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# Isoperimetric Upper Bounds and Reilly-Type Inequalities for the First Eigenvalue of the $p$-Biharmonic Operator

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**Abstract** In this paper, we give some isoperimetric upper bounds for the first eigenvalue of the $p$-biharmonic operator of an $n$-dimensional embedded closed hypersurface in an Euclidean space. We also give Reilly-type inequalities for the first eigenvalue of the $p$-biharmonic operator of an $n$-dimensional closed submanifold immersed into a higher dimensional manifold such as an Euclidean space, a unit sphere, a projective space.

**Keywords** eigenvalue; $p$-biharmonic operator; isoperimetric upper bounds; Reilly-type inequalities

**MR(2020) Subject Classification** 53C42; 58C40; 47J10

## 1. Introduction

In 1966, Mark Kac [1] wrote a paper named *Can one hear the shape of a drum* which tried to tell some geometric and topological properties from the information of eigenvalues. The spectrum consisting of the eigenvalues of a surface is often treated as an important characteristics to classify shapes of graphics appearing in engineering and medical science. In fact, the research on eigenvalues is also motivated by the phase transformation theory in physics and computer graphics, see [2] for some important physical interpretations of eigenvalues and eigenfunctions. In recent years, eigenvalue problems of some elliptic operators such as the usual Laplace operator, the $p$-Laplacian, the drifting Laplacian, the biharmonic operator and their weighted versions have been investigated for static Riemannian metrics under no boundary condition or various types of boundary conditions such as Dirichlet boundary condition, Neumann boundary condition, Robin boundary condition, Navier boundary condition, Steklov boundary condition, Wentzell boundary condition etc. (see [3–5] and the references therein). To avoid confusion of notations below, we denote the first nonzero eigenvalue of the Laplace operator as $\lambda_1^{\Delta}$. To our knowledge, Reilly [6] gave an upper bound for the first eigenvalue of the Laplace operator on a closed submanifold immersed in Euclidean space. Specifically, let $N^m$ be an $m$-dimensional closed manifold immersed in $k$-dimensional Euclidean space $\mathbb{R}^k$ $(k > m)$. Then the first nonzero eigenvalue of the Laplace

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operator on $N$ satisfies

$$\lambda_1^{\Delta}(N) \leq \frac{m}{A(N)} \int_N |\mathbf{H}|^2 \mathrm{d}\nu_g,$$

where $A(N) = \int_N \mathrm{d}\nu_g$ and $\mathbf{H}$ denote the area and inward mean curvature vector of the submanifold $N$ immersed in $\mathbb{R}^k$, respectively.

Du and Mao [7] generalized Reilly's work and obtained similar upper bounds for the $p$-Laplace operator. One can see [8] and the references therein for more research about Reilly-type inequalities for other operators such as the Schrödinger operator, Jacobi operator, linearized operator of the $r$-th mean curvature on submanifolds in special ambient manifolds such as space forms.

In addition, Wang and Xia [9] gave an isoperimetric upper bound for the first eigenvalue of the Laplace operator on a closed hypersurface embedded in $R^n$. Specifically, let $N$ be a connected closed hypersurface embedded in $n$-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 3)$, and $\Omega$ be a region bounded by $N$. Then the first nonzero eigenvalue of the Laplace operator on $N$ satisfies

$$\lambda_1^{\Delta}(N) \leq \frac{(n-1)A(N)}{nV(\Omega)} \left(\frac{\omega_n}{V(\Omega)}\right)^{\frac{1}{n}},$$

where $V(\Omega)$ denotes the volume of $\Omega$, $A(N)$ the area of the hypersurface $N$ embedded in $\mathbb{R}^n$, $\omega_n$ the volume of unit ball in $\mathbb{R}^n$.

Afterwards, Du and Wu [10] generalized their work and obtained isoperimetric upper bounds for the first eigenvalue of the Paneitz operator and the $p$-Laplace operator.

Motivated by these work above, in this paper, we wish to get similar inequalities for the $p$-biharmonic operator.

First we would like to give a sketch of the eigenvalue problem of the $p$-biharmonic operator.

The usual Laplace operator is defined as

$$\Delta = \frac{1}{\sqrt{G}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{G} g^{ij} \frac{\partial}{\partial x_j}\right),$$

in a local coordinate system $\{x_1, x_2, \ldots, x_n\}$, where the matrix $(g^{ij})$ is the inverse of the metric matrix $(g_{ij})$, $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, $G = \det(g_{ij})$.

