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On Heegaard Splittings with Finitely Many Pairs of Disjoint Compression Disks

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Abstract Suppose $V\cup_S W$ is a genus-g weakly reducible Heegaard splitting of a closed 3-manifold with finitely many pairs of disjoint compression disks on distinct sides up to isotopy and $g>2$. We show $V\cup_S W$ admits an untelescoping: $(V_1\cup_{S_1} W_1)\cup_F (W_2\cup_{S_2} V_2)$ such that W_i has a unique separating compressing disk and $d(S_i)\geq 2$, for $i=1,2$. If there exist more than one but finitely many pairs of disjoint compression disks, at least one of $d(S_i)$ is 2 and S is a critical Heegaard surface.

Keywords 3-manifolds; Heegaard splitting; weakly reducible; critical surfaces

MR(2020) Subject Classification 57K31; 57K20

1. Introduction

Let M be a connected, orientable, closed 3-manifold. Let $V\cup_S W$ be a Heegaard splitting of M, thus V and W are handlebodies which share the common boundary suface S. The Heegaard splitting $V\cup_S W$ is said to be weakly reducible if there are two essential disks $D\subset V$ and $E\subset W$, such that $D\cap E=\emptyset$. A Heegaard splitting which is not weakly reducible is said to be strongly irreducible. Casson and Gorden [1] proved that if a 3-manifold admits an irreducible but weakly reducible Heegaard splitting, it must contain an incompressible surface. Scharlemann and Thompson [2] proved that if a Heegaard splitting is weakly reducible, then it admits an untelescoping by reattaching handles.

In this paper, the authors are interested in a weakly reducible Heegaard splitting which admits finitely many pairs of disjoint compression disks on distinct sides up to isotopy. The first author [3] proved that for each $g>2$, there are infinitely many genus g Heegaard splittings which admit a unique pair of disjoint compression disks on distinct sides. Such a Heegaard splitting is said to be keen weakly reducible. Keen weakly reducible Heegaard surfaces are interesting because they are examples of topologically non-minimal surfaces which were introduced by David Bachman [4].

In Section 3, we will show if $V\cup_S W$ admits an untelescoping $(V_1\cup_{S_1} W_1)\cup_F (W_2\cup_{S_2} V_2)$, such that W_i has a unique separating compression disk and $d(S_i)\geq 3$, for $i=1,2$, then the

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amalgamated Heegaard surface S is keen weakly reducible. This improves our previous result in [3], which is under the similar assumption but $d(S_i) \geq 4$ is required.

Although we could not yet answer the question whether there exists a weakly reducible Heegaard splitting which admits more than one but finitely many pairs of disjoint compression disks on distinct sides, we will give some necessary conditions to describe properties of such Heegaard splittings in Section 4.

2. Preliminaries

A compression body V is a 3-manifold obtained from a closed surface S by attaching some 2-handles to $S \times \{0\} \subset S \times I$ and capping off any resulting 2-sphere boundary components with 3-balls. The surfaces $S \times \{1\}$ is denoted by $\partial_+ V$ and $\partial V - \partial_+ V$ is denoted by $\partial_- V$. If $\partial_- V = \emptyset$, then V is called a handlebody. For a 3-manifold M, a Heegaard splitting $V \cup_S W$ is a decomposition of M into two compression bodies V and W, where $\partial_+ V = \partial_+ W = S$. A disk D properly embedded in a compression body V is called a compression disk if ∂D is essential in $\partial_+ V$, that is, ∂D does not bound a disk in $\partial_+ V$. By abuse of notations, in the following statement, we will use the same letter for a compression disk (or a curve) and its isotopy class.

