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The Nullities of Signed Cycle-Spliced Graphs

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Abstract Let $\eta(\Gamma)$ and $c(\Gamma)$ be the nullity and the cyclomatic number of a signed graph Γ. A signed cycle-spliced graph is a connected signed graph in which every block is a cycle. In this paper, we prove that for every signed cycle-spliced graph Γ, $\eta(\Gamma)\le c(\Gamma)+1$ and the extremal graphs Γ with nullity $c(\Gamma)+1$ are characterized, which extend the related results of Wong, Zhou and Tian (2022) on simple cycle-spliced graphs. Moreover, we prove that for every signed cycle-spliced graph Γ, $\eta(\Gamma)\ne c(\Gamma)$. Some properties on signed cycle-spliced graphs Γ with $\eta(\Gamma)=c(\Gamma)-1$ are explored, as well as a structural characterization on signed cycle-spliced bipartite graphs Γ satisfying $\eta(\Gamma)=c(\Gamma)-1$.

Keywords signed graph; nullity; cyclomatic number

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1. Introduction

Let $G=(V(G),E(G))$ be a graph with vertex set $V(G)=\{v_1,\ldots,v_n\}$ and edge set $E(G)$. The adjacency matrix $A(G)=(a_{ij})$ of G is defined to be an $n\times n$ symmetric matrix such that $a_{ij}=1$ if vertices v_i and v_j are adjacent; and $a_{ij}=0$, otherwise. The multiplicity of the zero eigenvalues of $A(G)$ is called the nullity of G, and denoted by $\eta(G)$. A vertex is called a pendant vertex if its degree is 1. The number of pendant vertices of G is denoted by $p(G)$. The cyclomatic number of G is $c(G)=|E(G)|-|V(G)|+\theta(G)$, where $\theta(G)$ is the number of connected components of G. A block in a graph is a maximal connected subgraph with no cut vertex. A graph G is said to be a cycle-spliced graph if G is connected and every block in G is a cycle. A cycle-spliced bipartite graph is a cycle-spliced graph without odd cycle. An induced subgraph H of a graph G is called a pendant subgraph of G if H has at least two vertices and there is exactly one vertex in H, referred to as the root of H, that has at least one neighbor not in H. If, in addition, H is an induced cycle of G, then we refer to H as a pendant cycle of G.

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We call a pendant subgraph H of G a maximal pendant subgraph of G if there does not exist a pendant subgraph H' with $V(H') \supsetneqq V(H)$ (or, equivalently, $E(H') \supsetneqq E(H)$).

A signed graph $\Gamma = (G, \sigma)$ consists of a simple graph $G = (V(G), E(G))$, referred to as its underlying graph, and a mapping $\sigma : E \to \{-1, 1\}$, its edge labelling. The adjacency matrix of Γ is $A(G, \sigma) = (a^{\sigma}_{ij})$ with $a^{\sigma}_{ij} = \sigma(v_i v_j) a_{ij}$, where (a_{ij}) is the adjacency matrix of the underlying graph G. An edge e is said to be positive or negative if $\sigma(e) = 1$ or $\sigma(e) = -1$, respectively. And a simple graph can always be viewed as a signed graph with all positive edges. Let C be a cycle of Γ, the sign of C is defined by $\sigma(C) = \Pi_{e \in C} \sigma(e)$. A cycle C is said to be positive or negative if $\sigma(C) = 1$ or $\sigma(C) = -1$, respectively. Similarly, the nullity of a signed graph Γ, denoted by $\eta(\Gamma)$, is the multiplicity of the zero eigenvalues of its adjacency matrix $A(\Gamma)$. The cyclomatic number and the number of pendant vertices of Γ, denoted by $c(\Gamma)$ and $p(\Gamma)$, are the cyclomatic number and the number of pendant vertices of its underlying graph G, respectively.

The study on the nullity of graphs is a classical topic in spectral graph theory. For the nullity of simple graphs, lots of research work have been done on bounding the nullities of graphs with given order in terms of various graph parameters (or identifying the extremal graphs) such as: the matching number and the cyclomatic number [1–6]; the number of pendant vertices and the cyclomatic number [7–10], etc. Recently, corresponding work has been extended to the setting of signed graphs [11–16]. Wong et al. [17] proved that $\eta(G) \leq c(G) + 1$ for every cycle-spliced bipartite graph G, and characterized all cycle-spliced bipartite graphs G with $\eta(G) = c(G) + 1$. In this paper, we extend their work to the setting of signed cycle-spliced graphs. Moreover, we prove that there is no signed cycle-spliced graphs Γ of any order with nullity $\eta(\Gamma) = c(\Gamma)$ and also explore some structural characterization for signed cycle-spliced bipartite graphs Γ with nullity $\eta(\Gamma) = c(\Gamma) - 1$. Our main results can be read as follows.

Theorem 1.1 *Let Γ be a signed cycle-spliced graph with $c(\Gamma)$ cycles. Then $\eta(\Gamma) \leq c(\Gamma) + 1$ and the equality holds if and only if all cycles in Γ have nullity 2.*

Remark 1.2 In Theorem 1.1, when Γ is a simple cycle-spliced graph G, then we have $\eta(G) \leq c(G) + 1$, which also extends the result of Wong et al. [17] on cycle-spliced bipartite graphs.

Theorem 1.3 *For any signed cycle-spliced graph Γ of order n with $c(\Gamma)$ cycles, $\eta(\Gamma) \neq c(\Gamma)$.*

Theorem 1.4 *Let Γ be a signed cycle-spliced bipartite graph with $c(\Gamma) \geq 2$ and all pendant cycles have nullity 0. Then $\eta(\Gamma) = c(\Gamma) - 1$ if and only if the distance between any two cut vertices of Γ is even.*

Theorem 1.5 *For any signed cycle-spliced bipartite graph Γ with $c(\Gamma)$ cycles, $\eta(\Gamma) = c(\Gamma) - 1$ if and only if Γ is a signed graph obtained from a signed cycle-spliced bipartite graph (H, σ) with $\eta(H, \sigma) = c(H, \sigma) - 1$ in which every pendant cycle (if any) has nullity 0 by attaching $c(\Gamma) - c(H, \sigma)$ cycles with nullity 2 on arbitrary vertex of (H, σ).*

The rest of this paper is organized as follows. In Section 2, we give some notations and preliminary lemmas which will be used in our proofs. In Section 3, the proofs for Theorems

1.1 and 1.3 are presented. In Section 4, a number of auxiliary results involving properties on signed cycle-spliced graphs (or bipartite graphs) Γ with nullity $\eta(\Gamma) = c(\Gamma) - 1$ are presented. In Section 5, we establish Theorem 1.4, which provides a characterization for a signed cycle-spliced bipartite graph Γ with $c(\Gamma) \geq 2$ and all pendant cycles have nullity 0 satisfying $\eta(\Gamma) = c(\Gamma) - 1$. And the proof of Theorem 1.5 is presented, which provides a structural characterization for a signed cycle-spliced bipartite graph Γ satisfying $\eta(\Gamma) = c(\Gamma) - 1$.

2. Preliminaries

For $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ be the degree and the set of neighbors of v, respectively. Clearly, $d_G(v) = |N_G(v)|$. If H is an induced subgraph of G, we use $N_H(v)$ to denote the set of neighbors of v in H. We use P_n and C_n to denote a path and a cycle of order n, respectively.

Most of the concepts defined for graphs are directly extended to signed graphs. We call a signed graph connected if its underlying graph is connected, by the cyclomatic number of a signed graph we mean the cyclomatic number of its underlying graph, and by the degree of a vertex in a signed graph we mean its degree in the underlying graph, etc. So in the setting of a signed graph, the notations $V(\Gamma)$ and $V(G)$ (for the vertex set), $c(\Gamma)$ and $c(G)$ (for the cyclomatic number), $d_\Gamma(x)$ and $d_G(x)$ (for the degree of vertex x), etc. are used interchangeably.

If H is a subgraph (or, an induced subgraph) of the underlying graph G of a signed graph $\Gamma = (G, \sigma)$, then we can turn H into a signed graph by taking its signed function to be the restriction of σ to $E(H)$, refer to it as an signed subgraph (or, induced signed subgraph) of Γ, and, by a slight abuse of notation, we denote it by (H, σ). We call (H, σ) a pendant subgraph of Γ if H is a pendant subgraph of underlying graph G. If S is a nonempty subset of $V(G)$, then we use $\Gamma - S$ to denote the induced signed subgraph of Γ having $G - S$ as the underlying graph, while its sign function is the restriction of σ to $E(G - S)$. When H is an induced subgraph of G with $V(H) \bigcap S = \emptyset$, we use $(H + S, \sigma)$ to denote the induced signed subgraph of Γ having $H + S$ as its underlying graph. We also follow the standard practices and abbreviate $\Gamma - \{x\}$ as $\Gamma - x$, etc.

The following is a frequently useful result in this topic.