For a compact manifold $M$ without boundary, $\Delta$ is a self-adjoint operator, then it has discrete eigenvalues according to the spectral theory in functional analysis. Obviously, the smallest eigenvalue in the problem

$$\Delta f = -\lambda f, \quad \text{on} \quad M$$

is zero and the corresponding eigenfunctions should be nonzero constant functions. By Rayleigh's theorem and extreme principle, the first nonzero closed eigenvalue $\lambda_1(M)$ ($\lambda_1$ for short) can be defined by

$$\lambda_1^{\Delta} := \inf\Big\{\frac{\int_M (\nabla f)^2 \mathrm{d}\nu_g}{\int_M f^2 \mathrm{d}\nu_g} | f \neq 0, f \in W^{1,2}(M), \int_M f \mathrm{d}\nu_g = 0\Big\},$$

where $W^{1,2}(M)$ denotes the Sobolev space given by the completion of $C^{\infty}(M)$ for the norm

$$\|f\|_{W^{1,2}(M)} = \Big(\int_M f^2 \mathrm{d}\nu_g + \int_M |\nabla f|^2 \mathrm{d}\nu_g\Big)^{\frac{1}{2}}.$$

Now let us recall some facts of the $p$-biharmonic operator ($1 < p < \infty$). The eigenvalue problem of the $p$-biharmonic operator often concerns the partial differential equation

$$\Delta_p^2 f = \lambda_p |f|^{p-2} f, \quad \text{on} \quad M,$$

where $\Delta_p^2 f$ is given by

$$\Delta_p^2 f = \Delta(|\Delta f|^{p-2} \Delta f).$$

As shown in [11], let $M$ be a bounded region of Euclidean space $\mathbb{R}^n$, the above eigenvalue problem with the Navier boundary conditions $f = \Delta f = 0$ on $\partial M$ has at least a simple and isolated positive eigenvalue $\lambda_p^*$. Additionally, by [12], we know that the spectrum of the $p$-biharmonic operator with the Dirichlet or Navier boundary conditions contains at least a nondecreasing sequence of positive eigenvalues. By variational characterization, the first nonzero closed eigenvalue $\lambda_{1,p}^*(M)$ ($\lambda_p^*$ for short) can be defined by

$$\lambda_{1,p}^* := \inf\Big\{ \frac{\int_M |\Delta f|^p \mathrm{d}\nu_g}{\int_M |f|^p \mathrm{d}\nu_g} | f \neq 0, f \in W^{2,p}(M), \int_M |f|^{p-2} f \mathrm{d}\nu_g = 0 \Big\},$$

where $W^{2,p}(M)$ denotes the Sobolev space given by the completion of $C^\infty(M)$ for the norm

$$\|f\|_{W^{2,p}(M)} = \Big( \sum_{|\alpha| \leq 2} \int_M |\nabla^\alpha f|^p \mathrm{d}\nu_g \Big)^{\frac{1}{p}}.$$

The $p$-biharmonic operator is a natural generalization of the biharmonic operator for the fact that $p$-biharmonic operator is the biharmonic operator when $p = 2$, in this case, the above eigenvalue problem describes the clamped plate problem. The main difference between the two operators is that the $p$-biharmonic operator is nonlinear when $p \neq 2$ but the biharmonic operator is linear. As shown in [13, 14], the research of the $p$-biharmonic operator is used in fields of micro-electro-mechanical system, thin film theory, nonlinear surface diffusion on solids, interface dynamics, deformation of a nonlinear elastic beam and control of the nonlinearity artificial viscosity of diffusion surface of non-Newtonian fluids.

Here in this paper, we follow the arguments used in [7, 10, 15] and conclusions obtained in [16] to get the following isoperimetric upper bounds and Reilly-type inequalities for the first eigenvalue of the $p$-biharmonic operator.

## 2. Isoperimetric upper bounds inequalities for the first eigenvalue of the $p$-biharmonic operator

In this section, before giving conclusions and the corresponding proofs, we first introduce a formula, see [18] for more details. Let $y$ be the position vector of $p \in M^n$ in $\mathbb{R}^N$, define

$$y = (y_1(x_1, \ldots, x_n), \ldots, y_N(x_1, \ldots, x_n)),$$

where $(x_1, \ldots, x_n)$ is the local coordinates for a neighbourhood around $p$.