Let S be a closed orientable surface whose genus is at least 2. The distance between two essential simple closed curves α and β in S, denoted by $d_S(\alpha, \beta)$, is the smallest integer $n \geq 0$ such that there is a sequence of essential simple closed curves $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ in S where α_{i-1} is disjoint from α_i for $1 \leq i \leq n$. If S is an embedded surface in a 3-manifold, and D and E are two compression disks on distinct sides, sometimes $d_S(\partial D, \partial E)$ is denoted simply by $d_S(D, E)$. Let A and B be two sets of essential simple closed curves in S. The distance between A and B, which is denoted by $d_S(A, B)$, is defined to be $\min\{d_S(x, y) | x \in A, y \in B\}$. The Heegaard distance of a Heegaard splitting $V \cup_S W$ is defined to be $d(S) = d_S(\mathscr{D}_V, \mathscr{D}_W)$ where $\mathscr{D}_V$ and $\mathscr{D}_W$ are sets of compression disks in V and W, respectively. A Heegaard splitting $V \cup_S W$ is weakly reducible if and only if $d(S) \leq 1$. The Heegaard distance was first defined by Hempel [5]. Hempel also showed for any integers $g \geq 2$ and $n \geq 2$, there is a 3-manifold that admits a distance at least n Heegaard splitting of genus g. Therefore, suppose V is a handlebody whose genus is at least 2, and $\mathscr{D}_V$ is defined to be the set of compression disks of V, then for any integer $n \geq 2$, there exists an essential separating curve $\beta \in \partial V$ such that $d_{\partial V}(\beta, \mathscr{D}_V) \geq n$.

Lemma 2.1 *Let S be a closed orientable surface. Suppose that β and γ are two essential simple closed curves on S. If $d_S(\beta, \gamma) \geq 3$ and β is separating on S, then there are at least two essential sub-arcs of γ on each component of $S \setminus \beta$.*

Proof We assume that $|\beta \cap \gamma|$ is minimal in the isotopy classes of β and γ. Since β is separating on S, $|\beta \cap \gamma|$ is even. If $|\beta \cap \gamma| = 0$ then $d_S(\beta, \gamma) \leq 1$. If $|\beta \cap \gamma| = 2$ then one boundary component of the closure of $N(\beta \cup \gamma)$ is essential on S and disjoint from β and γ and we have $d_S(\beta, \gamma) = 2$. Therefore, $|\beta \cap \gamma| \geq 4$ and there are at least two essential sub-arcs of γ on each component of $S \setminus \beta$. □

Let V_1 be a handlebody whose genus is at least 2 and $\partial V_1 = S_1$. Let V be the handlebody obtained by attaching one 1-handle $D_0 \times I$ to V_1 along a pair of disjoint disks $D_1, D_2 \subset S_1$, where D_0 is the disk corresponding to the 1-handle. $\mathscr{D}_V$ is defined to be the set of compression disks of V. There is a partition of $\mathscr{D}_V$: $\mathscr{D}_V = \mathscr{D}_0 \cup \mathscr{D}_1 \cup \mathscr{D}_2 \cup \mathscr{D}_3$, such that, $\mathscr{D}_0 = \{D_0\}$, $\mathscr{D}_1 = \{D \mid D \cap D_0 = \emptyset;\ D \neq D_0, D \text{ is inessential in } V_1\}$, $\mathscr{D}_2 = \{D \mid D \cap D_0 = \emptyset;\ D \text{ is essential in } V_1\}$ and $\mathscr{D}_3 = \{D \mid D \cap D_0 \neq \emptyset\}$. It is clear that any compression disk D of V belongs to one and only one of the four subsets up to isotopy. For examples, in Figure 1, d_i is a compression disk of V where $d_i \in \mathscr{D}_i$. If $D \in \mathscr{D}_1$, then ∂D, ∂D_1 and ∂D_2 co-bound a pair of pants on S_1. Hence D is isotopic to a band-sum of D_1, D_2 along an arc and ∂D bounds a once-punctured torus. A compression disk $D \in \mathscr{D}_2$ if and only if $D \in \mathscr{D}_{V_1}$.

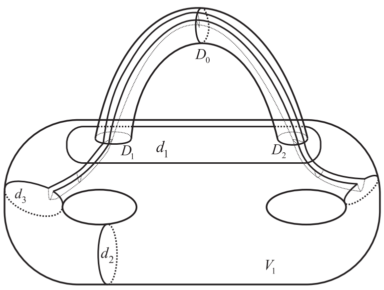

Figure 1 Compression disks in the handlebody V

Now we consider $D \in \mathscr{D}_3$, that is, D is an essential disk in V so that $D \cap D_0 \neq \emptyset$. Furthermore, D is isotoped in V such that $|D \cap D_0|$ is minimal. Let S_1^* be the surface $S_1 - (\text{int}(D_1) \cup \text{int}(D_2))$. Then S_1^* is a sub-surface of S_1 with two boundary components ∂D_1 and ∂D_2. By standard arguments, we have some observations as follows.