Lemma 2.1 ([18]) *For any $v \in V(\Gamma)$, we have $\eta(\Gamma) - 1 \leq \eta(\Gamma - v) \leq \eta(\Gamma) + 1$.*

By a pendant vertex of a signed graph we mean a pendant vertex of the underlying graph, i.e., a vertex of degree one. If u is a pendant vertex of a graph G and v is its unique neighbor in G, then the operation of obtaining $G - \{u, v\}$ from G is called a pendant K_2 deletion. A useful result in this topic says that upon performing a pendant K_2 deletion, the nullity of a graph is unchanged. The definition of a pendant K_2 deletion can be extended to a signed graph and the corresponding result also holds.

Lemma 2.2 ([11]) *Let Γ be a signed graph. If u is a pendant vertex of Γ and v is its unique neighbor, then $\eta(\Gamma) = \eta(\Gamma - \{u, v\})$.*

Lemma 2.3 ([16, 19]) *Let (P_n,σ) be a signed path. Then $\eta(P_n,\sigma)=1$ if n is odd, and $\eta(P_n,\sigma)=0$ if n is even.*

Lemma 2.4 ([16, 19]) *Let (C_n,σ) be a signed cycle. Then $\eta(C_n,\sigma)=2$ if and only if (C_n,σ) is positive and $n\equiv 0 \pmod 4$ or (C_n,σ) is negative and $n\equiv 2 \pmod 4$, $\eta(C_n,\sigma)=0$ otherwise.*

Lemma 2.5 ([15]) *Let Γ be a connected signed graph, v be a cut-vertex of Γ, and Γ' be a component of $\Gamma-v$.*

(i) If $\eta(\Gamma')=\eta(\Gamma'+v)-1$, then $\eta(\Gamma)=\eta(\Gamma')+\eta(\Gamma-\Gamma')$;

(ii) If $\eta(\Gamma')=\eta(\Gamma'+v)+1$, then $\eta(\Gamma)=\eta(\Gamma-v)-1$.

Recall that if Γ_1 is a pendant subgraph of a signed graph Γ with root u, then u is a cut vertex of Γ. Thus Lemma 2.5 can be reformulated as follows.

Lemma 2.6 *Let Γ_1 be a pendant subgraph of a signed graph Γ with root u.*

(i) If $\eta(\Gamma_1-u)=\eta(\Gamma_1)+1$, then $\eta(\Gamma)=\eta(\Gamma_1)+\eta(\Gamma-\Gamma_1)$;

(ii) If $\eta(\Gamma_1-u)=\eta(\Gamma_1)-1$, then $\eta(\Gamma)=\eta(\Gamma_1)+\eta(\Gamma-\Gamma_1+u)-1$.

Corollary 2.7 *Let C be a pendant cycle of a signed graph Γ with root u.*

(i) If C has nullity 2, then $\eta(\Gamma)=\eta(\Gamma-C+u)+1$;

(ii) If C is even signed cycle which has nullity 0, then $\eta(\Gamma)=\eta(\Gamma-C)$.

3. Proofs of Theorems 1.1 and 1.3

In order to prove Theorem 1.1, the following lemma is needed.

Lemma 3.1 *Let Γ be a signed cycle-spliced graph with $c(\Gamma)\ge 2$ cycles. Let C be a pendant cycle of Γ with root u and $(H,\sigma)=\Gamma-C+u$. Then $\eta(\Gamma)\le\eta(H,\sigma)+1$.*

Proof Let x be a vertex in C adjacent with u. By Lemma 2.1, we have $\eta(\Gamma)\le\eta(\Gamma-x)+1$. If C is an even cycle, applying pendant K_2 deletions on $\Gamma-x$, we have $\eta(\Gamma-x)=\eta(H,\sigma)$. Hence, $\eta(\Gamma)\le\eta(H,\sigma)+1$. Now we assume that C is an odd cycle. It follows from Lemma 2.1 that $\eta(H,\sigma)-1\le\eta((H,\sigma)-u)\le\eta(H,\sigma)+1$. We now consider the following three cases.

Case 1. $\eta((H,\sigma)-u)=\eta(H,\sigma)+1$.

Clearly, (H,σ) is a pendant subgraph of Γ with root u. By Lemma 2.6 (i) and Lemma 2.3, we have

$$\eta(\Gamma)=\eta(H,\sigma)+\eta(\Gamma-(H,\sigma))=\eta(H,\sigma)+\eta(C-u)=\eta(H,\sigma)<\eta(H,\sigma)+1.$$

Case 2. $\eta((H,\sigma)-u)=\eta(H,\sigma)$.

Lemmas 2.1 and 2.3 imply that

$$\eta(\Gamma)\le\eta(\Gamma-u)+1=\eta(C-u)+\eta((H,\sigma)-u)+1=\eta(H,\sigma)+1.$$

Case 3. $\eta((H,\sigma)-u)=\eta(H,\sigma)-1$.

Note that (H, σ) is a pendant subgraph of Γ with root u. Then Lemma 2.6 (ii) and Lemma 2.4 imply that

$$\eta(\Gamma) = \eta(H, \sigma) + \eta(C) - 1 = \eta(H, \sigma) - 1 < \eta(H, \sigma) + 1,$$

as desired. The proof is completed. □

Proof of Theorem 1.1 We proceed by induction on $c(\Gamma)$ to prove $\eta(\Gamma) \leq c(\Gamma)+1$. If $c(\Gamma) = 1$, then Γ is a signed cycle. It follows from Lemma 2.4 that $\eta(\Gamma) \leq 2 = c(\Gamma)+1$, as required. Assume the assertion $\eta(H, \sigma) \leq c(H, \sigma) + 1$ holds for signed cycle-spliced graphs with m cycles and Γ has $m+1 \geq 2$ cycles. Let C be a pendant cycle of Γ with root u and $(H, \sigma) = \Gamma - C + u$. By Lemma 3.1, we have $\eta(\Gamma) \leq \eta(H, \sigma) + 1$. As (H, σ) has one cycle less than Γ, the induction hypothesis implies that $\eta(H, \sigma) \leq c(H, \sigma) + 1$. Hence, $\eta(\Gamma) \leq \eta(H, \sigma) + 1 \leq (c(H, \sigma) + 1) + 1 = c(\Gamma) + 1$, which completes the proof for $\eta(\Gamma) \leq c(\Gamma) + 1$.

Suppose every cycle in Γ has nullity 2. We proceed by induction on $c(\Gamma)$ to prove $\eta(\Gamma) = c(\Gamma)+1$. If $c(\Gamma) = 1$, then Γ is a signed cycle. It follows from Lemma 2.4 that $\eta(\Gamma) = 2 = c(\Gamma)+1$, as required. Assume that $\eta(H, \sigma) = c(H, \sigma) + 1$ holds for signed cycle-spliced graphs with m cycles and all cycles have nullity 2, and Γ has $m + 1 \geq 2$ such cycles. Let C be a pendant cycle of Γ with root u and $(H, \sigma) = \Gamma - C + u$. Recall that C has nullity 2, by Corollary 2.7 (i), we have $\eta(\Gamma) = \eta(H, \sigma) + 1$. As (H, σ) has m cycles and all cycles have nullity 2, the induction hypothesis implies that $\eta(H, \sigma) = c(H, \sigma) + 1 = m + 1$. Hence, $\eta(\Gamma) = \eta(H, \sigma) + 1 = c(\Gamma) + 1$, which completes the proof for sufficiency part.

Now, we proceed by induction on $c(\Gamma)$ to prove $\eta(\Gamma) < c(\Gamma) + 1$ if Γ has at least one cycle with nullity 0. If $c(\Gamma) = 1$, then Γ is a signed cycle. It follows from Lemma 2.4 that $\eta(\Gamma) = 0 < c(\Gamma) + 1$, as required. Now we assume that $c(\Gamma) \geq 2$. Then Γ has at least two pendant cycles. If a pendant cycle, say C, has nullity 2, then by a similar discussion as above, we have $\eta(\Gamma) = \eta(H, \sigma) + 1$, where $(H, \sigma) = \Gamma - C + u$ and u is the root of C. Noting that (H, σ) has at least one cycle with nullity 0 and it has one cycle less than Γ, thus the induction hypothesis implies that $\eta(H, \sigma) < c(H, \sigma) + 1$. Hence, $\eta(\Gamma) = \eta(H, \sigma) + 1 < c(\Gamma) + 1$, as required. Now suppose all pendant cycles of Γ have nullity 0. Let C and (H, σ) be as in the first paragraph. Then (H, σ) has at least one cycle with nullity 0. The induction hypothesis implies that $\eta(H, \sigma) < c(H, \sigma)+1$. Furthermore, it follows from Lemma 3.1, we have $\eta(\Gamma) \leq \eta(H, \sigma)+1$. Thus, $\eta(\Gamma) \leq \eta(H, \sigma) + 1 < (c(H, \sigma) + 1) + 1 = c(\Gamma) + 1$, which completes the proof for necessity part. □

For $u, v \in V(\Gamma)$, the distance between u and v, denoted by $d_\Gamma(u, v)$, is the shortest length of paths between u and v. The notation $d_\Gamma(v, S)$ stands for the distance between a vertex $v \in V(\Gamma)$ and a subset $S \subseteq V(\Gamma)$, i.e., the length of the shortest path from v to a vertex of S. In order to prove Theorem 1.3, the following property on a signed cycle-spliced graph with $\eta(\Gamma) = c(\Gamma) + 1$ is needed.