Let $g$ be the metric induced by $\mathbb{R}^N$, then

$$\sum_{\alpha=1}^N |\Delta y_\alpha|^2 = (\Delta y)^2 = n^2 |\mathbf{H}|^2. \tag{2.1}$$

**Theorem 2.1** *Let $M^n$ $(n \geq 3)$ be a connected closed orientable hypersurface embedded in $\mathbb{R}^{n+1}$. Let $\Omega$ be the region bounded by $M^n$, $M^n = \partial\Omega$. Denote by $V(\Omega)$ the volume of $\Omega$. Then the first nonzero eigenvalue $\lambda^*_{1,p}$ of the $p$-biharmonic operator acting on functions on $M^n$ satisfies*

$$\lambda^*_{1,p} \leq n^p(n+1)^{-1}\Big(\frac{\omega_{n+1}}{V(\Omega)}\Big)^{\frac{p-1}{n+1}} \frac{\int_M |\mathbf{H}|^p \mathrm{d}\nu_g}{V(\Omega)} \begin{cases} (n+1)^{\frac{p-2}{2}}, & p \geq 2, \\ (n+1)^{\frac{2-p}{2}}, & 1 < p \leq 2, \end{cases} \tag{2.2}$$

*where $\omega_{n+1}$ is the volume of the unit ball in $\mathbb{R}^{n+1}$, $\mathbf{H}$ the inward mean curvature vector of $M$ embedded in $\mathbb{R}^{n+1}$, $|\mathbf{H}|$ the length of the vector $\mathbf{H}$. The equality in (2.2) holds if and only if $M^n$ is a sphere of radius $r_0$ of $\mathbb{R}^{n+1}$ with $r_0 = (\frac{n}{\lambda_1^\Delta})^{\frac{1}{2}}$ and $p = 2$.*

**Proof** Here we follow the argument used in [10]. Let $y$ be the position vector of point $p \in M^n$ in $\mathbb{R}^{n+1}$, define

$$y = (y_1(x_1, \dots, x_n), \dots, y_{n+1}(x_1, \dots, x_n)),$$

without loss of generality, set

$$\int_M |y_\alpha|^{p-2} y_\alpha \mathrm{d}\nu_g = 0,$$

where $1 \leq \alpha \leq n+1$, we can substitute $|y_\alpha|^{p-2}y_\alpha$ by $|y_\alpha|^{p-2}y_\alpha - \frac{\int_M |y_\alpha|^{p-2}y_\alpha \mathrm{d}\nu_g}{A(M)}$ if necessary, where $A(M) = \int_M \mathrm{d}\nu_g$ denotes the area of $M$. It follows from the Rayleigh-Ritz inequality that

$$\lambda^*_{1,p} \int_M |y_\alpha|^p \mathrm{d}\nu_g \leq \int_M |\Delta y_\alpha|^p \mathrm{d}\nu_g. \tag{2.3}$$

Summing the above inequality (2.3) with respect to $\alpha$ from 1 to $n+1$, we obtain

$$\lambda^*_{1,p} \int_M \sum_{\alpha=1}^{n+1} |y_\alpha|^p \mathrm{d}\nu_g \leq \int_M \sum_{\alpha=1}^{n+1} |\Delta y_\alpha|^p \mathrm{d}\nu_g. \tag{2.4}$$

Similar to the discussion in [10], we divide the subsequent discussion into two cases.

Case 1. If $p \geq 2$, it follows from the Hölder's inequality that

$$|y|^2 = \sum_{\alpha=1}^{n+1} |y_\alpha|^2 \leq (n+1)^{\frac{p-2}{p}} \Big(\sum_{\alpha=1}^{n+1} |y_\alpha|^p\Big)^{\frac{2}{p}}, \tag{2.5}$$

then if $p = 2$, the equality in (2.5) holds obviously. If $p > 2$, the equality in (2.5) holds if and only if $|y_1| = \cdots = |y_{n+1}|$.

Furthermore, it follows from the Minkowski's inequality

$$\Big(\sum_{i=1}^n a_i^r\Big)^{\frac{1}{r}} \leq \Big(\sum_{i=1}^n a_i^s\Big)^{\frac{1}{s}}, \ 1 \leq s \leq r, \ a_i \geq 0, \ 1, \dots, n,$$

and (2.1) that

$$\sum_{\alpha=1}^{n+1} |\Delta y_\alpha|^p \leq \Big(\sum_{\alpha=1}^{n+1} |\Delta y_\alpha|^2\Big)^{\frac{p}{2}} = n^p |\mathbf{H}|^p, \tag{2.6}$$

then if $p = 2$, the equality in (2.6) holds obviously. If $p > 2$, the equality in (2.6) holds if and only if $\Delta y_\alpha = 0$ for some $1 \leq \alpha \leq n+1$. Hence, when $p > 2$, the equalities in (2.5) and (2.6) hold simultaneously if and only if $\Delta y_\alpha = 0$, $\alpha = 1, \dots, n+1$. Whereas, substituting $\Delta y_\alpha = 0$,

$\alpha = 1, \ldots, n+1$ into (2.4), we have $\lambda_{1,p}^* = 0$, this is a contradiction. Therefore, the equalities in (2.5) and (2.6) hold simultaneously if and only if $p = 2$.