Lemma 2.2 ([6]) *(1) Each component of $D \cap D_0$ is a properly embedded arc in both D and D_0.*

(2) Each component of $\partial D \cap S_1^*$ is essential on S_1^*.

(3) Each component of $D \cap (\partial D_0 \times I)$ is an arc with its two endpoints lying in distinct boundary components of the annulus $\partial D_0 \times I$.

Let γ be an outermost component of $D \cap (D_1 \cup D_2)$ on D. This means that γ, together with an arc $\gamma_1 \subset \partial D$, bounds a sub-disk in D, say D_γ, such that $D_\gamma \cap (D_1 \cup D_2) = \gamma$. We call γ_1 an outermost arc related to γ and call D_γ an outermost disk related to γ. See Figure 2.

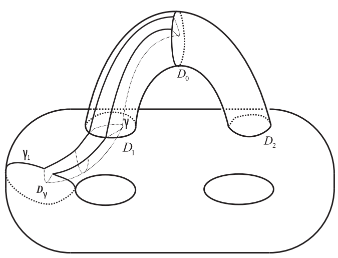

Figure 2 The sub-disk D_γ in D

Lemma 2.3 ([6]) *(1) The arc γ_1, whose end points lie in one of D_1 and D_2, is strongly essential in S_1^*.*

(2) The disk $D_\gamma \in \mathscr{D}_2$, that is, D_γ is an essential disk both in V_1 and V.

We explain that an essential arc in S_1^* is called strongly essential if both boundary points lie in ∂D_i and it is an essential arc on $S_1^* \cup D_j$, where $\{i, j\} = \{1, 2\}$.

The definition of untelescoping was first introduced in [2]. Let $V_1 \cup_{S_1} W_1$ and $W_2 \cup_{S_2} V_2$ be Heegaard splittings of 3-manifolds M_1 and M_2, respectively. Suppose that M_1 and M_2 are glued together along some homomorphic boundary components $F_1 \subset \partial_- W_1$ and $F_2 \subset \partial_- W_2$. Let $M = M_1 \cup_F M_2$ be the resulting manifold and F be the image of F_1 and F_2 in M. Now collapse $(F_1 \cup F_2) \times [0, 1]$ to F and regard the 1-handles of W_1 and W_2 are attached to F. Let $V = V_1 \cup \{$1-handles in $W_2\}$ and $W = V_2 \cup \{$1-handles in $W_1\}$ and $S = V \cap W$. Then $V \cup_S W$ is a weakly reducible Heegaard splitting and is called an amalgamation of $V_1 \cup_{S_1} W_1$ and $W_2 \cup_{S_2} V_2$. Conversely, $(V_1 \cup_{S_1} W_1) \cup_F (W_2 \cup_{S_2} V_2)$ is called an untelescoping of $V \cup_S W$. See Figure 3. We also say $(V_1 \cup_{S_1} W_1) \cup_F (W_2 \cup_{S_2} V_2)$ is a generalized Heegaard splitting of M.

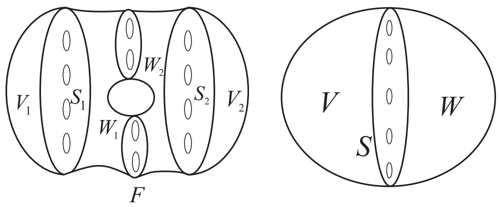

Figure 3 The untelescoping of $V \cup_S W$

3. Keen weakly reducible Heegaard splittings

We show a sufficient condition for a Heegaard splitting to be keen weakly reducible.

Theorem 3.1 *Let $V \cup_S W$ be a Heegaard splitting of a closed 3-manifold M. Suppose it admits an untelescoping:*

$$M = V \cup_S W = (V_1 \cup_{S_1} W_1) \cup_F (W_2 \cup_{S_2} V_2),$$

such that W_i has only one separating compressing disk and $d(S_i) \geq 3$, for $i = 1, 2$, then S is keen weakly reducible.

Proof The untelescoping of $V \cup_S W$ could be realized by reattaching handles as follows: Let V_i be a genus $g > 1$ handlebody and $S_i = \partial V_i$, for $i = 1, 2$. There exists an essential separating curve $\beta \subset S_1$, such that $d_{S_1}(\beta, \mathscr{D}_{V_1}) \geq 3$ and we attach a 2-handle $E_0 \times I$ to V_1 along $\beta \times I$ so that β bounds the unique disk $E_0 \subset W_1$. Let $M_1 = V_1 \cup_{\beta \times I} (E_0 \times I) = V_1 \cup_{S_1} W_1$ and $F = \partial M_1$ which has two components.