Lemma 3.2 *Let Γ be a signed cycle-spliced graph with $c(\Gamma)$ cycles. If $\eta(\Gamma) = c(\Gamma) + 1$, then $\eta(\Gamma - x) = \eta(\Gamma) - 1$ for any $x \in V(\Gamma)$.*

Proof We proceed by induction on $c(\Gamma)$ to prove $\eta(\Gamma - x) = \eta(\Gamma) - 1$ for any $x \in V(\Gamma)$. If $c(\Gamma) = 1$, then Γ is a signed cycle with nullity 2 since $\eta(\Gamma) = c(\Gamma)+1 = 2$. By Lemma 2.4, we know that Γ is an even cycle since Γ has nullity 2. Then by Lemma 2.3, we have $\eta(\Gamma-x) = 1 = \eta(\Gamma)-1$ for any $x \in V(\Gamma)$, as required. Assume the assertion holds for signed cycle-spliced graphs with $c(\Gamma) = m$ cycles and Γ has $m+1 \geq 2$ cycles. Theorem 1.1 implies that all cycles in Γ have nullity 2. Let C be a pendant cycle of Γ with root u and $(H, \sigma) = \Gamma - C + u$. Then Corollary 2.7 (i) implies that $\eta(\Gamma) = \eta(H, \sigma) + 1$ since C is a cycle with nullity 2. By calculations, we know that (H, σ) is a signed cycle-spliced graph with $\eta(H, \sigma) = c(H, \sigma) + 1$. Note that $c(H, \sigma) = m$. Then by the induction hypothesis, we have $\eta((H, \sigma) - v) = \eta(H, \sigma) - 1$ for any $v \in V(H, \sigma)$. Let x be an arbitrary vertex in Γ. We now consider the following two cases according to the position of x in Γ.

Case 1. x does not lie on C.

In this case, C is also a pendant cycle of $\Gamma - x$ with root u. Let $(H, \sigma) - x = (\Gamma - x) - C + u$. Then Corollary 2.7 (i) implies that $\eta(\Gamma - x) = \eta((H, \sigma) - x) + 1$. Hence, $\eta(\Gamma - x) = (\eta(H, \sigma) - 1) + 1 = \eta(\Gamma) - 1$, as desired.

Case 2. x lies on C.

Recall that C is an even cycle since C has nullity 2. If $d_\Gamma(x, u)$ is even (possibly zero), applying pendant K_2 deletions on $\Gamma-x$, we have $\eta(\Gamma-x) = \eta((H, \sigma)-u)+1 = (\eta(H, \sigma)-1)+1 = \eta(\Gamma)-1$; if $d_\Gamma(x, u)$ is odd, applying pendant K_2 deletions on $\Gamma-x$, we have $\eta(\Gamma-x) = \eta(H, \sigma) = \eta(\Gamma)-1$, which completes the proof. □

Proof of Theorem 1.3 We proceed by induction on $c(\Gamma)$ to prove $\eta(\Gamma) \neq c(\Gamma)$. If $c(\Gamma) = 1$, then Γ is a signed cycle. It follows from Lemma 2.4 that $\eta(\Gamma) \neq 1 = c(\Gamma)$, as required. Assume the assertion holds for signed cycle-spliced graphs with $c(\Gamma) = m$ cycles and Γ has $m + 1 \geq 2$ cycles. Let C be a pendant cycle of Γ with root u and $(H, \sigma) = \Gamma - C + u$. Note that (H, σ) is a signed cycle-spliced graph and $c(H, \sigma) = m$. Then by the induction hypothesis, we have $\eta(H, \sigma) \neq c(H, \sigma)$. Thus Theorem 1.1 implies that $\eta(H, \sigma) = c(H, \sigma)+1$ or $\eta(H, \sigma) \leq c(H, \sigma)-1$. We now consider the following two cases.

Case 1. $\eta(H, \sigma) = c(H, \sigma) + 1$.

By Lemma 3.2, $\eta((H, \sigma)-u) = \eta(H, \sigma)-1$ since $\eta(H, \sigma) = c(H, \sigma)+1$. It follows from Lemma 2.6 (ii), $\eta(\Gamma) = \eta(H, \sigma) + \eta(C) - 1$. If C has nullity 2, we have $\eta(\Gamma) = \eta(H, \sigma) + 1 = (c(H, \sigma) + 1) + 1 = c(\Gamma) + 1$. If C has nullity 0, we have $\eta(\Gamma) = \eta(H, \sigma) - 1 = (c(H, \sigma) + 1) - 1 = c(\Gamma) - 1$.

Case 2. $\eta(H, \sigma) \leq c(H, \sigma) - 1$.

By Lemma 3.1, we have $\eta(\Gamma) \leq \eta(H, \sigma) + 1$. Hence, $\eta(\Gamma) \leq \eta(H, \sigma) + 1 \leq (c(H, \sigma) - 1) + 1 = c(\Gamma) - 1$.

From the above arguments, we see that $\eta(\Gamma) \neq c(\Gamma)$. □

4. Properties on signed cycle-spliced graphs Γ with $\eta(\Gamma) = c(\Gamma) - 1$

In this section, we present some properties on signed cycle-spliced graphs Γ with $\eta(\Gamma) = c(\Gamma) - 1$.

Lemma 4.1 *Let Γ be a signed graph obtained from two signed cycle-spliced graphs Γ_1 and Γ_2 by identifying the unique common vertex u. If one of Γ_i $(i = 1, 2)$, say Γ_1, satisfies $\eta(\Gamma_1) \leq c(\Gamma_1)-2$, then $\eta(\Gamma) \leq c(\Gamma) - 2$.*

Proof Firstly, Theorem 1.1 implies that $\eta(\Gamma_2) \leq c(\Gamma_2) + 1$. If $\eta(\Gamma_2) = c(\Gamma_2) + 1$, by Lemma 3.2, $\eta(\Gamma_2 - u) = \eta(\Gamma_2) - 1$. Then by Lemma 2.6 (ii), we have $\eta(\Gamma) = \eta(\Gamma_1) + \eta(\Gamma_2) - 1 \leq (c(\Gamma_1) - 2) + (c(\Gamma_2) + 1) - 1 = c(\Gamma) - 2$, as required. If $\eta(\Gamma_2) \neq c(\Gamma_2) + 1$, by Theorem 1.3, we have $\eta(\Gamma_2) \leq c(\Gamma_2) - 1$. Lemma 2.1 implies that $\eta(\Gamma_2) - 1 \leq \eta(\Gamma_2 - u) \leq \eta(\Gamma_2) + 1$. Now we consider the following three cases.

Case 1. $\eta(\Gamma_2 - u) = \eta(\Gamma_2) + 1$.

Clearly, Γ_2 is a pendant subgraph of Γ with root u. By Lemma 2.6 (i), we have

$$\begin{aligned}\eta(\Gamma) &= \eta(\Gamma_2) + \eta(\Gamma - \Gamma_2) = \eta(\Gamma_2) + \eta(\Gamma_1 - u)\\ &\leq \eta(\Gamma_2) + (\eta(\Gamma_1) + 1) \leq (c(\Gamma_2) - 1) + (c(\Gamma_1) - 2) + 1 = c(\Gamma) - 2.\end{aligned}$$

Case 2. $\eta(\Gamma_2 - u) = \eta(\Gamma_2)$.

Clearly, Γ_1 is also a pendant subgraph of Γ with root u. If $\eta(\Gamma_1 - u) = \eta(\Gamma_1) + 1$, then by Lemma 2.6 (i), we have

$$\begin{aligned}\eta(\Gamma) &= \eta(\Gamma_1) + \eta(\Gamma - \Gamma_1) = \eta(\Gamma_1) + \eta(\Gamma_2 - u)\\ &= \eta(\Gamma_1) + \eta(\Gamma_2) \leq (c(\Gamma_1) - 2) + (c(\Gamma_2) - 1) < c(\Gamma) - 2;\end{aligned}$$

if $\eta(\Gamma_1 - u) \leq \eta(\Gamma_1)$, then Lemma 2.1 implies that

$$\begin{aligned}\eta(\Gamma) &\leq \eta(\Gamma - u) + 1 = \eta(\Gamma_1 - u) + \eta(\Gamma_2 - u) + 1\\ &\leq \eta(\Gamma_1) + \eta(\Gamma_2) + 1 \leq (c(\Gamma_1) - 2) + (c(\Gamma_2) - 1) + 1 = c(\Gamma) - 2.\end{aligned}$$

Case 3. $\eta(\Gamma_2 - u) = \eta(\Gamma_2) - 1$.