Combining (2.4)–(2.6), we have

$$\lambda_{1,p}^*(n+1)^{\frac{2-p}{2}} \int_M |y|^p \mathrm{d}\nu_g \leq n^p \int_M |\mathbf{H}|^p \mathrm{d}\nu_g. \tag{2.7}$$

Choose a ball $B$ of radius $r$ centered at the origin such that $V(B) = V(\Omega)$ ($V(B)$ denotes the volume of $B$), then

$$r = (\frac{V(\Omega)}{\omega_{n+1}})^{\frac{1}{n+1}}. \tag{2.8}$$

It follows from the weighted isoperimetric inequality in [19] and (2.8) that

$$\int_M |y|^p \mathrm{d}\nu_g \geq \int_{\partial B} |y|^p \mathrm{d}\nu = A(\partial B) r^p = (n+1)V(\Omega)(\frac{V(\Omega)}{\omega_{n+1}})^{\frac{p-1}{n+1}}, \tag{2.9}$$

where $A(\partial B)$ denotes the surface area of the ball $B$. Substituting (2.9) into (2.7), we get the desired inequality (2.2).

Case 2. If $1 < p \leq 2$, it follows from the Minkowski's inequality and the inverse Hölder's inequality that

$$|y|^p = \Big(\sum_{\alpha=1}^{n+1} |y_\alpha|^2\Big)^{\frac{p}{2}} \leq \sum_{\alpha=1}^{n+1} |y_\alpha|^p, \tag{2.10}$$

$$(n+1)^{\frac{p-2}{p}} \Big(\sum_{\alpha=1}^{n+1} |\Delta y_\alpha|^p\Big)^{\frac{2}{p}} \leq \sum_{\alpha=1}^{n+1} |\Delta y_\alpha|^2 = n^2 |\mathbf{H}|^2, \tag{2.11}$$

similar to the discussion in Case 1, the equalities in (2.10) and (2.11) hold simultaneously if and only if $p = 2$.

Substituting (2.10) and (2.11) into (2.4), we have

$$\lambda_{1,p}^* \int_M |y|^p \mathrm{d}\nu_g \leq n^p (n+1)^{\frac{2-p}{2}} \int_M |\mathbf{H}|^p \mathrm{d}\nu_g. \tag{2.12}$$

Substituting (2.9) into (2.12), we get the desired inequality (2.2).

From the discussion above, the equality in (2.2) holds if and only if $p = 2$, by [9] and [16], the equality in (2.2) holds if and only if $M^n$ is a sphere of radius $r_0$ of $\mathbb{R}^{n+1}$ with $r_0 = (\frac{n}{\lambda_1^\Delta})^{\frac{1}{2}}$. □

**Theorem 2.2** *Same assumptions as in Theorem 2.1. Denote by $V(\Omega)$ and $A(M) = \int_M \mathrm{d}\nu_g$ the volume of $\Omega$ and area of $M$, respectively. Then the first nonzero eigenvalue $\lambda_{1,p}^*$ of the p-biharmonic operator acting on functions on $M^n$ satisfies*

$$\lambda_{1,p}^* \leq n^p (n+1)^{-p} \frac{(A(M))^{p-1} \int_M |\mathbf{H}|^p \mathrm{d}\nu_g}{(V(\Omega))^p} \begin{cases} (n+1)^{\frac{p-2}{2}}, & p \geq 2, \\ (n+1)^{\frac{2-p}{2}}, & 1 < p \leq 2, \end{cases} \tag{2.13}$$

*where $\mathbf{H}$ is the inward mean curvature vector of $M$ embedded in $\mathbb{R}^{n+1}$, $|\mathbf{H}|$ the length of the vector $\mathbf{H}$. The equality in (2.13) holds if and only if $M^n$ is a sphere of radius $r_0$ of $\mathbb{R}^{n+1}$ with $r_0 = (\frac{n}{\lambda_1^\Delta})^{\frac{1}{2}}$ and $p = 2$.*

**Proof** Here we follow the argument used in [15]. Let $y$ be defined as in Theorem 2.1. Since

$\mathrm{div}(y) = n + 1$, by the divergence theorem, we have

$$(n+1)V(\Omega) = \int_{\Omega} \mathrm{div}(y)\mathrm{d}\mu = \int_M \langle y, \nu\rangle \mathrm{d}\nu_g \le \int_M |y|\mathrm{d}\nu_g, \tag{2.14}$$

where $\nu$ denotes the outward unit normal vector field.