Next, we attach a 1-handle $D_0 \times I$ to M_1 such that $(D_0 \times I) \cap (E_0 \times I) = \emptyset$ and the gluing disks denoted by $D_1 \cup D_2 = V_1 \cap (D_0 \times I)$ are on distinct components of F. D_0, corresponding to the 1-handle, is the unique disk in W_2 and we denote ∂D_0 by α. Let $M_2 = M_1 \cup_{D_1 \cup D_2} (D_0 \times I) = (V_1 \cup_{S_1} W_1) \cup_F W_2$ be the resulted 3-manifold.

Finally, we attach V_2 to M_2 via some orientation preserving homeomorphism $f : \partial M_2 \to S_2$ such that $d_{S_2}(f(\alpha), \mathscr{D}_{V_2}) \geq 3$. Since α is separating on ∂M_2, $f(\alpha)$ is separating on S_2 and after the handlebody-attaching, we will use α instead of $f(\alpha)$. $V_2 \cap (E_0 \times I)$ are two disks, denoted by E_1,E_2. The result is the closed 3-manifold M, that is

$$M = (V_1 \cup_{S_1} W_1) \cup_F (W_2 \cup_{S_2} V_2) = V_1 \cup_{\beta\times I} (E_0 \times I) \cup_{D_1\cup D_2} (D_0 \times I) \cup_{S_2} V_2.$$

Now let $V = V_1 \cup_{D_1\cup D_2} (D_0 \times I)$, $W = V_2 \cup_{E_1\cup E_2} (E_0 \times I)$ and $S = V \cap W$. Then both V and W are genus $g + 1$ handlebodies and $M = V \cup_S W$ is a Heegaard splitting. α bounds the disk D_0 in V and β bounds the disk E_0 in W. $D_0 \cap E_0 = \emptyset$ implies that $M = V \cup_S W$ is weakly reducible. See Figure 4.

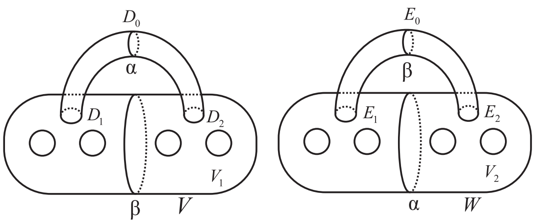

Figure 4 V and W

In order to prove (D_0, E_0) is the unique pair of disjoint compression disks of S, we divide $\mathscr{D}_V$ and $\mathscr{D}_W$ as mentioned in Section 2: $\mathscr{D}_V = \mathscr{D}_0 \cup \mathscr{D}_1 \cup \mathscr{D}_2 \cup \mathscr{D}_3$, such that $\mathscr{D}_0 = \{D_0\}$. $\mathscr{D}_1 = \{D \mid D \cap D_0 = \emptyset;\ D \neq D_0; D$ is inessential in $V_1\}$. $\mathscr{D}_2 = \{D \mid D \cap D_0 = \emptyset;\ D$ is essential in $V_1\} = \mathscr{D}_{V_1}$. $\mathscr{D}_3 = \{D \mid D \cap D_0 \neq \emptyset\}$. Similarly, $\mathscr{D}_W = \mathscr{D}^0 \cup \mathscr{D}^1 \cup \mathscr{D}^2 \cup \mathscr{D}^3$, such that $\mathscr{D}^0 = \{E_0\}$. $\mathscr{D}^1 = \{E \mid E \cap E_0 = \emptyset;\ E \neq E_0; E$ is inessential in $V_2\}$. $\mathscr{D}^2 = \{E \mid E \cap E_0 = \emptyset;\ E$ is essential in $V_2\} = \mathscr{D}_{V_2}$. $\mathscr{D}^3 = \{E \mid E \cap E_0 \neq \emptyset\}$. We will show that for each $D \in \mathscr{D}_i$, $E \in \mathscr{D}^j$, where $i, j = 0, 1, 2, 3$ and $(i, j) \neq (0, 0)$, $D \cap E \neq \emptyset$ holds. Notice that if it holds for some (i, j), then by the symmetric construction of the Heegaard splitting and the similar partitions of the $\mathscr{D}_V$ and $\mathscr{D}_W$, it also holds for (j, i). Thus it is sufficient to prove the case $i \leq j$.