Note that Γ_2 is a pendant subgraph of Γ with root u. Then Lemma 2.6 (ii) implies that

$$\eta(\Gamma) = \eta(\Gamma_1) + \eta(\Gamma_2) - 1 \leq (c(\Gamma_1) - 2) + (c(\Gamma_2) - 1) - 1 < c(\Gamma) - 2,$$

as desired. This completes the proof. □

Lemma 4.2 *Let Γ be a signed graph obtained from two signed cycle-spliced graphs Γ_1 and Γ_2 by identifying the unique common vertex u and $\eta(\Gamma) = c(\Gamma) - 1$.*

(i) If $\eta(\Gamma_1) = c(\Gamma_1) + 1$, then $\eta(\Gamma_2) = c(\Gamma_2) - 1$;

(ii) If $\eta(\Gamma_i) \leq c(\Gamma_i) - 1$ for $i = 1, 2$, then $\eta(\Gamma_i) = c(\Gamma_i) - 1$ for $i = 1, 2$.

Proof (i) Firstly, Lemma 3.2 implies that $\eta(\Gamma_1 - u) = \eta(\Gamma_1) - 1$ since $\eta(\Gamma_1) = c(\Gamma_1) + 1$. Moreover, by Lemma 2.6 (ii), we have

$$\eta(\Gamma) = \eta(\Gamma_1) + \eta(\Gamma_2) - 1 \quad \text{and} \quad c(\Gamma) - 1 = c(\Gamma_1) + 1 + \eta(\Gamma_2) - 1.$$

It follows that $\eta(\Gamma_2) = c(\Gamma_2) - 1$ since $\eta(\Gamma) = c(\Gamma) - 1$.

(ii) Since $\eta(\Gamma) = c(\Gamma) - 1$ and $\eta(\Gamma_i) \leq c(\Gamma_i) - 1$ for $i = 1, 2$, by Lemma 4.1, we have $\eta(\Gamma_i) = c(\Gamma_i) - 1$ for $i = 1, 2$. □

Lemma 4.3 *Let Γ be a signed graph obtained from two signed cycle-spliced graphs Γ_1 and Γ_2 by identifying the unique common vertex u. Then $\eta(\Gamma) = c(\Gamma) - 1$ if and only if one of the following conditions is satisfied:*

(i) There is one of Γ_i $(i = 1, 2)$, say Γ_1, such that $\eta(\Gamma_1) = c(\Gamma_1) + 1$ and $\eta(\Gamma_2) = c(\Gamma_2) - 1$;

(ii) $\eta(\Gamma_i) = c(\Gamma_i) - 1$ and $\eta(\Gamma_i) = \eta(\Gamma_i - u) + 1$ for $i = 1, 2$;

(iii) $\eta(\Gamma_i) = c(\Gamma_i) - 1$ and $\eta(\Gamma_i) = \eta(\Gamma_i - u)$ for $i = 1, 2$, and $\eta(\Gamma) = \eta(\Gamma - u) + 1$.

Proof "Only if" part: If there is a signed graph, say Γ_1, such that $\eta(\Gamma_1) = c(\Gamma_1) + 1$, then Lemma 4.2 (i) implies that $\eta(\Gamma_2) = c(\Gamma_2) - 1$, (i) holds.

If $\eta(\Gamma_i) \neq c(\Gamma_i) + 1$ for $i = 1, 2$, then Theorems 1.1 and 1.3 imply that $\eta(\Gamma_i) \leq c(\Gamma_i) - 1$. Moreover, by Lemma 4.2 (ii), we have $\eta(\Gamma_i) = c(\Gamma_i) - 1$ for $i = 1, 2$. Lemma 2.1 implies that $\eta(\Gamma_1) - 1 \leq \eta(\Gamma_1 - u) \leq \eta(\Gamma_1) + 1$. Now we consider the following three cases.

Case 1. $\eta(\Gamma_1 - u) = \eta(\Gamma_1) + 1$.

Clearly, Γ_1 is a pendant subgraph of Γ with root u. By Lemma 2.6 (i), we have $\eta(\Gamma) = \eta(\Gamma_1) + \eta(\Gamma - \Gamma_1) = \eta(\Gamma_1) + \eta(\Gamma_2 - u)$. If $\eta(\Gamma_2 - u) \leq \eta(\Gamma_2)$, then $\eta(\Gamma) = \eta(\Gamma_1) + \eta(\Gamma_2 - u) \leq \eta(\Gamma_1) + \eta(\Gamma_2) = (c(\Gamma_1) - 1) + (c(\Gamma_2) - 1) = c(\Gamma) - 2$, which is a contradiction. Then $\eta(\Gamma_2 - u) = \eta(\Gamma_2) + 1$, (ii) holds.

Case 2. $\eta(\Gamma_1 - u) = \eta(\Gamma_1)$.

Clearly, Γ_2 is also a pendant subgraph of Γ with root u. If $\eta(\Gamma_2 - u) = \eta(\Gamma_2) + 1$, then by Lemma 2.6 (i), we have

$$\begin{aligned}\eta(\Gamma) &= \eta(\Gamma_2) + \eta(\Gamma - \Gamma_2) = \eta(\Gamma_2) + \eta(\Gamma_1 - u)\\ &= \eta(\Gamma_2) + \eta(\Gamma_1) = (c(\Gamma_2) - 1) + (c(\Gamma_1) - 1) = c(\Gamma) - 2,\end{aligned}$$

which is a contradiction. If $\eta(\Gamma_1 - u) = \eta(\Gamma_1) - 1$, then Lemma 2.6 (ii) implies that

$$\eta(\Gamma) = \eta(\Gamma_1) + \eta(\Gamma_2) - 1 = (c(\Gamma_1) - 1) + (c(\Gamma_2) - 1) - 1 = c(\Gamma) - 3,$$

which is a contradiction. Then we have $\eta(\Gamma_2 - u) = \eta(\Gamma_2)$. Moreover, note that

$$\begin{aligned}\eta(\Gamma - u) &= \eta(\Gamma_1 - u) + \eta(\Gamma_2 - u) = \eta(\Gamma_1) + \eta(\Gamma_2)\\ &= (c(\Gamma_1) - 1) + (c(\Gamma_2) - 1) = c(\Gamma) - 2 = \eta(\Gamma) - 1.\end{aligned}$$

Then (iii) holds.

Case 3. $\eta(\Gamma_1 - u) = \eta(\Gamma_1) - 1$.

Note that Γ_1 is a pendant subgraph of Γ with root u. Then Lemma 2.6 (ii) implies that

$$\eta(\Gamma) = \eta(\Gamma_1) + \eta(\Gamma_2) - 1 = (c(\Gamma_1) - 1) + (c(\Gamma_2) - 1) - 1 = c(\Gamma) - 3,$$

which is a contradiction.

"If" part: (i) By Lemma 3.2, $\eta(\Gamma_1 - u) = \eta(\Gamma_1) - 1$ since $\eta(\Gamma_1) = c(\Gamma_1) + 1$. It follows from Lemma 2.6 (ii), we have $\eta(\Gamma) = \eta(\Gamma_1) + \eta(\Gamma_2) - 1 = (c(\Gamma_1) + 1) + (c(\Gamma_2) - 1) = c(\Gamma) - 1$.

(ii) It follows from Lemma 2.6 (i), we have $\eta(\Gamma) = \eta(\Gamma_1) + \eta(\Gamma_2 - u) = \eta(\Gamma_1) + \eta(\Gamma_2) + 1 = (c(\Gamma_1) - 1) + (c(\Gamma_2) - 1) + 1 = c(\Gamma) - 1$.

(iii) $\eta(\Gamma) = \eta(\Gamma - u) + 1 = \eta(\Gamma_1 - u) + \eta(\Gamma_2 - u) + 1 = \eta(\Gamma_1) + \eta(\Gamma_2) + 1 = (c(\Gamma_1) - 1) + (c(\Gamma_2) - 1) + 1 = c(\Gamma) - 1$. □

It seems somewhat difficult to give a full characterization on signed cycle-spliced graphs Γ with $\eta(\Gamma) = c(\Gamma) - 1$. In what follows, we further explore some properties on Γ with $\eta(\Gamma) = c(\Gamma) - 1$ when Γ is a signed cycle-spliced bipartite graph.

Lemma 4.4 *Let Γ be a signed cycle-spliced bipartite graph with $c(\Gamma)$ cycles. If $\eta(\Gamma) = c(\Gamma) - 1$, then $\eta(\Gamma - x) \neq \eta(\Gamma)$ for any $x \in V(\Gamma)$.*

Proof We proceed by induction on $c(\Gamma)$ to prove $\eta(\Gamma - x) \neq \eta(\Gamma)$ for any $x \in V(\Gamma)$. If $c(\Gamma) = 1$, then Γ is a signed cycle which has nullity 0 since $\eta(\Gamma) = c(\Gamma) - 1 = 0$. Clearly, Γ is an even cycle since Γ is bipartite. It follows from Lemma 2.4 that $\eta(\Gamma - x) = \eta(\Gamma) + 1 \neq \eta(\Gamma)$ for any $x \in V(\Gamma)$, as required. Assume that the assertion holds for signed cycle-spliced bipartite graphs (H, σ) with $\eta(H, \sigma) = c(H, \sigma) - 1$ which have less cycles than Γ. Now we assume that $c(\Gamma) \geq 2$. Hence, Γ has at least one pendant cycle.