Then by the Hölder's inequality of the integral form, since $p > 1$, we have

$$\int_M |y|\mathrm{d}\nu_g \le \Big(\int_M |y|^p \mathrm{d}\nu_g\Big)^{\frac{1}{p}} (A(M))^{\frac{p-1}{p}}. \tag{2.15}$$

Therefore, by (2.14) and (2.15), we have

$$\int_M |y|^p \mathrm{d}\nu_g \ge \frac{(n+1)^p (V(\Omega))^p}{(A(M))^{p-1}}. \tag{2.16}$$

Substituting (2.16) into (2.7) and (2.12), respectively, we get the desired inequality (2.13). By the proof of Theorem 2.1 and [15, Theorem 1.3] and [16], the characterization of equality in (2.13) can be obtained. □

## 3. Reilly-type inequalities for the first eigenvalue of the $p$-biharmonic operator

In 1977, Reilly [6] gave an inequality for the first eigenvalue of the Laplace operator, for the $p$-biharmonic operator, we can prove the following Reilly-type inequalities.

**Theorem 3.1** *Let $M^n$ $(n \ge 3)$ be an $n$-dimensional connected closed orientable Riemannian submanifold immersed in $\mathbb{R}^N$ $(N \ge n+1)$. Then the first nonzero eigenvalue $\lambda^*_{1,p}$ of the $p$-biharmonic operator acting on functions on $M^n$ satisfies*

*(i) If $N = n+1$ and $r \in \{0, 1, \ldots, n-1\}$,*

$$\lambda^*_{1,p}\Big|\int_M H_r \mathrm{d}\nu_g\Big|^p \le n^p \Big(\int_M |\mathbf{H}|^p \mathrm{d}\nu_g\Big)\Big(\int_M |\mathbf{H}_{r+1}|^{\frac{p}{p-1}} \mathrm{d}\nu_g\Big)^{p-1} \begin{cases} N^{\frac{p-2}{2}}, & p \ge 2,\\ N^{\frac{2-p}{2}}, & 1 < p \le 2.\end{cases} \tag{3.1}$$

*(ii) If $N > n+1$ and $r$ is even and $r \in \{0, 1, \ldots, n-1\}$,*

$$\lambda^*_{1,p}\Big|\int_M H_r \mathrm{d}\nu_g\Big|^p \le n^p \Big(\int_M |\mathbf{H}|^p \mathrm{d}\nu_g\Big)\Big(\int_M |\mathbf{H}_{r+1}|^{\frac{p}{p-1}} \mathrm{d}\nu_g\Big)^{p-1} \begin{cases} N^{\frac{p-2}{2}}, & p \ge 2,\\ N^{\frac{2-p}{2}}, & 1 < p \le 2,\end{cases} \tag{3.2}$$

*where $H_r$, $\mathbf{H}$ and $\mathbf{H}_{r+1}$ (see [6,7] for more details) are the $r$-th mean curvature, the inward mean curvature vector and $(r+1)$-th mean curvature vector field of $M$ immersed in $\mathbb{R}^N$, respectively. Particularly, $H_0 = 1$, $\mathbf{H}_1 = \mathbf{H}$. If $H_{r+1}$ does not vanish identically, the equality in (3.1) holds if and only if $M^n$ is a sphere of radius $r_0$ of $\mathbb{R}^{n+1}$ with $r_0 = (\frac{n}{\lambda_1^\Delta})^{\frac{1}{2}}$ and $p = 2$ when $N = n+1$ and the equality in (3.2) holds if and only if $M^n$ is minimally immersed in a sphere of radius $r_0$ of $\mathbb{R}^N$ with $r_0 = (\frac{n}{\lambda_1^\Delta})^{\frac{1}{2}}$ and $p = 2$ when $N > n+1$.*

**Proof** Here we follow the argument used in [7]. Let $y$ be the position vector of $p \in \Omega \subset M^n$ in $\mathbb{R}^N$, and define

$$y = (y_1(x_1, \ldots, x_n), \ldots, y_N(x_1, \ldots, x_n)).$$

Without loss of generality, set

$$\int_M |y_\beta|^{p-2} y_\beta \mathrm{d}\nu_g = 0,$$

where $1 \leq \beta \leq N$, we can substitute $|y_\beta|^{p-2}y_\beta$ by $|y_\beta|^{p-2}y_\beta - \frac{\int_M |y_\beta|^{p-2}y_\beta \mathrm{d}\nu_g}{A(M)}$ if necessary. It follows from the Rayleigh-Ritz inequality that

$$\lambda_{1,p}^* \int_M |y_\beta|^p \mathrm{d}\nu_g \leq \int_M |\Delta y_\beta|^p \mathrm{d}\nu_g. \tag{3.3}$$

Summing the above inequality (3.3) with respect to $\beta$ from 1 to $N$, we obtain

$$\lambda_{1,p}^* \int_M \sum_{\beta=1}^N |y_\beta|^p \mathrm{d}\nu_g \leq \int_M \sum_{\beta=1}^N |\Delta y_\beta|^p \mathrm{d}\nu_g. \tag{3.4}$$

Similar to the discussion in [7], we divide the subsequent discussion into two cases.

