^\epsilon)|^{p'(\cdot)}\le(\beta+1)^{(p')^+}|\nabla u^\epsilon|^{p(\cdot)}$, which implies that the sequence $\{A(x,t,\nabla u^\epsilon)\}_\epsilon$ is bounded in $(L^{p'(\cdot)}(Q_T))^N$. Hence there exists a function

$\xi \in (L^{p'(\cdot)}(Q_T))^N$ such that

$$A(x,t,\nabla u^\epsilon) \rightharpoonup \xi \ \text{ weakly in } \ (L^{p'(\cdot)}(Q_T))^N. \tag{2.7}$$

Now we claim that

$$\{g^\epsilon(x,t,u^\epsilon)\}_\epsilon \ \text{ is equi-integrable in } \ L^1(Q_T). \tag{2.8}$$

In fact, given any $\varepsilon > 0$ and any measurable subset $E \subset Q_T$, by means of (1.5), (1.6) and (2.5),

$$\begin{aligned}
&\int_E |g^\epsilon(x,t,u^\epsilon)|\mathrm{d}x\mathrm{d}t \\
&= \int_{E\cap\{(x,t)\in Q_T:|u^\epsilon(x,t)|\le k\}} |g^\epsilon(x,t,u^\epsilon)|\mathrm{d}x\mathrm{d}t + \int_{E\cap\{(x,t)\in Q_T:|u^\epsilon(x,t)|> k\}} |g^\epsilon(x,t,u^\epsilon)|\mathrm{d}x\mathrm{d}t \\
&\le \int_E \sup_{|s|\le k} |g(x,t,s)|\mathrm{d}x\mathrm{d}t + \frac{1}{k}\int_{E\cap\{(x,t)\in Q_T:|u^\epsilon(x,t)|> k\}} g^\epsilon(x,t,u^\epsilon)u^\epsilon \mathrm{d}x\mathrm{d}t \\
&\le \int_E h_k(x,t)\mathrm{d}x\mathrm{d}t + \frac{C}{k}.
\end{aligned}$$

First, we choose $k \ge \hat{k}$ sufficiently large so that $\frac{C}{k} < \frac{\varepsilon}{2}$. Secondly, by the absolute continuity of the integral, and since $h_{\hat{k}}(x,t) \in L^1(Q_T)$ is assumed in (1.6), $\int_E h_{\hat{k}}(x,t)\mathrm{d}x\mathrm{d}t$ can be smaller than $\frac{\varepsilon}{2}$ when $|E|$ is small enough.

Thanks to the Dunford-Pettis Theorem, the equi-integrability of $\{g^\epsilon(x,t,u^\epsilon)\}_\epsilon$ in (2.8) implies that it is weak compact in $L^1(Q_T)$. In addition, $\{g^\epsilon(x,t,u^\epsilon)\}_\epsilon$ is bounded in $L^1(Q_T)$.

Based upon the previous analysis, we deduce from $\partial_t u^\epsilon = \mathrm{div}A(x,t,\nabla u^\epsilon) - g^\epsilon(x,t,u^\epsilon) + f^\epsilon(x,t)$ that

$$\{\partial_t u^\epsilon\}_\epsilon \ \text{ is bounded in } \ \mathbb{V}^* + L^1(Q_T). \tag{2.9}$$

From the embedding relationship $\mathbb{V}^* \hookrightarrow L^1(0,T;W^{-1,p'(x)}(\Omega))$ and by (2.9), we know that $\{\partial_t u^\epsilon\}_\epsilon$ is bounded in $L^1(0,T;W^{-1,(p^+)'}(\Omega)) + L^1(Q_T)$. Furthermore, we have that

$$\|\partial_t u^\epsilon\|_{L^1(0,T;W^{-s,(p^+)'}(\Omega))} \le C \tag{2.10}$$

with the choice of $s > \max\{\frac{N}{p^+}, 1\}$, where the positive constant C is independent of ϵ.