Case 1. There is a pendant cycle C which has nullity 2.

Let u be the root of pendant cycle C and $(H, \sigma) = \Gamma - C + u$. Then Corollary 2.7 (i) implies that $\eta(\Gamma) = \eta(H, \sigma) + 1$. Moreover, Lemma 4.2 (i) implies that (H, σ) is a signed cycle-spliced bipartite graph with $\eta(H, \sigma) = c(H, \sigma) - 1$. As (H, σ) has one cycle less than Γ, the induction hypothesis implies that $\eta((H, \sigma) - v) \neq \eta(H, \sigma)$ for any $v \in V(H, \sigma)$. Let x be an arbitrary vertex in Γ. We consider the following two subcases according to the position of x in Γ.

Subcase 1.1. x does not lie on C.

Similarly to the proof of Case 1 in Lemma 3.2, we have

$$\eta(\Gamma - x) = \eta(H - x) + 1 \neq \eta(H) + 1 = \eta(\Gamma).$$

Subcase 1.2. x lies on C.

If $d_\Gamma(x, u)$ is even (possibly zero), applying pendant K_2 deletions on $\Gamma - x$, we have $\eta(\Gamma - x) = \eta(H - u) + 1 \neq \eta(H) + 1 = \eta(\Gamma)$; if $d_\Gamma(x, u)$ is odd, applying pendant K_2 deletions on $\Gamma - x$, we have $\eta(\Gamma - x) = \eta(H) = \eta(\Gamma) - 1 \neq \eta(\Gamma)$.

Case 2. All pendant cycles have nullity 0.

Let C be a pendant cycle of Γ with root u and $(H, \sigma) = \Gamma - C + u$. Note that C is an even cycle which has nullity 0, by Corollary 2.7 (ii), we have $\eta(\Gamma) = \eta((H, \sigma) - u)$. On the other hand, since all pendant cycles of Γ have nullity 0, (H, σ) contains at least one cycle with nullity 0. By Theorems 1.1 and 1.3, we have $\eta(H, \sigma) \leq c(H, \sigma) - 1$. Moreover, by Lemma 4.2 (ii), we have $\eta(H, \sigma) = c(H, \sigma) - 1$ since $\eta(C) = c(C) - 1$ and $\eta(\Gamma) = c(\Gamma) - 1$. As (H, σ) has one cycle less than Γ, the induction hypothesis implies that $\eta((H, \sigma) - v) \neq \eta(H, \sigma)$ for any $v \in V(H, \sigma)$. Let x be an arbitrary vertex in Γ. We consider the following two subcases according to the position of x in Γ.

Subcase 2.1. x lies on one pendant cycle, say C.

Let u be the root of pendant cycle C and $(H, \sigma) = \Gamma - C + u$. If $d_\Gamma(x, u)$ is even (possibly zero), applying pendant K_2 deletions on $\Gamma - x$, we have $\eta(\Gamma - x) = \eta((H, \sigma) - u) + 1 = \eta(\Gamma) + 1 \neq \eta(\Gamma)$;

if $d_\Gamma(x,u)$ is odd, applying pendant K_2 deletions on $\Gamma - x$, we have $\eta(\Gamma - x) = \eta(H,\sigma) \neq \eta((H,\sigma) - u) = \eta(\Gamma)$.

Subcase 2.2. x does not lie on any pendant cycle.

Suppose x lies on a cycle C'. We can assume that all cut vertices in C' are $u_1,\dots,u_k$, where $k \geq 2$ since C' is not a pendant cycle. Let Γ_i be the maximal pendant subgraph of Γ with root u_i $(i = 1,\dots,k)$. Then Γ can be seen as a signed graph obtained by a signed cycle C' with Γ_i attached at u_i $(i = 1,\dots,k)$, respectively (See Figure 1). Let $(H_i,\sigma) = \Gamma - \Gamma_i + u_i$.

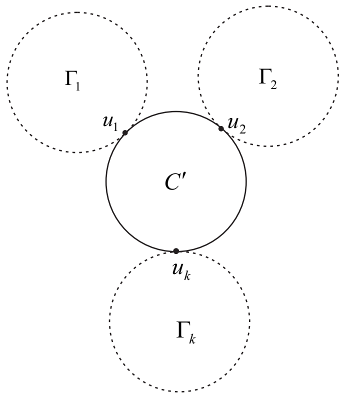

Figure 1 Signed graph Γ with a cycle C' which is not a pendant cycle

Since all pendant cycles of Γ have nullity 0, Γ_i and (H_i,σ) both contain at least one cycle which has nullity 0. By Theorems 1.1 and 1.3, we have $\eta(\Gamma_i) \leq c(\Gamma_i) - 1$ and $\eta(H_i,\sigma) \leq c(H_i,\sigma) - 1$. Moreover, by Lemma 4.2 (ii), we have $\eta(\Gamma_i) = c(\Gamma_i) - 1$ and $\eta(H_i,\sigma) = c(H_i,\sigma) - 1$. As Γ_i has less cycles than Γ, the induction hypothesis implies that $\eta(\Gamma_i - u_i) \neq \eta(\Gamma_i)$. We contend that $\eta(\Gamma_i - u_i) = \eta(\Gamma_i) + 1$. Otherwise, there is a signed graph Γ_i such that $\eta(\Gamma_i - u_i) = \eta(\Gamma_i) - 1$. By Lemma 2.6 (ii), we have $\eta(\Gamma) = \eta(\Gamma_i) + \eta(H_i,\sigma) - 1 = (c(\Gamma_i) - 1) + (c(H_i,\sigma) - 1) - 1 = c(\Gamma) - 3$, which is a contradiction. Then Lemma 2.6 (i) implies that

$$\begin{aligned}\eta(\Gamma) &= \eta(\Gamma_1) + \cdots + \eta(\Gamma_k) + \eta(\Gamma - \Gamma_1 - \cdots - \Gamma_k)\\ &= \sum_{i=1}^{k} \eta(\Gamma_i) + \eta(C' - u_1 - \cdots - u_k) = \sum_{i=1}^{k} (c(\Gamma_i) - 1) + \eta(C' - u_1 - \cdots - u_k)\\ &= c(\Gamma) - 1 - k + \eta(C' - u_1 - \cdots - u_k).\end{aligned}$$

It follows that $\eta(C' - u_1 - \cdots - u_k) = k$ since $\eta(\Gamma) = c(\Gamma) - 1$. It means that $d_\Gamma(u_i,u_j)$ is even for any $i,j = 1,\dots,k$ and $\eta(\Gamma) = \sum_{i=1}^{k} \eta(\Gamma_i) + k$.

If $x = u_i$, then we have $\eta(\Gamma - x) = \eta(\Gamma_i - u_i) + \eta((H_i,\sigma) - u_i)$. In this case, Γ_j $(j \neq i)$ is also a pendant subgraph of $(H_i,\sigma) - u_i$ with root u_j and $\eta(\Gamma_j - u_j) = \eta(\Gamma_j) + 1$. Applying Lemma 2.6 (i), we have

$$\begin{aligned}\eta(\Gamma - x) &= \eta(\Gamma_i - u_i) + \eta((H_i,\sigma) - u_i) = (\eta(\Gamma_i) + 1) + \sum_{j \neq i} \eta(\Gamma_j) + \eta(C' - u_1 - \cdots - u_k)\\ &= \sum_{i=1}^{k} \eta(\Gamma_i) + 1 + k = \eta(\Gamma) + 1 \neq \eta(\Gamma).\end{aligned}$$

Similarly, if $x \neq u_i$, then we have

$$\eta(\Gamma - x) = \sum_{i=1}^{k} \eta(\Gamma_i) + \eta((C' - x) - u_1 - \cdots - u_k) \neq \sum_{i=1}^{k} \eta(G_i) + k = \eta(G).$$

The inequality holds because $\eta(C' - x - u_1 - \cdots - u_k) \neq k$ since $d_\Gamma(u_i, u_j)$ is even. □

Lemma 4.5 *Let Γ be a signed cycle-spliced bipartite graph with $c(\Gamma) \geq 2$ and all pendant cycles have nullity 0. If $\eta(\Gamma) = c(\Gamma) - 1$, then*

(i) $\eta(\Gamma - u) = \eta(\Gamma) + 1$ for any cut vertex u of Γ;

(ii) $d_\Gamma(u, v)$ is even for any two cut vertices u and v in Γ;

(iii) $\eta(\Gamma - v) = \eta(\Gamma) + 1$ for $v \in V(\Gamma)$ such that the distance between v and any cut vertex of Γ is even;

(iv) $\eta(\Gamma - w) = \eta(\Gamma) - 1$ for $w \in V(\Gamma)$ such that the distance between w and any cut vertex of Γ is odd.