In the following argument, we suppose to the contrary that $D \cap E = \emptyset$ and assume that $|D \cap D_0| + |E \cap E_0|$ is minimal in the isotopy classes of D and E.

Case 1. $D \in \mathscr{D}_0$.

In this case $D = D_0$. If $E \in \mathscr{D}^1$, E is a band-sum of E_1 and E_2 along an arc on S_2. Since α separates E_1 and E_2 in S_2, any arc connecting them must intersect $\alpha = \partial D_0$. It follows that $D \cap E \neq \emptyset$. If $E \in \mathscr{D}^2$, E is an essential disk in V_2. Since $d_{S_2}(D_0, \mathscr{D}_{V_2}) = d_{S_2}(f(\alpha), \mathscr{D}_{V_2}) \geq 3 > 2$, D_0 intersects each compression disk of V_2. It follows that $D \cap E \neq \emptyset$. If $E \in \mathscr{D}^3$, $E \cap E_0 \neq \emptyset$. By applying Lemma 2.3 to W and V_2, there is an outermost disk of E, say E_γ, such that $E_\gamma \in \mathscr{D}^2$. By the above discussion, $D \cap E = D_0 \cap E \supset D_0 \cap E_\gamma \neq \emptyset$.

Case 2. $D \in \mathscr{D}_1$.

In this case, D is a band-sum of D_1 and D_2 along an arc. We denote the arc by c. β separates D_1 and D_2 in S_1 implies that β intersects c as well as the band $N(c)$. Notice that ∂D bounds a once-punctured torus $T_D = (\alpha \times I) \cup N(c)$ on S. Since $\alpha \cap \beta = \emptyset$, $\beta \cap T_D = \beta \cap N(c)$. If β

intersects c only once, then $\overline{T_D \setminus \beta}$ is isotopic to $\alpha \times I$; Otherwise, $\overline{T_D \setminus \beta}$ contains a component isotopic to $\alpha \times I$ and some disks.

Since we suppose to the contrary that $D \cap E = \emptyset$, $\partial E \subset T_D$ or $\partial E \subset (S \setminus T_D)$. If $\partial E \subset (S \setminus T_D)$ then $E \cap D_0 = \emptyset$ which contradicts Case 1. Thus $\partial E \subset T_D$.

If $E \in \mathscr{D}^1$, ∂E also bounds a once punctured torus T_E. $\partial E \subset T_D$ implies that ∂E is isotopic to ∂D. It follows that α and β are isotopic because $\alpha \subset T_D$, $\beta \subset T_E$ and $\alpha \cap \beta = \emptyset$. But α and β are not isotopic, a contradiction.

If $E \in \mathscr{D}^2$, $E \cap E_0 = \partial E \cap \beta = \emptyset$ and ∂E is essential in S. It follows that $\partial E \subset \overline{T_D \setminus \beta}$ and ∂E cannot lie in any disk component of $\overline{T_D \setminus \beta}$. It follows that after isotopy, $\partial E \subset \alpha \times I \subset S$. In this case ∂E is isotopic to α which implies that $d_{S_2}(f(\alpha), \mathscr{D}_{V_2}) = 0$, a contradiction.

If $E \in \mathscr{D}^3$, $E \cap E_0 \neq \emptyset$. By applying Lemma 2.3 to W and V_2, there is an outermost arc, say γ, whose two endpoints lie in ∂E_i, where $i = 1$ or 2. Furthermore, there is an outermost disk of E, say E_γ, such that E_γ is essential in V_2. $E_\gamma \cap E_i$ is an arc denoted by e_i. On the other hand, D is a band-sum of D_1 and D_2 which lie distinct sides of β. Hence we may find a sub-arc γ_1 of ∂D such that $\partial\gamma_1 \subset \partial E_j$, where $j = 3 - i$. γ_1, together with a sub-arc of ∂E_j, say e_j, bounds a disk isotopic to D_0. $D \cap E = \emptyset$ implies that $\gamma \cap \gamma_1 = \emptyset$. Since $e_i \subset E_i$, $e_j \subset E_j$ and $E_i \cap E_j = \emptyset$, we have $e_i \cap e_j = \emptyset$. Thus $(\gamma \cup e_i) \cap (\gamma_1 \cup e_j) = \emptyset$ and it means that $E_\gamma \cap D_0 = \emptyset$. This contradicts Case 1. See Figure 5.