Using the embedding $\mathbb{V} \hookrightarrow L^{p^-}(0,T;W_0^{1,p^-}(\Omega))$ and by (2.4), we have that

$$\{u^\epsilon\}_\epsilon \ \text{ is bounded in } \ L^1(0,T;W_0^{1,p^-}(\Omega)). \tag{2.11}$$

Since $W_0^{1,p^-}(\Omega) \overset{\text{compact}}{\hookrightarrow} L^{p^-}(\Omega) \hookrightarrow W^{-s,(p^+)'}(\Omega)$, by virtue of (2.10) and (2.11), we may apply the Simon Compactness Theorem in [13] and infer that $\{u^\epsilon\}_\epsilon$ is compact in $L^1(0,T;L^{p^-}(\Omega))$. By extracting a subsequence of $\{u^\epsilon\}_\epsilon$ if necessary, we obtain that as ϵ tends to zero,

$$u^\epsilon \to u \ \text{ a.e. in } \ Q_T. \tag{2.12}$$

Observe that g is a Carathéodory function; thus, by (2.12),

$$g^\epsilon(x,t,u^\epsilon) \to g(x,t,u) \ \text{ a.e. in } \ Q_T. \tag{2.13}$$

It follows from (2.13), (2.8) and the Vitali Theorem that

$$g^\epsilon(x,t,u^\epsilon) \to g(x,t,u) \ \text{ strongly in } \ L^1(Q_T). \tag{2.14}$$

From (2.5), (1.5), (2.12), Fatou Lemma yields

$$\int_{Q_T} g(x,t,u)u\mathrm{d}x\mathrm{d}t \le \liminf_{\epsilon\to 0} \int_{Q_T} g^\epsilon(x,t,u^\epsilon)u^\epsilon \mathrm{d}x\mathrm{d}t \le C.$$

Noting (1.5) again, we find that $g(x,t,u)u \in L^1(Q_T)$. Thus, (1.7) is obtained.

Denote $T_k(s) = \min\{k, \max\{-k,s\}\}$, $G_k(s) = s - T_k(s)$. We now take $G_k(u^\epsilon)$ as a test function in (2.1) to obtain, by (1.3), that

$$\begin{aligned}
&\frac{1}{2}\operatorname*{ess\,sup}_{\tau\in(0,T)} \int_\Omega |G_k(u^\epsilon)|^2(\tau)\mathrm{d}x + \alpha\int_{Q_T} |\nabla G_k(u^\epsilon)|^{p(\cdot)}\mathrm{d}x\mathrm{d}t + \int_{Q_T} g^\epsilon(x,t,u^\epsilon)G_k(u^\epsilon)\mathrm{d}x\mathrm{d}t \\
&\le \frac{3}{2}\|G_k(u_0)\|_{2,\Omega}^2 + 3\int_{Q_T} f^\epsilon G_k(u^\epsilon)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{2.15}$$

From $|G_k(u^\epsilon)| \le |u^\epsilon|$ and (2.4), we have that $\frac{1}{2}\min\{\frac{1}{2},\alpha\}\mathbb{E}(G_k(u^\epsilon);0,T) \le C$, which implies that $\{G_k(u^\epsilon)\}_\epsilon$ is bounded in $L^{p^-(1+\frac{2}{N})}(Q_T)$ by parabolic embedding. This, combined with the almost everywhere convergence of u^ϵ in (2.12), results in that $G_k(u^\epsilon) \rightharpoonup G_k(u)$ weakly in $L^{p^-(1+\frac{2}{N})}(Q_T)$, as $\epsilon \to 0$. Thus, dropping the nonnegative terms and taking limit in (2.15), we obtain that

$$\limsup_{\epsilon\to 0}\int_{Q_T} |\nabla G_k(u^\epsilon)|^{p(\cdot)}\mathrm{d}x\mathrm{d}t \le \frac{3}{2\alpha}\|G_k(u_0)\|_{2,\Omega}^2 + \frac{3}{\alpha}\int_{Q_T} fG_k(u)\mathrm{d}x\mathrm{d}t, \tag{2.16}$$

for every $k > 0$.