Case 1. If $p \geq 2$, similar to the proof in Theorem 2.1, we can obtain

$$\lambda_{1,p}^* N^{\frac{2-p}{2}} \int_M |y|^p \mathrm{d}\nu_g \leq n^p \int_M |\mathbf{H}|^p \mathrm{d}\nu_g,$$

then

$$(\lambda_{1,p}^*)^{\frac{1}{p}} N^{\frac{2-p}{2p}} \Big( \int_M |y|^p \mathrm{d}\nu_g \Big)^{\frac{1}{p}} \leq n \Big( \int_M |\mathbf{H}|^p \mathrm{d}\nu_g \Big)^{\frac{1}{p}}. \tag{3.5}$$

Multiplying both sides of (3.5) with $(\int_M |\mathbf{H}_{r+1}|^{\frac{p}{p-1}} \mathrm{d}\nu_g)^{\frac{p-1}{p}}$ (if $N > n+1$, let $r$ be even and $r \in \{0, 1, \ldots, n-1\}$), then we get

$$(\lambda_{1,p}^*)^{\frac{1}{p}} N^{\frac{2-p}{2p}} \Big( \int_M |y|^p \mathrm{d}\nu_g \Big)^{\frac{1}{p}} \Big( \int_M |\mathbf{H}_{r+1}|^{\frac{p}{p-1}} \mathrm{d}\nu_g \Big)^{\frac{p-1}{p}} \leq n \Big( \int_M |\mathbf{H}|^p \mathrm{d}\nu_g \Big)^{\frac{1}{p}} \Big( \int_M |\mathbf{H}_{r+1}|^{\frac{p}{p-1}} \mathrm{d}\nu_g \Big)^{\frac{p-1}{p}}. \tag{3.6}$$

Since by the Hölder's inequality of the integral form,

$$\Big( \int_M |y|^p \mathrm{d}\nu_g \Big)^{\frac{1}{p}} \Big( \int_M |\mathbf{H}_{r+1}|^{\frac{p}{p-1}} \mathrm{d}\nu_g \Big)^{\frac{p-1}{p}} \geq \Big| \int_M \langle y, \mathbf{H}_{r+1} \rangle \mathrm{d}\nu_g \Big|, \tag{3.7}$$

where the equality holds if and only if $|y|^p = c|\mathbf{H}_{r+1}|^{\frac{p}{p-1}}$ for some constant $c$.

Moreover, by the Minkowski formula [6, Proposition 1],

$$\int_M \langle y, \mathbf{H}_{r+1} \rangle \mathrm{d}\nu_g = -\int_M H_r \mathrm{d}\nu_g. \tag{3.8}$$

By (3.6)–(3.8), we get the desired inequalities (3.1) and (3.2) for the case $p \geq 2$.

Case 2. If $1 < p \leq 2$, similar to the proof in Theorem 2.1, we can obtain

$$(\lambda_{1,p}^*)^{\frac{1}{p}} \Big( \int_M |y|^p \mathrm{d}\nu_g \Big)^{\frac{1}{p}} \leq n N^{\frac{2-p}{2p}} \Big( \int_M |\mathbf{H}|^p \mathrm{d}\nu_g \Big)^{\frac{1}{p}}. \tag{3.9}$$

Noticing that (3.7) holds also for $1 < p \leq 2$, similar to the process of the case of $p \geq 2$, together with (3.9), we can get the desired inequalities (3.1) and (3.2) for the case $1 < p \leq 2$.

Since we have conducted some process similar to the proof of Theorem 2.1 for both two cases above, we can easily infer that the equalities in (3.1) and (3.2) hold if and only if $p = 2$. By [16], the characterization of equalities in (3.1) and (3.2) can be obtained. □

**Remark 3.2** From the proof of Theorem 2.1 in [16] and Theorems 2.1, 2.2 and 3.1, we know the inequalities (2.2), (2.13), (3.1) and (3.2) still hold for $n = 2$, one can turn to [16] for the corresponding characterizations of equalities for special cases.

Let $r = 0$ in Theorem 3.1. Noticing that $H_0 = 1$, $\mathbf{H}_1 = \mathbf{H}$, we can derive the following inequalities.