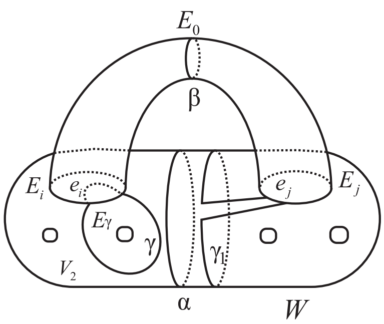

Figure 5 $D \in \mathscr{D}_1$ and $E \in \mathscr{D}^3$

Case 3. $D \in \mathscr{D}_2$.

In this case, D is essential in V_1. If $E \in \mathscr{D}^2$, E is essential in V_2. Since $\partial D \cap \beta \neq \emptyset$ and β is separating in S_1, there is a sub-arc γ_1 of ∂D such that $\partial\gamma_1 \subset \partial E_1$ and γ_1 together with an arc of E_1 forms an essential closed curve γ on S_2. Moreover, $D_0 \cap D = \emptyset$ and $D_0 \cap E_1 = \emptyset$ mean that $D_0 \cap \gamma = \emptyset$. $E \cap D = \emptyset$ and $E \cap E_1 = \emptyset$ mean that $E \cap \gamma = \emptyset$. See Figure 6. Hence $d_{S_2}(f(\alpha), \mathscr{D}_{V_2}) \leq d_{S_2}(D_0, E) \leq d_{S_2}(D_0, \gamma) + d_{S_2}(\gamma, E) = 1 + 1 = 2$, a contradiction.

If $E \in \mathscr{D}^3$, $E \cap E_0 \neq \emptyset$. By applying Lemma 2.3 to W and V_2, there is an outermost arc, say γ_1, whose two endpoints lie in ∂E_i, where $i = 1$ or 2. Furthermore, there is an outermost disk of E, say E_{γ_1}, such that E_{γ_1} is essential in V_2. $E_{\gamma_1} \cap E_i$ is an arc denoted by e_i. On the other hand, since ∂D is essential in S_1, $d_{S_1}(\partial D, \beta) \geq 3$ and β separates S_1, by Lemma 2.1 there is a sub-arc γ_2 of ∂D such that $\partial\gamma_2 \subset \partial E_j$ where $j = 3 - i$ and γ_2 together with an arc of

$e_j \subset E_j$ forms an essential closed curve γ on S_2 such that $D_0 \cap \gamma = \emptyset$. See Figure 7. $D \cap E = \emptyset$ implies that $\gamma_1 \cap \gamma_2 = \emptyset$. Since $e_i \subset E_i$, $e_j \subset E_j$ and $E_i \cap E_j = \emptyset$, we have $e_i \cap e_j = \emptyset$. Hence $E_{\gamma_1} \cap \gamma = \partial E_{\gamma_1} \cap \gamma = (\gamma_1 \cup e_i) \cap (\gamma_2 \cup e_j) = \emptyset$. Thus $d_{S_2}(f(\alpha), \mathscr{D}_{V_2}) \le d_{S_2}(D_0, E_{\gamma_1}) \le d_{S_2}(D_0, \gamma) + d_{S_2}(\gamma, E_{\gamma_1}) = 1 + 1 = 2$, a contradiction.

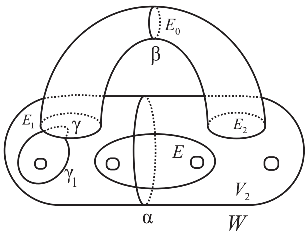

Figure 6 $D \in \mathscr{D}_2$ and $E \in \mathscr{D}^2$

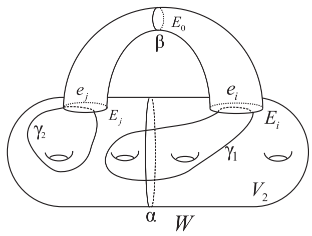

Figure 7 $D \in \mathscr{D}_2$ and $E \in \mathscr{D}^3$

Case 4. $D \in \mathscr{D}_3$.