Notice that u and u_0 are both finite almost everywhere; thus, as k tends to infinity, $G_k(u) \to 0$ a.e. in Q_T and $G_k(u_0) \to 0$ a.e. in Ω. Using the Lebesgue Dominated Convergence Theorem in (2.16), we get that

$$\limsup_{k\to\infty}\limsup_{\epsilon\to 0}\int_{\{(x,t)\in Q_T:|u^\epsilon(x,t)|>k\}} |\nabla u^\epsilon|^{p(\cdot)}\mathrm{d}x\mathrm{d}t = 0. \tag{2.17}$$

Step 2. Landes time regularization and its related limit.

Recall a special time regularization of $T_j(u)$ in [1]. Consider $(T_j(u))_\mu \in \mathbb{V}\cap L^\infty(Q_T)$ as a solution to

$$\begin{cases} \partial_t[(T_j(u))_\mu] = \mu[T_j(u) - (T_j(u))_\mu] \ \text{ in } \ \mathcal{D}'(Q_T), \\ (T_j(u))_\mu(x,0) = v_0^\mu \ \text{ in } \ \Omega, \end{cases}$$

where $v_0^\mu \to T_j(u_0)$ a.e. in Ω as $\mu\to\infty$. $v_0^\mu \in W_0^{1,p(\cdot)}\cap L^\infty(\Omega)$, $\|v_0^\mu\|_{L^\infty(\Omega)} \le j$, and $\partial_t(T_j(u))_\mu \in \mathbb{V}$; moreover, as $\mu\to\infty$, $(T_j(u))_\mu \to T_j(u)$ a.e. in Q_T, weakly* in $L^\infty(Q_T)$, and strongly in $\mathbb{V}$; furthermore, $\|(T_j(u))_\mu\|_{L^\infty(Q_T)} \le j$. This regularization is known as the Landes time regularization. For its role in the parabolic equations, see [7].

Let us choose $T_i[u^\epsilon - (T_j(u))_\mu]$ as a test function in Problem (2.1); then

$$\int_{Q_T} A(x,t,\nabla u^\epsilon)\cdot\nabla T_i[u^\epsilon - (T_j(u))_\mu]\mathrm{d}x\mathrm{d}t = -\int_0^T \langle\partial_t u^\epsilon, T_i[u^\epsilon - (T_j(u))_\mu]\rangle\mathrm{d}t$$

$$-\int_{Q_T} g^\epsilon(x,t,u^\epsilon)T_i[u^\epsilon-(T_j(u))_\mu]\mathrm{d}x\mathrm{d}t+\int_{Q_T} f^\epsilon T_i[u^\epsilon-(T_j(u))_\mu]\mathrm{d}x\mathrm{d}t. \tag{2.18}$$

Here, the bracket $\langle\cdot,\cdot\rangle$ is the duality pairing between $W^{-1,p'(\cdot)}(\Omega)+L^1(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$.

In order to deal with the time derivative term, we use a scheme that goes back to [14]. The method is modified according to a nonzero initial value function u_0.

$$\begin{aligned}
&\int_0^T \langle \partial_t u^\epsilon, T_i[u^\epsilon-(T_j(u))_\mu]\rangle\mathrm{d}t\\
&=\int_0^T \langle \partial_t[u^\epsilon-(T_j(u))_\mu], T_i[u^\epsilon-(T_j(u))_\mu]\rangle\mathrm{d}t+\int_0^T \langle \partial_t(T_j(u))_\mu, T_i[u^\epsilon-(T_j(u))_\mu]\rangle\mathrm{d}t\\
&=\overbrace{\int_\Omega \widetilde{T}_i[u^\epsilon(T)-(T_j(u))_\mu(T)]\mathrm{d}x}^{\geq 0}-\int_\Omega \widetilde{T}_i[u_0-(T_j(u))_\mu(0)]\mathrm{d}x+\\
&\quad\mu\int_{Q_T}[T_j(u)-(T_j(u))_\mu]T_i[u^\epsilon-(T_j(u))_\mu]\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $\widetilde{T}_k(s)=\int_0^s T_k(\sigma)\mathrm{d}\sigma$ is the primitive function of $T_k(s)$. It is obvious that $\widetilde{T}_k(s)$ is nonnegative. As $\epsilon\to 0$, $T_i[u^\epsilon-(T_j(u))_\mu]$ converges to $T_i[u-(T_j(u))_\mu]$ a.e. in Q_T and weakly* in $L^\infty(Q_T)$. Employing the Lebesgue Dominated Convergence Theorem, it yields that