Proof (i) Since $c(\Gamma) \geq 2$, Γ has at least one cut vertex. Let u be an arbitrary cut vertex of Γ. Then Γ can be seen as a signed graph obtained from two signed cycle-spliced bipartite graphs Γ_1 and Γ_2 by identifying the unique common vertex u. Since all pendant cycles of Γ have nullity 0, Γ_i $(i = 1, 2)$ contains at least one cycle with nullity 0. By Theorems 1.1 and 1.3, we have $\eta(\Gamma_i) \leq c(\Gamma_i) - 1$. Moreover, by Lemma 4.3 (ii), we have $\eta(\Gamma_i) = c(\Gamma_i) - 1$ and $\eta(\Gamma_i - u) = \eta(\Gamma_i) + 1$ for $i = 1, 2$ since $\eta(\Gamma) = c(\Gamma) - 1$. It follows from Lemma 2.6 (i), we have $\eta(\Gamma) = \eta(\Gamma_1) + \eta(\Gamma - \Gamma_1) = \eta(\Gamma_1) + \eta(\Gamma_2 - u) = \eta(\Gamma_1) + \eta(\Gamma_2) + 1$. Thus we have $\eta(\Gamma - u) = \eta(\Gamma_1 - u) + \eta(\Gamma_2 - u) = \eta(\Gamma_1) + 1 + \eta(\Gamma_2) + 1 = \eta(\Gamma) + 1$.

(ii) There is nothing to prove when Γ has only one cut vertex. Suppose Γ has at least two cut vertices. Let u and v be two arbitrary cut vertices of Γ. In order to prove that $d_\Gamma(u, v)$ is even, we only consider u and v lie on same cycle, say C'. It means that C' is not a pendant cycle. We can assume that all cut vertices in C' are $u_1, \ldots, u_k$ $(k \geq 2)$. Clearly, $u, v \in \{u_1, \ldots, u_k\}$. Let Γ_i be the maximal pendant subgraph of Γ with root u_i $(i = 1, \ldots, k)$. Then Γ can be seen as a signed graph obtained by a signed cycle C' with Γ_i attached at u_i $(i = 1, \ldots, k)$, respectively. By the argument given in the proof of Case 2 of Lemma 4.4, we have $d_\Gamma(u_i, u_j)$ $(i, j = 1, \ldots, k)$ is even. It means that $d_\Gamma(u, v)$ is even.

(iii) Let v be a vertex such that the distance between v and any cut vertex of Γ is even. Without loss of generality, we assume that v and a cut vertex u of Γ lie on the same cycle C. We consider the following two cases.

Case 1. C is a pendant cycle of Γ.

Clearly, u is the unique cut vertex in C. Since C is an even cycle with nullity 0, by Corollary 2.7 (ii), we have $\eta(\Gamma) = \eta(\Gamma - C)$. Applying pendant K_2 deletions on $\Gamma - v$, we have $\eta(\Gamma - v) = 1 + \eta(\Gamma - C) = \eta(\Gamma) + 1$.

Case 2. C is not a pendant cycle of Γ.

In this case, there are at least two cut vertices in C. We can assume that all cut vertices in C are $u_1, \ldots, u_k$ $(k \geq 2)$. Let Γ_i be the maximal pendant subgraph of Γ with root u_i $(i =$

$1,\dots,k$). Then Γ can be seen as a signed graph obtained by a signed cycle C with Γ_i attached at u_i ($i=1,\dots,k$), respectively. By the argument given in the proof of Case 2 of Lemma 4.4, we have $\eta(\Gamma)=\sum_{i=1}^{k}\eta(\Gamma_i)+k$. Since the distance between any two cut vertices is even, applying pendant K_2 deletions on $\Gamma-v$, we have $\eta(\Gamma-v)=1+\sum_{i=1}^{k}\eta(\Gamma_i)+k=\eta(\Gamma)+1$.

(iv) Let w be a vertex such that the distance between w and any cut vertex of Γ is odd. Then Lemma 4.4 implies that $\eta(\Gamma-w)\neq\eta(\Gamma)$. To establish (iv), assume to the contrary that $\eta(\Gamma-w)=\eta(\Gamma)+1$. Let Γ' be a signed graph obtained from Γ with an even signed cycle C with nullity 0 attached at w. By Lemma 4.3 (ii), we have $\eta(\Gamma')=c(\Gamma')-1$. Thus Γ' is a signed cycle-spliced bipartite graph with $\eta(\Gamma')=c(\Gamma')-1$ and all pendant cycles have nullity 0. Note that w and u are cut vertices of G'. Then Lemma 4.5 (ii) implies that $d_{\Gamma'}(w,u)$ is even. Also $d_{\Gamma}(w,u)$ is even, which contradicts the assumption. Then $\eta(\Gamma-w)=\eta(\Gamma)-1$, as desired. □

Note 4.6 The condition that all pendant cycles of Γ have nullity 0 in Lemma 4.5 is necessary. See Figure 2, where the signed graph Γ is obtained by attaching a positive cycle C_4 (has nullity 2) and a positive cycle C_6 (has nullity 0) at vertices u_1 and u_2 of a positive cycle C_4, respectively. It is easy to see that $\eta(\Gamma)=c(\Gamma)-1$. But for u_2, we have $\eta(\Gamma-u_2)=\eta(\Gamma)-1$ and the distance between the two cut vertices (u_1 and u_2) is odd.

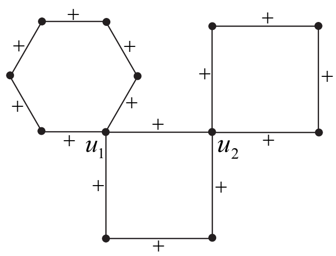

Figure 2 An example of a signed cycle-spliced bipartite graph with $\eta(\Gamma)=c(\Gamma)-1$

Lemma 4.7 *Let Γ be a signed cycle-spliced bipartite graph in which every non-pendant cycle has exactly two cut vertices (See Figure 3). If exactly one pendant cycle has nullity 0, the other cycles all have nullity 2 and the distance between any two cut vertices of Γ is even, then*

(i) $\eta(\Gamma)=c(\Gamma)-1$;

(ii) $\eta(\Gamma-u)=\eta(\Gamma)+1$ for any cut vertex u of Γ;

(iii) $\eta(\Gamma-v)=\eta(\Gamma)+1$ for $v\in V(\Gamma)$ such that the distance between v and any cut vertex of G is even;

(iv) $\eta(\Gamma-w)=\eta(\Gamma)-1$ for $w\in V(\Gamma)$ such that the distance between w and any cut vertex of Γ is odd.

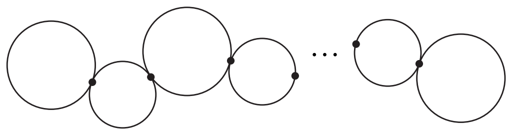

Figure 3 A signed cycle-spliced bipartite graph Γ in which every non-pendant cycle has exactly two cut vertices

Proof (i) Let C be the pendant cycle of Γ with nullity 0. Clearly, $\eta(C) = 0 = c(C) - 1$. Let u be the root of C and $(H, \sigma) = \Gamma - C + u$. Then Γ can be seen as a signed graph obtained from a signed cycle C and a signed cycle-spliced bipartite graph (H, σ) by identifying a unique common vertex u. Theorem 1.1 implies that $\eta(H, \sigma) = c(H, \sigma) + 1$ since all cycles in (H, σ) have nullity 2. Then by Lemma 4.3 (i), we immediately have $\eta(\Gamma) = c(\Gamma) - 1$.