In this case, $D \cap D_0 \neq \emptyset$. We only need to discuss when $E \in \mathscr{D}^3$, that is, $E \cap E_0 \neq \emptyset$. By Lemma 2.3, there exists an outermost arc γ_1 of ∂D and an outermost disk of D, say D_{γ_1} which is essential in V_1. Since $d_{S_1}(\beta, D_{\gamma_1}) \geq d_{S_1}(\beta, \mathscr{D}_{V_1}) \geq 3$ and β separates S_1, by Lemma 2.1, there exist two sub-arcs of γ_1, say γ_{11} and γ_{12}, such that $\partial\gamma_{11} \subset \partial E_1$ and $\partial\gamma_{12} \subset \partial E_2$. For each $i = 1, 2$, γ_{1i} together with an arc $e_i \subset E_i$ forms a closed curve, say γ_{e_i}, which is essential in S_2. Since the outermost arc of D and E_i are disjoint from α, $D_0 \cap \gamma_{e_i} = \emptyset$. By applying Lemma 2.3 to W and V_2, there is an outermost arc of ∂E, say γ_2, where $\partial\gamma_2 \subset \partial E_j$, $j = 1$ or 2. γ_2, together with an arc $e'_j \subset E_j$, bounds an outermost disk, say E_{γ_2}, which is essential in S_2. $D \cap E = \emptyset$ means that $\gamma_{11} \cup \gamma_{12}$ and γ_2 are disjoint. Therefore, γ_{e_i} and E_{γ_2} are disjoint, where $i = 3 - j$. It follows that $d_{S_2}(f(\alpha), \mathscr{D}_{V_2}) \le d_{S_2}(D_0, E_{\gamma_2}) \le d_{S_2}(D_0, \gamma_{e_i}) + d_{S_2}(\gamma_{e_i}, E_{\gamma_2}) \le 2$ which is a contradiction. See Figure 8.

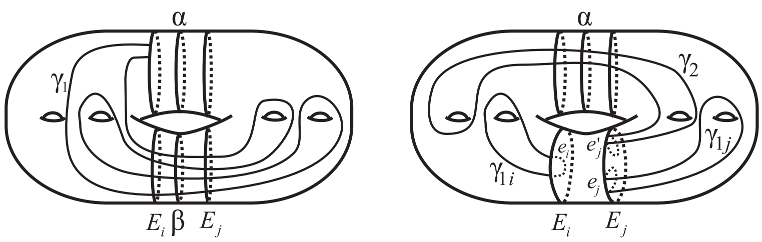

Figure 8 $D \in \mathscr{D}_3$ and $E \in \mathscr{D}^3$

Hence for each $D \in \mathscr{D}_i$, $E \in \mathscr{D}^j$, where $(i, j) \neq (0, 0)$, $D \cap E = \partial D \cap \partial E \neq \emptyset$ holds. It means that (D_0, E_0) is the unique disjoint pair of compression disks on distinct sides of S. This completes the proof. □

4. Heegaard splittings with finitely many pairs of disjoint compression disks

In this section, we discuss a weakly reducible Heegaard splitting with more than one but finitely many pairs of disjoint compression disks. It is uncertain whether there exists such a weakly reducible Heegaard splitting but we will give some necessary conditions to describe some properties of such a Heegaard splitting if it does exist.

Lemma 4.1 *Let $V \cup_S W$ be a Heegaard splitting of a closed 3-manifold M, where $g > 2$. Suppose there are only finitely many pairs of disjoint compression disks*

$$\{(D_i, E_i) | D_i \subset V, E_i \subset W, D_i \cap E_i = \emptyset, \quad i = 1, 2, \dots, n\}.$$

Then for each i, ∂D_i and ∂E_i are not isotopic, moreover, each is non-separating on S, and $\partial D_i \cup \partial E_i$ is separating on S. If $D_i = D_j$ for some i, j, then $E_i \cap E_j \neq \emptyset$. Similarly, If $E_i = E_j$ for some i, j, then $D_i \cap D_j \neq \emptyset$.

Proof If for some i, ∂D_i and ∂E_i and $\partial D_i \cup \partial E_i$ are all non-separating on S, then any band-sum of two copies of D_i along an arc which is disjoint from ∂E_i is disjoint from E_i. There are infinitely many such arcs, hence there are infinity many pairs of disjoint disks, a contradiction.