**Corollary 3.3** *Same assumptions as in Theorem 3.1. Denote by $A(M) = \int_M \mathrm{d}\nu_g$ the area of $M^n$. Then the first nonzero eigenvalue $\lambda_p^*$ of the $p$-biharmonic operator acting on functions on $M^n$ satisfies*

$$\lambda_{1,p}^* \le \Big(\frac{n}{A(M)}\Big)^p \Big(\int_M |\mathbf{H}|^p \mathrm{d}\nu_g\Big)\Big(\int_M |\mathbf{H}|^{\frac{p}{p-1}} \mathrm{d}\nu_g\Big)^{p-1} \begin{cases} N^{\frac{p-2}{2}}, & p \ge 2, \\ N^{\frac{2-p}{2}}, & 1 < p \le 2, \end{cases} \tag{3.10}$$

*where $\mathbf{H}$ is the inward mean curvature vector of $M$ immersed in $\mathbb{R}^N$. The equality in (3.10) holds if and only if $M^n$ is a sphere of radius $r_0$ of $\mathbb{R}^{n+1}$ with $r_0 = (\frac{n}{\lambda_1^\Delta})^{\frac{1}{2}}$ and $p = 2$ when $N = n+1$ or $M^n$ is minimally immersed in a sphere of radius $r_0$ of $\mathbb{R}^N$ with $r_0 = (\frac{n}{\lambda_1^\Delta})^{\frac{1}{2}}$ and $p = 2$ when $N > n+1$.*

**Remark 3.4** When $p = 2$, then the $p$-biharmonic operator becomes biharmonic operator whose eigenvalues are the square of corresponding ones of the Laplace operator, the special case when $p = 2$ in Corollary 3.3 coincides with the case when $r = 1$ in [6, Theorem A] or the case when $p = 2$ in [7, Theorem 1.2] or the case when $q \equiv 0$, $c = 0$ in [16, Theorem 2.1]. Moreover, it follows from the special case when $H$ is constant in Corollary 3.3 that $\lambda_{1,p}^* \le (nH^2)^p$, when $p = 2$, this coincides with the case when $r = 0$ in [15, Corollary 1.2].

By referring to Corollary 3.3, the proof of Theorems 1.5 and 1.6 in [7] and Theorem 2.1 in [16], we can obtain the following conclusions.

**Corollary 3.5** *Let $M^n$ $(n \ge 3)$ be an $n$-dimensional closed orientable Riemannian submanifold immersed in the unit sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ $(N > n+1)$. Denote by $A(M) = \int_M \mathrm{d}\nu_g$ the area of $M^n$. Then the first nonzero eigenvalue $\lambda_{1,p}^*$ of the $p$-biharmonic operator acting on functions on $M^n$ satisfies*

$$\lambda_{1,p}^* \le \Big(\frac{n}{A(M)}\Big)^p \Big(\int_M (|\mathbf{H}|^2+1)^{\frac{p}{2}} \mathrm{d}\nu_g\Big)\Big(\int_M (|\mathbf{H}|^2+1)^{\frac{p}{2(p-1)}} \mathrm{d}\nu_g\Big)^{p-1} \begin{cases} N^{\frac{p-2}{2}}, & p \ge 2, \\ N^{\frac{2-p}{2}}, & 1 < p \le 2, \end{cases} \tag{3.11}$$

*where $\mathbf{H}$ is the inward mean curvature vector of $M$ immersed in $\mathbb{S}^{N-1}$. The equality in (3.11) holds if and only if $M^n$ is a geodesic sphere of radius $r_1$ of $\mathbb{S}^{N-1}$ with $r_1 = \arcsin r_0$ and $p = 2$ when $N = n+2$ or $M^n$ is minimally immersed in a geodesic sphere of radius $r_1$ of $\mathbb{S}^{N-1}$ with $r_1 = \arcsin r_0$ and $p = 2$ when $N > n+2$, where $r_0 = (\frac{n}{\lambda_1^\Delta})^{\frac{1}{2}}$.*

**Remark 3.6** The special case when $p = 2$ in Corollary 3.5 coincides with the case when $q \equiv 0$, $c = 1$ in [16, Theorem 2.1].