(ii) We proceed by induction on $c(\Gamma)$ to prove $\eta(\Gamma - u) = \eta(\Gamma) + 1$ for any cut vertex u of Γ. If $c(\Gamma) = 2$, then Γ is a signed graph obtained from two signed cycles C_1 and C_2 by identifying a unique common vertex u. Without loss of generality, we assume that C_1 has nullity 0, C_2 has nullity 2. Clearly, we have $\eta(\Gamma - u) = \eta(\Gamma) + 1$. Assume the assertion holds for signed cycle-spliced bipartite graphs satisfying the above assumptions with less cycles than Γ. Now we assume that $c(\Gamma) \geq 3$. Let u be an arbitrary cut vertex of Γ. Then Γ can be seen as a signed graph obtained from two signed cycle-spliced bipartite graphs Γ_1 and Γ_2 by identifying the unique common vertex u. Recall that Γ has exactly one pendant cycle with nullity 0, the other cycles all have nullity 2. Without loss of generality, we can assume that Γ_1 contains the pendant cycle of nullity 0. Note that Γ_1 has less cycles than Γ and the distance between any two cut vertices of Γ_1 is even. Then by the induction hypothesis, we have $\eta(\Gamma_1 - x) = \eta(\Gamma_1) + 1$ for any cut vertex x in Γ_1. Let x' be the cut vertex of Γ_1 which lies on same cycle with u. Then $d_{\Gamma_1}(x', u)$ is even since x' and u are cut vertices of G. Applying pendant K_2 deletions on $\Gamma_1 - u$, we have $\eta(\Gamma_1 - u) = \eta(\Gamma_1 - x') = \eta(\Gamma_1) + 1$. Moreover, Theorem 1.1 implies that $\eta(\Gamma_2) = c(\Gamma_2) + 1$ since every cycle in Γ_2 has nullity 2. By Lemma 3.2, we have $\eta(\Gamma_2 - u) = \eta(\Gamma_2) - 1$. Then Lemma 2.6 (ii) implies that $\eta(\Gamma) = \eta(\Gamma_1) + \eta(\Gamma_2) - 1$. Thus we have $\eta(\Gamma - u) = \eta(\Gamma_1 - u) + \eta(\Gamma_2 - u) = \eta(\Gamma_1) + 1 + \eta(\Gamma_2) - 1 = \eta(\Gamma) + 1$.

(iii) Let v be a vertex such that the distance between v and any cut vertex of Γ is even. Without loss of generality, we assume that v and the cut vertex, say u, lie on same cycle C.

Case 1. C is a pendant cycle of Γ.

Since C is even cycle, applying pendant K_2 deletions on $\Gamma - v$, we have $\eta(\Gamma - v) = \eta(\Gamma - u) = \eta(\Gamma) + 1$.

Case 2. C is not a pendant cycle of Γ.

In this case, there are exactly two cut vertices in C, say u_1 and u_2. Without loss of generality, we assume that $u_1 = u$. Let Γ_i be the maximal pendant subgraph of Γ with root u_i $(i = 1, 2)$. Then Γ can be seen as a signed graph obtained by a signed cycle C with Γ_i attached at u_i $(i = 1, 2)$, respectively. Recall that Γ has exactly one pendant cycle with nullity 0, the other cycles all have nullity 2. Without loss of generality, we can assume that Γ_1 contains the pendant cycle which has nullity 0. Note that Γ_1 is a signed cycle-spliced bipartite graph and the distance between any two cut vertices of Γ_1 is even. Then Lemma 4.7 (ii) implies that $\eta(\Gamma_1 - x) = \eta(\Gamma_1) + 1$ for any cut vertex x in Γ_1. Let x' be the cut vertex of Γ_1 which lies on same cycle with u_1. Then $d_{\Gamma_1}(x', u_1)$ is even since x' and u_1 are cut vertices of Γ. Applying pendant K_2 deletions on $\Gamma_1 - u_1$, we have $\eta(\Gamma_1 - u_1) = \eta(\Gamma_1 - x') = \eta(\Gamma_1) + 1$. Moreover, Theorem 1.1 implies that $\eta(\Gamma_2) = c(\Gamma_2) + 1$ since all cycles in Γ_2 have nullity 2. By Lemma 3.2, we have $\eta(\Gamma_2 - u_2) = \eta(\Gamma_2) - 1$. Then Lemma 2.6 (ii) implies that $\eta(\Gamma) = \eta(\Gamma_1 + C) + \eta(\Gamma_2) - 1 = \eta(\Gamma_1) + 1 + \eta(\Gamma_2) - 1$. Since

$d_\Gamma(v,u)$ and $d_\Gamma(u,u_2)$ are even, applying pendant K_2 deletions on $\Gamma - v$, we have $\eta(\Gamma - v) = 1 + \eta(\Gamma_1 - u_1) + \eta(\Gamma_2 - u_2) = 1 + \eta(\Gamma_1) + 1 + \eta(\Gamma_2) - 1 = \eta(\Gamma) + 1$.

(iv) Let w be a vertex such that the distance between w and any cut vertex is odd. Then Lemma 4.4 implies that $\eta(\Gamma - w) \neq \eta(\Gamma)$ since $\eta(\Gamma) = c(\Gamma) - 1$. To establish (iv), assume to the contrary that $\eta(\Gamma - w) = \eta(\Gamma) + 1$. Let Γ' be a signed graph obtained from Γ with an even signed cycle C with nullity 0 attached at w. By Lemma 4.3 (ii), we have $\eta(\Gamma') = c(\Gamma') - 1$. Thus Γ' is a signed cycle-spliced bipartite graph with $\eta(\Gamma') = c(\Gamma') - 1$ and all pendant cycles have nullity 0. Note that w and u are cut vertices of Γ'. Then Lemma 4.5 (ii) implies that $d_{\Gamma'}(w,u)$ is even. Also $d_\Gamma(w,u)$ is even, which contradicts the assumption. Then $\eta(\Gamma - w) = \eta(\Gamma) - 1$, as desired. □

To identify signed cycle-spliced graphs Γ with $\eta(\Gamma) = c(\Gamma) - 1$, the case when Γ has both even and odd cycles seems more complicated. But for some special cases when Γ is nonbipartite and all cycles in Γ are of nullity 0, we have the following result.

Lemma 4.8 *Let Γ be a nonbipartite signed bicyclic graph obtained from two signed cycles C_1 and C_2 both with nullity 0 by identifying a unique common vertex u. If $\eta(\Gamma) = c(\Gamma) - 1$, then C_1 and C_2 both are odd cycles and $\eta(\Gamma - x) = \eta(\Gamma) - 1$ for any $x \in V(\Gamma)$.*

Proof At least one cycle of Γ is odd since Γ is nonbipartite. Without loss of generality, we can assume that C_1 is odd cycle. If C_2 is an even cycle. Recall that C_2 is even signed cycle with nullity 0, by Corollary 2.7 (ii), $\eta(\Gamma) = \eta(C_1 - u) = 0$, which contradicts $\eta(\Gamma) = c(\Gamma) - 1 = 1$. Then C_2 is also an odd cycle. Let x be an arbitrary vertex in Γ. We consider the position of x in Γ. Without loss of generality, we can assume that x lies on C_1. Whether $d_\Gamma(x,u)$ is even (possibly zero) or odd, applying pendant K_2 deletions on $\Gamma - x$, we have $\eta(\Gamma - x) = \eta((C_2,\sigma) - u) = 0 = c(\Gamma) - 2 = \eta(\Gamma) - 1$. □

Lemma 4.9 *Let Γ be a nonbipartite signed cycle-spliced graph with $c(\Gamma) \geq 3$ and all cycles of Γ have nullity 0. Then $\eta(\Gamma) \leq c(\Gamma) - 2$.*

Proof Clearly, Γ has at least one pendant cycle since $c(\Gamma) \geq 3$. Let C be a pendant cycle of Γ with root u and $(H,\sigma) = \Gamma - C + u$. On the other hand, since all cycles of Γ have nullity 0, by Theorems 1.1 and 1.3, we have $\eta(H,\sigma) \leq c(H,\sigma) - 1$.

Now we proceed by induction on $c(\Gamma)$ to prove $\eta(\Gamma) \leq c(\Gamma) - 2$. If $c(\Gamma) = 3$, then $c(H,\sigma) = 2$. When $\eta(H,\sigma) \leq c(H,\sigma) - 2$, by Lemma 3.1, $\eta(\Gamma) \leq \eta(H,\sigma) + 1 \leq (c(H,\sigma) - 2) + 1 = c(\Gamma) - 2$. When $\eta(H,\sigma) = c(H,\sigma) - 1$, now we consider the following two cases.

Case 1. C is an even cycle.

(H,σ) must be nonbipartite since Γ is nonbipartite. By Lemma 4.8, $\eta((H,\sigma) - u) = \eta(H,\sigma) - 1$. It follows from Lemma 2.6 (ii), $\eta(\Gamma) = \eta(H,\sigma) + \eta(C) - 1 = (c(H,\sigma) - 1) - 1 = c(\Gamma) - 3$.

Case 2. C is an odd cycle.

If (H,σ) is nonbipartite, then by similar discussion as above, we have $\eta(\Gamma) = c(\Gamma) - 3$. If (H,σ) is bipartite, by Lemma 4.4, we have $\eta((H,\sigma) - u) \neq \eta(H,\sigma)$. When $\eta((H,\sigma) - u) = \eta(H,\sigma) + 1$, by Lemma 2.6 (i), we have $\eta(\Gamma) = \eta(H,\sigma) + \eta(C - u) = (c(H,\sigma) - 1) + 0 = c(\Gamma) - 2$. When

$\eta((H,\sigma)-u)=\eta(H,\sigma)-1$, by Lemma 2.6 (ii), we have $\eta(\Gamma)=\eta(H,\sigma)+\eta(C)-1=(c(H,\sigma)-1)-1=c(\Gamma)-3$.