If for some i, ∂D_i is separating on S, then we may find a non-separating disk D'_i such that $D'_i \cap E_i = \emptyset$. If ∂E_i is also separating on S, then we may find a non-separating disk E'_i such that $D'_i \cap E'_i = \emptyset$. Since $D'_i \cup E_i$ (or $D'_i \cup E'_i$) could not be separating on S, by the above discussion, there are infinity many pairs of disjoint disks, a contradiction.

If for some i, ∂D_i and ∂E_i are isotopic, $V \cup_S W$ is reducible and we may also find two separating disk $\partial D'_i$ and $\partial E'_i$ which are disjoint, By the above discussion, there are infinity many pairs of disjoint disks, a contradiction.

If for some i, j, $D_i = D_j$ and $E_i \cap E_j = \emptyset$, then $E_i \cup E_j$ bounds a sub-surface S_1 of S. Therefore S_1 is not an annulus and $D_i \cap S_1 = \emptyset$. Any band-sum of $E_i \cup E_j$ along an arc on S_1 is disjoint from D_i. There are infinitely many such arcs, hence there are infinity many pairs of disjoint disks, a contradiction. □

Theorem 4.2 *Suppose $V \cup_S W$ is a genus-g weakly reducible Heegaard splitting of a closed 3-manifold with finitely many pairs of disjoint compression disks on distinct sides up to isotopy and $g(S) > 2$. Then it admits an untelescoping:*

$$M = V \cup_S W = (V_1 \cup_{S_1} W_1) \cup_F (W_2 \cup_{S_2} V_2),$$

such that W_i has a unique separating compressing disk and $d(S_i) \geq 2$, for $i = 1, 2$. If there exist more than one but finitely many pairs of disjoint compression disks, at least one of $d(S_i)$ is 2.

Proof For any pair of disjoint disk D_i and E_i, ∂D_i and ∂E_i are not isotopic, moreover, each is non-separating on S, and $\partial D_i \cup \partial E_i$ is separating on S. E_i is corresponding to the 2-handle attaching to $V_1 = \overline{V \setminus N(D_i)}$ and D_i is corresponding to the 2-handle attaching to $\overline{W \setminus N(E_i)}$. If $d(S_1) \leq 1$, there exists a disk D in V_1 such that $D \cap E_i = \emptyset$. Since $D_i \cap E_i = \emptyset$, by Lemma 4.1 we have $D \cap D_i \neq \emptyset$. But $D \subset V_1$ and D_i is corresponding to the 1-handle attached to V_1, so that $D \cap D_i = \emptyset$, a contradiction. If there exist more than one but finitely many pairs of disjoint compression disk, by Theorem 3.1, at least one of $d(S_i)$ is 2. □

An embedded surface S is said to be critical if it is an index 2 topologically minimal surface.

There is an equivalent definition of critical surface [4]: the compression disks for S can be partitioned into two sets C_0 and C_1, such that there exists at least one pair of disks $D_i, E_i \in C_i$ on opposite sides of S, such that $D_i \cap E_i = \emptyset$, for $i = 0, 1$. On the other hand, if $D \in C_i$ and $E \in C_{1-i}$ lie on opposite sides of S, then $D \cap E \neq \emptyset$. See [7–9] for many examples.

Corollary 4.3 *Suppose $V \cup_S W$ is a genus-g weakly reducible Heegaard splitting of a closed 3-manifold M with more than one but finitely many disjoint compression disks on distinct sides and $g(S) > 2$. Then S is a critical Heegaard surface.*

Proof It follows that $M = V \cup_S W = (V_1 \cup_{S_1} W_1) \cup_F (W_2 \cup_{S_2} V_2)$, such that W_i has only one separating compressing disk and $d(S_i) \geq 2$, for $i = 1,\ 2$ while at least one of $d(S_i)$ is 2. Let C_0 be $\{D_0, E_0\}$ and C_1 be other essential disks of S. The Case 1 in the proof of Theorem 3.1 shows that any disk in $\mathscr{D}_W \backslash E_0$ intersects D_0, and any disk in $\mathscr{D}_V \backslash D_0$ intersects E_0. Since $(D_0,\ E_0)$ is not the unique disjoint pair of compression disks, there exist $D \in \mathscr{D}_V \backslash D_0$ and $E \in \mathscr{D}_W \backslash E_0$, such that $D \cap E = \emptyset$. Therefore, C_0 and C_1 satisfy the definition of criticality and S is a critical Heegaard surface. □

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