**Corollary 3.7** *Let $M^n$ be an $n$-dimensional closed orientable Riemannian submanifold immersed in the projective space $\mathbb{F}P^m$ (see [7, 8, 17] for more details). Denote by $A(M) = \int_M \mathrm{d}\nu_g$ the area of $M^n$. Then the first nonzero eigenvalue $\lambda_{1,p}^*$ of the $p$-biharmonic operator acting on* *functions on $M^n$ satisfies*

$$\lambda^*_{1,p} \leq (\frac{n}{A(M)})^p \Big( \int_M (|\mathbf{H}|^2 + \frac{2(n+d_{\mathbb{F}})}{n})^{\frac{p}{2}} \mathrm{d}\nu_g \Big) \Big( \int_M (|\mathbf{H}|^2 + \frac{2(n+d_{\mathbb{F}})}{n})^{\frac{p}{2(p-1)}} \mathrm{d}\nu_g \Big)^{p-1}$$
$$\begin{cases} (\frac{m(m+1)d_{\mathbb{F}}}{2} + m)^{\frac{p-2}{2}}, \ p \geq 2, \\ (\frac{m(m+1)d_{\mathbb{F}}}{2} + m)^{\frac{2-p}{2}}, \ 1 < p \leq 2, \end{cases} \tag{3.12}$$

*where*

$$d_{\mathbb{F}} = \dim_{\mathbb{R}} \mathbb{F} = \begin{cases} 1, \ \mathbb{F} = \mathbb{R}, \\ 2, \ \mathbb{F} = \mathbb{C}, \\ 4, \ \mathbb{F} = \mathbb{H}, \end{cases}$$

$\mathbf{H}$ *is the inward mean curvature vector of $M$ immersed in $\mathbb{F}P^m$.*

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## References


[1] M. KAC. *Can one hear the shape of a drum?* Amer. Math. Monthly, 1966, **73**(4): 1–23.

[2] I. BABUŠKA, J. OSBORN. *Eigenvalue Problems*. Handbook of Numerical Analysis, Vol.II, 641-787, Handb. Numer. Anal., II, North-Holland, Amsterdam, 1991.

[3] I. CHAVEL. *Eigenvalues in Riemannian Geometry*. Academic Press, New York, 1984.

[4] A. GRIGORYAN, S. T. YAU. *Surveys in Differential Geometry: Eigenvalues of Laplacians and Other Geometric Operators. Vol. 9*. International Press of Boston, Boston, 2004.

[5] Jing MAO, Chuanxi WU, Feng DU. *Eigenvalue Problems on Manifolds*. Science Press, Beijing, 2017.

[6] R. REILLY. *On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space*. Comm. Math. Helv., 1977, **52**(4): 525–533.

[7] Feng DU, Jing MAO. *Reilly-type inequalities for p-Laplacian on compact Riemannian manifolds*. Front. Math. China, 2015, **10**(3): 583–594.

[8] Hang CHEN, Xianfeng WANG. *Sharp Reilly-type inequalities for a class of elliptic operators on submanifolds*. Differential Geom. Appl., 2019, **63**: 1–29.

[9] Qiaoling WANG, Changyu XIA. *Isoperimetric bounds for the first eigenvalue of the Laplacian*. Z. Angew. Math. Phys., 2010, **61**(1): 171–175.

[10] Feng DU, Chuanxi WU. *Isoperimetric upper bounds for the first eigenvalue of the Paneitz operator and the p-Laplace operator*. Scientia Sinica Mathematica, 2017, **47**(9): 1047–1054.

[11] P. DRABEK, M. OTANI. *Global bifurcation result for the p-biharmonic operator*. Electron. J. Differential Equations, 2001, **48**: 1–19.

[12] M. TALBI, N. TSOULI. *On the spectrum of the weighted p-biharmonic operator with weight*. Mediterr. J. Math., 2007, **4**(1): 73–86.

[13] M. LAGHZAL, A. TOUZANI. *Eigenvalues for the p-biharmonic Dirichlet-Steklov problem with weights*. Gulf J. Math., 2020, **9**(2): 1–9.

[14] A. E. KHALIL, M. D. M. ALAOUI, A. TOUZANI. *On the p-biharmonic operator with critical Sobolev exponent and nonlinear Steklov boundary condition*. Int. J. Anal., 2014, Art. ID 498386, 8 pp.

[15] H. ALENCAR, M. D. CARMO, H. ROSENBERG. *On the first eigenvalue of the Linearized operator of the r-th mean curvature of a hypersurface*. Ann. Global Anal. Geom., 1993, **11**(4): 387–395.

[16] A. E. SOUFI, S. ILIAS. *Second eigenvalue of Schrödinger operators and mean curvature*. Comm. Math. Phys., 2000, **208**(3): 761–770.

[17] Bang-Yen CHEN. *Total Mean Curvature and Submanifolds of Finite Type*. World Scientific Publishing Co., Singapore, 1984.

[18] Daguang CHEN, Qingming CHENG. *Extrinsic estimates for eigenvalues of the Laplace operator*. J. Math. Soc. Japan, 2008, **60**(2): 325–339.

[19] M. F. BETTA, F. BROCK, A. MERCALDO, et al. *A weighted isoperimetric inequality and applications to symmetrization*. J. Inequal. Appl., 1999, **4**(3): 215–240.