Journal of Mathematical Research with Applications
Nov., 2023, Vol. 43, No. 6, pp. 665–672
DOI:10.3770/j.issn:2095-2651.2023.06.003
Http://jmre.dlut.edu.cn

A Periodicity Property of Symmetric Algebras with Actions of Metacyclic Groups in the Modular Case

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Abstract In this paper, we consider the decomposition of the symmetric algebra $\mathbb{F}[V]$ into indecomposables with linear actions of a metacyclic group $G = C_p \times H$, where H is a p'-group, and prove a periodicity property of the symmetric algebra $\mathbb{F}[V]$ if V is a direct sum of indecomposable G-module such that the norm polynomial of the simple H-module is the power of the product of the basis elements of the dual.

Keywords indecomposable module; symmetric algebra; periodicity property

MR(2020) Subject Classification 20C20; 13A50

1. Introduction

Let V denote a finite dimensional representation of a finite group G over a field $\mathbb{F}$ in characteristic p such that $p \mid |G|$. Then V is called a modular representation. We choose a basis $\{x_1, \ldots, x_n\}$ for the dual space V^*. The action of G on V induces an action on V^* and it extends to an action by algebra automorphisms on the symmetric algebra $S(V^*)$ which is equivalent to the polynomial ring $\mathbb{F}[V] = \mathbb{F}[x_1, \ldots, x_n]$. The ring of invariants, $\mathbb{F}[V]^G := \{f \in \mathbb{F}[V] | g(f) = f, \forall\, g \in G\}$, is an $\mathbb{F}$-subalgebra of $\mathbb{F}[V]$.

The essential case of modular representation theory is the study of modules of cyclic groups $G = C_p$ of order p over a field $\mathbb{F}$ in characteristic p, and there are only a finite number of indecomposable modules over $\mathbb{F}C_p$, the Jordan blocks of degree n with 1's in the diagonal for $1 \leqslant n \leqslant p$.

We would like to know the decomposition of the polynomial ring $\mathbb{F}[V]$ into indecomposables with a linear action of a finite group G. The periodicity property shows that for any large enough degree d, there is a natural number r such that $\mathbb{F}[V]_d$ can be decomposed into a direct sum of $\mathbb{F}[V]_r$ and some projective $\mathbb{F}G$-modules. The number r is relevant to the order of the Sylow p-subgroup of G. Almkvist and Fossum [1] obtained complete information about the decomposition of $\mathbb{F}[V]$ with action of cyclic groups of prime order C_p, and derived formulas for the Hilbert series of the ring of invariants $\mathbb{F}[V]^{C_p}$ in the case that V is an indecomposable $\mathbb{F}C_p$-module. Campbell

Received November 14, 2022; Accepted June 1, 2023
Supported by the National Natural Science Foundation of China (Grant No. 12171194) and the Natural Science Foundation for Young Scientists of Shanxi Province (Grant No. 201901D211184).
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and Wehlau show a periodicity property of the symmetric algebra for any finite-dimensional decomposable $\mathbb{F}C_p$-module, see Chapter 7 of the textbook [2]. Symonds [3] not only shows a periodicity property of the cyclic group C_{p^n} for general n (more generally for a group G with a normal cyclic Sylow p-subgroup C_{p^n}), but the decomposition of symmetric algebras of degrees less than p^n into indecomposable modules by reducing the problem to the calculation of the exterior powers of V, where $\mathrm{Res}^G_{C_{p^n}} V$ is a indecomposable C_{p^n}-module. Chen [4,5] considers the indecomposable representations of the dihedral group which is a semidirect product of C_p and C_2, and the dicyclic group which is neither a direct product nor a semi-product of finite groups. Furthermore, Chen proves the periodicity property of the dihedral group.

Note that the indecomposable modules of $G \cong C_p \times H$ are the tensor product of indecomposables of C_p and H, and it is enough to consider the simple H-modules as $p \nmid |H|$. For any simple H-module W, we find that the norm polynomial $N^H(w)$ is the power of the product of $h(w)$, where $h \in H$, for a fixed w which is a generator of W^*. In this paper, we extend the periodicity property of the symmetric algebra $\mathbb{F}[V]$ to the case $G \cong C_p \times H$ if V is a direct sum of indecomposable G-module such that the norm polynomial of the simple H-module is the power of the product of the basis elements of the dual.

2. Indecomposable $\mathbb{F}G$-modules

Let G be a finite group, whose form is a direct sum of the cyclic group of order p, C_p, and a p'-group H. Let F be a field of characteristic p. The complete set of indecomposable modules of C_p is well-known, which are exactly the Jordan blocks. Let V_n be the Jordan block of degree n. Let Sim(H) be the complete set of non-isomorphic simple $\mathbb{F}H$-modules.