$$\lim_{\epsilon\to 0}\int_{Q_T}[T_j(u)-(T_j(u))_\mu]T_i[u^\epsilon-(T_j(u))_\mu]\mathrm{d}x\mathrm{d}t=\overbrace{\int_{Q_T}[T_j(u)-(T_j(u))_\mu]T_i[u-(T_j(u))_\mu]\mathrm{d}x\mathrm{d}t}^{\geq 0\ \text{ was proved in [14]}}.$$

As a consequence,

$$\limsup_{\epsilon\to 0}\Big[-\int_0^T \langle \partial_t u^\epsilon, T_i[u^\epsilon-(T_j(u))_\mu]\rangle\mathrm{d}t\Big]\leq\int_\Omega \widetilde{T}_i(u_0-v_0^\mu)\mathrm{d}x. \tag{2.19}$$

Noting (2.19), (2.14) and the strong compactness of f^ϵ in $L^1(Q_T)$ and taking the limit in (2.18), we discover that

$$\begin{aligned}
&\limsup_{\epsilon\to 0}\int_{Q_T} A(x,t,\nabla u^\epsilon)\cdot\nabla T_i[u^\epsilon-(T_j(u))_\mu]\mathrm{d}x\mathrm{d}t\\
&\leq\int_\Omega \widetilde{T}_i(u_0-v_0^\mu)\mathrm{d}x-\int_{Q_T} g(x,t,u)T_i[u-(T_j(u))_\mu]\mathrm{d}x\mathrm{d}t+\\
&\quad\int_{Q_T} fT_i[u-(T_j(u))_\mu]\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

From the Lebesgue Dominated Convergence Theorem and the properties of $(T_j(u))_\mu$, it follows that

$$\begin{aligned}
&\limsup_{\mu\to\infty}\limsup_{\epsilon\to 0}\int_{Q_T} A(x,t,\nabla u^\epsilon)\cdot\nabla T_i[u^\epsilon-(T_j(u))_\mu]\mathrm{d}x\mathrm{d}t\\
&\leq\int_\Omega \widetilde{T}_i(G_j(u_0))\mathrm{d}x-\int_{Q_T} g(x,t,u)\theta_j^i(u)\mathrm{d}x\mathrm{d}t+\int_{Q_T} f\theta_j^i(u)\mathrm{d}x\mathrm{d}t,
\end{aligned}\tag{2.20}$$

where $\theta_j^i(s)=T_i[s-T_j(s)]$.

As $j\to\infty$, $\widetilde{T}_i(G_j(u_0))\to 0$ a.e. in Ω; while $|\widetilde{T}_i(G_j(u_0))|\leq i|u_0|$. Thus, utilizing the Lebesgue Dominated Convergence Theorem, one has $\lim_{j\to\infty}\int_\Omega \widetilde{T}_i(G_j(u_0))\mathrm{d}x=0$, for every

$i>0$. Similarly, as $j\to\infty$, noting that $\theta_j^i(u)\to 0$ a.e. in Q_T and $|\theta_j^i(u)|\le i$, the last two terms in (2.20) tend to zero. As a result, for every $i>0$,

$$\limsup_{j\to\infty}\limsup_{\mu\to\infty}\limsup_{\epsilon\to 0}\int_{Q_T}A(x,t,\nabla u^\epsilon)\cdot\nabla T_i[u^\epsilon-(T_j(u))_\mu]\mathrm{d}x\mathrm{d}t\le 0. \tag{2.21}$$