Assume the assertion holds for the nonbipartite signed cycle-spliced graphs which satisfy the above assumptions with less cycles than Γ. Now we assume that $c(\Gamma)\geq 4$.

Case 1. C is an even cycle.

(H,σ) must be nonbipartite since Γ is nonbipartite. Note that (H,σ) has one cycle less than Γ, the induction hypothesis implies that $\eta(H,\sigma)\leq c(H,\sigma)-2$. By Lemma 3.1, $\eta(\Gamma)\leq \eta(H,\sigma)+1\leq (c(H,\sigma)-2)+1=c(\Gamma)-2$.

Case 2. C is an odd cycle.

If (H,σ) is nonbipartite, then by similar discussion as above, we have $\eta(\Gamma)\leq c(\Gamma)-2$. Now we consider when (H,σ) is bipartite. Recall that $\eta(H,\sigma)\leq c(H,\sigma)-1$. When $\eta(H,\sigma)\leq c(H,\sigma)-2$, by Lemma 3.1, $\eta(\Gamma)\leq\eta(H,\sigma)+1\leq(c(H,\sigma)-2)+1=c(\Gamma)-2$. When $\eta(H,\sigma)=c(H,\sigma)-1$, by Lemma 4.4, we have $\eta((H,\sigma)-u)\neq\eta(H,\sigma)$. When $\eta((H,\sigma)-u)=\eta(H,\sigma)+1$, by Lemma 2.6 (i), we have $\eta(\Gamma)=\eta(H,\sigma)+\eta(C-u)=(c(H,\sigma)-1)+0=c(\Gamma)-2$. When $\eta((H,\sigma)-u)=\eta(H,\sigma)-1$, by Lemma 2.6 (ii), we have $\eta(\Gamma)=\eta(H,\sigma)+\eta(C)-1=(c(H,\sigma)-1)-1=c(\Gamma)-3$.

From the above arguments, we see that $\eta(\Gamma)\leq c(\Gamma)-2$, which completes the proof. □

5. Proofs of Theorems 1.4 and 1.5

Now we give the proofs of Theorems 1.4 and 1.5, respectively.

Proof of Theorem 1.4 "If" part: We proceed by induction on $c(\Gamma)$ to prove $\eta(\Gamma)=c(\Gamma)-1$. If $c(\Gamma)=2$, then Γ is a signed graph obtained from two signed cycles C_1 and C_2 by identifying a unique common vertex u. And C_1 and C_2 both have nullity 0 since they are pendant cycles. It is easy to calculate that $\eta(\Gamma)=c(\Gamma)-1$. Assume the assertion holds for signed cycle-spliced bipartite graphs satisfying the above assumptions with less cycles than Γ. Now we assume that $c(\Gamma)\geq 3$. Let C be an arbitrary pendant cycle of Γ with root u. Then $\Gamma-C+u$ has at most one pendant cycle with nullity 2. If all pendant cycles of $\Gamma-C+u$ have nullity 0, then let $\Gamma_0=C$ and $(H,\sigma)=\Gamma-\Gamma_0+u$. Otherwise, $\Gamma-C+u$ has exactly one pendant cycle which has nullity 2, say C_1 with root u_1. Let $\Gamma_1=C+C_1$. Then $\Gamma-\Gamma_1+u_1$ has at most one pendant cycle with nullity 2. If all pendant cycles of $\Gamma-\Gamma_1+u_1$ have nullity 0 or $\Gamma-\Gamma_1+u_1$ is a signed cycle with nullity 0, then let $(H,\sigma)=\Gamma-\Gamma_1+u$. In a similar way, we must obtain

$$\Gamma_k=C+C_1+\cdots+C_k \text{ and } (H,\sigma)=\Gamma-\Gamma_k+u_k.$$

It is easy to see that Γ_k is a signed cycle-spliced bipartite graph in which each non-pendant cycle of Γ_k has exactly two cut vertices, the pendant cycle C of Γ_k has nullity 0 and the other cycles of Γ_k all have nullity 2. Moreover, the distance between any two cut vertices of Γ_k is even since Γ_k is induced subgraph of Γ. Then Lemma 4.7 (i) implies that $\eta(\Gamma_k)=c(\Gamma_k)-1$. Since u_k is a vertex such that the distance between u_k and the cut vertex u_{k-1} of Γ_k is even, by Lemma 4.7 (iii), we have

$$\eta(\Gamma_k-u_k)=\eta(\Gamma_k)+1.$$

If $c(H,\sigma) \geq 2$, then $(H,\sigma) = \Gamma - \Gamma_k + u_k$ is a signed cycle-spliced bipartite graph with $c(H,\sigma) = c(\Gamma) - (k+1) < c(\Gamma)$ and all pendant cycles of (H,σ) have nullity 0. The distance between any two cut vertices of (H,σ) is even since (H,σ) is induced subgraph of Γ. The induction hypothesis implies that $\eta(H,\sigma) = c(H,\sigma) - 1$. Since u_k is a vertex such that the distance between u_k and any cut vertex of (H,σ) is even, by Lemma 4.5 (iii), we have

$$\eta((H,\sigma) - u_k) = \eta(H,\sigma) + 1.$$

Then Lemma 4.3 (ii) implies that $\eta(\Gamma) = c(\Gamma) - 1$. If $c(H,\sigma) = 1$, then $(H,\sigma) = \Gamma - \Gamma_k + u_k$ is a signed cycle with nullity 0. Clearly, $\eta(H,\sigma) = c(H,\sigma) - 1$ and $\eta((H,\sigma) - u_k) = \eta(H,\sigma) + 1$. By Lemma 4.3 (ii), we have $\eta(\Gamma) = c(\Gamma) - 1$.

"Only if" part: Follows from Lemma 4.5 (ii). □

Proof of Theorem 1.5 "If" part: When $c(\Gamma) = 1$, $\Gamma = (H,\sigma)$ is a signed cycle with nullity 0 by attaching no cycles with nullity 2 on arbitrary vertex. Clearly, $\eta(\Gamma) = c(\Gamma) - 1$. When $c(\Gamma) \geq 2$, if all pendant cycles of Γ have nullity 0, then $\Gamma = (H,\sigma)$ is a signed graph obtained from (H,σ) by attaching no cycles with nullity 2 on arbitrary vertex. Clearly, $\eta(\Gamma) = \eta(H,\sigma) = c(H,\sigma) - 1 = c(\Gamma) - 1$. Otherwise, Γ must contain a pendant cycle which has nullity 2. By contracting all pendant cycles with nullity 2 into a vertex, finally we have the graph (H,σ) which is a signed cycle with nullity 0 or a signed graph with all pendant cycles of nullity 0. Then by Corollary 2.7 (i), we have $\eta(\Gamma) = \eta(C) + (c(\Gamma) - 1) = c(\Gamma) - 1$ or $\eta(\Gamma) = \eta(H,\sigma) + (c(\Gamma) - c(H,\sigma)) = (c(H,\sigma) - 1) + (c(\Gamma) - c(H,\sigma)) = c(\Gamma) - 1$.

"Only if" part: When $c(\Gamma) = 1$, Γ is a signed cycle which has nullity 0 since $\eta(\Gamma) = c(\Gamma) - 1$. Then Γ is a signed cycle which has nullity 0 by attaching no cycles with nullity 2 on arbitrary vertex. When $c(\Gamma) \geq 2$, if all pendant cycles of Γ have nullity 0, then Γ is a signed graph obtained from $\Gamma = (H,\sigma)$ by attaching no cycles with nullity 2 on arbitrary vertex. Otherwise, Γ must contain a pendant cycle with nullity 2. By contracting all pendant cycles with nullity 2 into a vertex, finally we have the graph (H,σ) which is a signed cycle with nullity 0 or a signed graph with all pendant cycles of nullity 0. Then by Lemma 4.3 (i), we have $\eta(H,\sigma) = c(H,\sigma) - 1$. Thus Γ is a signed graph obtained from a signed cycle-spliced bipartite graph (H,σ) satisfying $\eta(H,\sigma) = c(H,\sigma) - 1$ with all pendant cycles of nullity 0 or a signed cycle of nullity 0 by attaching $c(\Gamma) - c(H,\sigma)$ cycles with nullity 2 on arbitrary vertex of (H,σ). □

Concluding remarks. In this paper, we considered the nullity of a signed cycle-spliced graph and proved that for every signed cycle-spliced graph Γ, $\eta(\Gamma) \leq c(\Gamma) + 1$ and $\eta(\Gamma) \neq c(\Gamma)$. The extremal graphs Γ which have nullity $c(\Gamma) + 1$ were characterized. We also explored some properties on signed cycle-spliced graphs Γ with $\eta(\Gamma) = c(\Gamma) - 1$, especially, a structural characterization on signed cycle-spliced bipartite graphs Γ with $\eta(\Gamma) = c(\Gamma) - 1$. But it seems somewhat difficult to give a full characterization on signed cycle-spliced graphs Γ with $\eta(\Gamma) = c(\Gamma) - 1$, and we leave it for the further study.

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