Now we focus on the integral

$$\begin{aligned}&\int_{Q_T}A(x,t,\nabla u^\epsilon)\cdot\nabla(u^\epsilon-u)\mathrm{d}x\mathrm{d}t\\ &=\overbrace{\int_{Q_T}A(x,t,\nabla u^\epsilon)\cdot\nabla T_i[u^\epsilon-(T_j(u))_\mu]\mathrm{d}x\mathrm{d}t}^{(A1)}+\\ &\quad\overbrace{\int_{\{(x,t)\in Q_T:|u^\epsilon(x,t)-(T_j(u))_\mu(x,t)|>i\}}A(x,t,\nabla u^\epsilon)\cdot\nabla[u^\epsilon-(T_j(u))_\mu]\mathrm{d}x\mathrm{d}t}^{(A2)}\\ &\quad\overbrace{-\int_{Q_T}A(x,t,\nabla u^\epsilon)\cdot\nabla[u-(T_j(u))_\mu]\mathrm{d}x\mathrm{d}t}^{(A3)}.\end{aligned}$$

$1^\diamond$. By (2.21), $\limsup_{j\to\infty}\limsup_{\mu\to\infty}\limsup_{\epsilon\to 0}(A1)\le 0$.

$2^\diamond$. Let $i>j>0$, then

$$\{(x,t)\in Q_T:|u^\epsilon(x,t)-(T_j(u))_\mu(x,t)|>i\}\subset\{(x,t)\in Q_T:|u^\epsilon(x,t)|>i-j\}.$$

In view of (1.4) and by the Young inequality, $(A2)$ can be estimated in the following manner:

$$\begin{aligned}|(A2)|\le&\int_{\{(x,t)\in Q_T:|u^\epsilon(x,t)|>i-j\}}\beta|\nabla u^\epsilon|^{p(\cdot)-1}(|\nabla u^\epsilon|+|\nabla(T_j(u))_\mu|)\mathrm{d}x\mathrm{d}t\\ \le&\beta\Big(1+\frac{1}{(p')^-}\Big)\int_{\{(x,t)\in Q_T:|u^\epsilon(x,t)|>i-j\}}|\nabla u^\epsilon|^{p(\cdot)}\mathrm{d}x\mathrm{d}t+\\ &\frac{\beta}{p^-}\int_{\{(x,t)\in Q_T:|u^\epsilon(x,t)|>i-j\}}|\nabla(T_j(u))_\mu|^{p(\cdot)}\mathrm{d}x\mathrm{d}t.\end{aligned}$$

First, let $\epsilon\to 0$, and then let $\mu\to\infty$; using the property that $\nabla(T_j(u))_\mu\to\nabla T_j(u)$ strongly in $(L^{p(\cdot)}(Q_T))^N$, we obtain that

$$\begin{aligned}\limsup_{\mu\to\infty}\limsup_{\epsilon\to 0}|(A2)|\le&\beta\Big(1+\frac{1}{(p')^-}\Big)\limsup_{\epsilon\to 0}\int_{\{(x,t)\in Q_T:|u^\epsilon(x,t)|>i-j\}}|\nabla u^\epsilon|^{p(\cdot)}\mathrm{d}x\mathrm{d}t+\\ &\frac{\beta}{p^-}\int_{\{(x,t)\in Q_T:|u(x,t)|\ge i-j\}}|\nabla u|^{p(\cdot)}\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Observing (2.17), and the absolute continuity of the integral ($|\nabla u|^{p(\cdot)}\in L^1(Q_T)$), we conclude that $\limsup_{\mu\to\infty}\limsup_{\epsilon\to 0}|(A2)|$ can be sufficiently small when $i-j$ is large enough.

$3^\diamond$. For fixed $\mu>0$, it follows from (2.7) that $\lim_{\epsilon\to 0}(A3)=-\int_{Q_T}\xi\cdot\nabla[u-(T_j(u))_\mu]\mathrm{d}x\mathrm{d}t$. Taking into account the strong convergence of $\nabla(T_j(u))_\mu$ in $(L^{p(\cdot)}(Q_T))^N$, we see that for fixed $j>0$, $\lim_{\mu\to\infty}\lim_{\epsilon\to 0}(A3)=-\int_{Q_T}\xi\cdot\nabla G_j(u)\mathrm{d}x\mathrm{d}t$. As $j\to\infty$, $G_j(u)\to 0$ a.e. in Q_T, while $\{G_j(u)\}_j$ is bounded in the reflexive Banach space $\mathbb{V}$, we have that $\nabla G_j(u)\rightharpoonup 0$ weakly in $(L^{p(\cdot)}(Q_T))^N$. Consequently, $\lim_{j\to\infty}\lim_{\mu\to\infty}\lim_{\epsilon\to 0}(A3)=0$.

In summary, the above limit analyses and (2.7) establish that

$$\limsup_{\epsilon\to 0}\int_{Q_T} A(x,t,\nabla u^\epsilon)\cdot\nabla u^\epsilon \mathrm{d}x\mathrm{d}t \le \int_{Q_T}\xi\cdot\nabla u\mathrm{d}x\mathrm{d}t. \tag{2.22}$$

Step 3. Almost everywhere convergence of the gradient sequence $\{\nabla u^\epsilon\}_\epsilon$.

Inspired by the idea in [15], in order to prove the almost everywhere convergence of the gradient sequence, we take advantage of the Young measures method.

We infer from (2.6) that $\nabla u^\epsilon \rightharpoonup \int_{\mathbb{R}^{N+1}}\lambda\mathrm{d}\nu_{t,x}(\lambda)$ weakly in $(L^1(Q_T))^N$, where $\nu_{t,x}$ is the Young measure generated by $\{\nabla u^\epsilon\}_\epsilon$. Therefore,

$$\nabla u = \int_{\mathbb{R}^{N+1}}\lambda\mathrm{d}\nu_{t,x}(\lambda)\ \text{ a.e. in }\ Q_T. \tag{2.23}$$

It follows from (2.7) that $\{A(x,t,\nabla u^\epsilon)\}_\epsilon$ is weakly compact in $(L^1(Q_T))^N$. Note also that A is a Carathéodory function; thus

$$A(x,t,\nabla u^\epsilon)\rightharpoonup\int_{\mathbb{R}^{N+1}}A(x,t,\lambda)\mathrm{d}\nu_{t,x}(\lambda)\ \text{ weakly in }\ (L^1(Q_T))^N.$$

Hence for the weak limit ξ in (2.7) we have

$$\xi=\int_{\mathbb{R}^{N+1}}A(x,t,\lambda)\mathrm{d}\nu_{t,x}(\lambda)\ \text{ a.e. in }\ Q_T. \tag{2.24}$$

Since $\nu_{t,x}$ is a probability measure, it is obvious that $\int_{\mathbb{R}^{N+1}}\mathrm{d}\nu_{t,x}(\lambda)=1$. Using (2.23), we see that

$$\begin{aligned}&\int_{Q_T}\int_{\mathbb{R}^{N+1}}A(x,t,\nabla u)\cdot(\lambda-\nabla u)\mathrm{d}\nu_{t,x}(\lambda)\mathrm{d}x\mathrm{d}t\\ &=\int_{Q_T}A(x,t,\nabla u)\cdot\int_{\mathbb{R}^{N+1}}\lambda\mathrm{d}\nu_{t,x}(\lambda)\mathrm{d}x\mathrm{d}t-\\ &\quad\int_{Q_T}A(x,t,\nabla u)\cdot\nabla u\int_{\mathbb{R}^{N+1}}\mathrm{d}\nu_{t,x}(\lambda)\mathrm{d}x\mathrm{d}t=0.\end{aligned} \tag{2.25}$$

The assumption (1.3) says that $A(x,t,\nabla u^\epsilon)\cdot\nabla u^\epsilon\ge 0$. Obviously, the negative part $[A(x,u^\epsilon,\nabla u^\epsilon)\cdot\nabla u^\epsilon]^-$ is weakly compact in $L^1(Q_T)$. Thanks to the Fatou type Lemma in [16], we obtain

$$\liminf_{\epsilon\to 0}\int_{Q_T}A(x,t,\nabla u^\epsilon)\cdot\nabla u^\epsilon\mathrm{d}x\mathrm{d}t\ge\int_{Q_T}\int_{\mathbb{R}^{N+1}}A(x,t,\lambda)\cdot\lambda\mathrm{d}\nu_{t,x}(\lambda)\mathrm{d}x\mathrm{d}t. \tag{2.26}$$

Combining (2.22), (2.26) with (2.24), we conclude that

$$\int_{Q_T}\int_{\mathbb{R}^{N+1}}A(x,t,\lambda)\cdot\lambda\mathrm{d}\nu_{t,x}(\lambda)\mathrm{d}x\mathrm{d}t\le\int_{Q_T}\xi\cdot\nabla u\mathrm{d}x\mathrm{d}t=\int_{Q_T}\int_{\mathbb{R}^{N+1}}A(x,t,\lambda)\mathrm{d}\nu_{t,x}(\lambda)\cdot\nabla u\mathrm{d}x\mathrm{d}t.$$

Consequently,

$$\int_{Q_T}\int_{\mathbb{R}^{N+1}}A(x,t,\lambda)\cdot(\lambda-\nabla u)\mathrm{d}\nu_{t,x}(\lambda)\mathrm{d}x\mathrm{d}t\le 0. \tag{2.27}$$

From (2.25) and (2.27), it follows that

$$\int_{Q_T}\int_{\mathbb{R}^{N+1}}[A(x,t,\lambda)-A(x,t,\nabla u)]\cdot(\lambda-\nabla u)\mathrm{d}\nu_{t,x}(\lambda)\mathrm{d}x\mathrm{d}t\le 0. \tag{2.28}$$

Moreover, due to the monotonicity of the operator A in (1.2), estimate (2.28) implies that

$$\int_{\mathbb{R}^{N+1}} [A(x,t,\lambda) - A(x,t,\nabla u)] \cdot (\lambda - \nabla u) \mathrm{d}\nu_{t,x}(\lambda) = 0 \ \text{ a.e. in } \ Q_T.$$

Since $\nu_{t,x} \geq 0$ is a probability measure and A is strictly monotone, we conclude that $[A(x,t,\lambda) - A(x,t,\nabla u)] \cdot (\lambda - \nabla u)$ is strictly positive for all $\lambda \neq \nabla u$ and thus supp $\nu_{t,x} := \{\lambda : \nu_{t,x}(\lambda) \neq 0\} = \{\nabla u(x,t)\}$ a.e. $(x,t) \in Q_T$. In other words, $\nu_{t,x} = \delta_{\nabla u(x,t)}$. Since $|Q_T| < \infty$, according to [17, Proposition 1], we have that $\nabla u^\epsilon \to \nabla u$ in measure. Hence one can extract a subsequence of $\{\nabla u^\epsilon\}_\epsilon$, denoted by itself for the sake of simplicity, such that

$$\nabla u^\epsilon \to \nabla u \ \text{ a.e. in } \ Q_T. \tag{2.29}$$

Let $\phi \in \mathcal{D}([0,T) \times \Omega)$ be a test function to Problem (2.1). In (2.7), once the weak limit of $A(x,t,\nabla u^\epsilon)$ is identified as $A(x,t,\nabla u)$ through (2.29), the weak formulation (1.8) can be obtained through a standard limit process. This finally proves that u is a weak solution to Problem (1.1). $\square$

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