Proposition 2.1 *The $\mathbb{F}G$-modules*

$$V_n \otimes W, \quad 1 \leqslant n \leqslant p, \ W \in \mathrm{Sim(H)}$$

form a complete set of non-isomorphic indecomposable $\mathbb{F}G$-modules.

Proof Since $p \nmid |H|$, there is no difference between irreducibility and indecomposibility of $\mathbb{F}H$-modules. Then the result follows from Huppert and Blackburn [6, Chapter VII, Theorem 9.15]. □

Proposition 2.2 *$V_n \otimes W$ is projective if and only if $n = p$.*

Proof The result follows from Alperin [7, Corollar 7, p.33; Corollary 3, p.66]. □

Proposition 2.3 *The decomposition of $\mathbb{F}G$ as $\mathbb{F}G$-module is isomorphic to*

$$\sum_{W \in \mathrm{Sim}(H)} \dim_{\mathbb{F}}(W)\, V_p \otimes W.$$

Proof Note that $V_1 \otimes W \cong W$ is a simple $\mathbb{F}G$-module for any $W \in \mathrm{Sim}(H)$ via $G \to G/C_p \cong H$, the result follows from Alperin [7, Corollary 4, p.31]. □

Proposition 2.4 *$V_n \otimes W$ are uniserial with composition factors isomorphic to W.*

Proof The fact that $V_n \otimes W$ are uniserial follows from Alperin [7, p.42]. By Proposition 2.3, let $\{B_W = \dim_{\mathbb{F}}(W)\, V_p \otimes W | W \in \mathrm{Sim}(H)\}$ be the blocks of $\mathbb{F}G$. Then

$$\mathbb{F}G = \sum_{W \in \mathrm{Sim}(H)} B_W$$

is the unique decomposition of $\mathbb{F}G$ as an algebra. Since W is the unique isomorphic class of simple modules lying in B_W, composition factors of $V_p \otimes W$ are simple $\mathbb{F}G$-modules isomorphic to W by Alperin [7, Proposition 3, p.94], so is $V_n \otimes W$. □

3. The periodicity property of the symmetric algebras

In this section, we will consider the structure of symmetric algebra $\mathbb{F}[V]$ as a graded $\mathbb{F}G$-module. First we treat the case that $V = V_n \otimes W$ is an indecomposable $\mathbb{F}G$-module, and next consider the decomposable $\mathbb{F}G$-modules V.

$\mathbb{F}[V]$ as a graded ring has degree decomposition $\mathbb{F}[V] = \bigoplus_{d=0}^{\infty} \mathbb{F}[V]_d$ and $\dim_{\mathbb{F}}\mathbb{F}[V]_d < \infty$ for each d. The group action preserves degree, so $\mathbb{F}[V]_d$ is also a finite dimensional $\mathbb{F}G$-module. By Krull-Schmidt theorem, we want to decompose each $\mathbb{F}[V]_d$ into

$$\mathbb{F}[V]_d = \bigoplus_{\substack{1 \leqslant n \leqslant p \\ W \in \mathrm{Sim}(H)}} m_{d,n} V_n \otimes W.$$

3.1. The $\mathbb{F}G$-module structure of $\mathbb{F}[V_n \otimes W]$

Notice that the $\mathbb{F}H$-module W is simple if and only if W^* is simple, i.e., $\forall w \in W^*$, w generates W^* as an $\mathbb{F}H$-module. For a fixed $w \in W^*$, let $H = \{h_1 = e, h_2, \ldots, h_{|H|}\}$, and $\{w_1 = w, w_2 = h_2 w, \ldots, w_k = h_k w\}$ be a basis of W^* with $\dim_{\mathbb{F}} W^*$=k. The norm polynomial $N^H(w) = \prod_{h \in H} h(w)$ is an H-invariant polynomial.

Lemma 3.1 *For a simple $\mathbb{F}H$-module W^* with basis $\{w_1, w_2, \ldots, w_k\}$, the norm polynomial $N^H(w_1)$ is of the form $(w_1 w_2 \cdots w_k)^l f$, where $l \geqslant 1$ and f is a polynomial in $\mathbb{F}[w_1, w_2, \ldots, w_k]$ such that $w_i \nmid f$ for $1 \leqslant i \leqslant k$.*

Proof Note that $N^H(w_1) = w_1(h_2 w_1) \cdots (h_{|H|} w_1)$. If $N^H(w_1) = w_1^l f'$ such that $w_1 \nmid f'$, then

$$N^H(w_1) = h_j N^H(w_1) = (h_j w_1^l)(h_j f') = (h_j w_1)^l (h_j f')$$

for $1 \leqslant j \leqslant |H|$. Then $(h_j w_1)^l | w_1^l$ or $(h_j w_1)^l | f'$. Therefore, we have

$$N^H(w_1) = (w_1 (h_{j_2} w_1) \cdots (h_{j_q} w_1))^l$$

for some q such that $ql = |H|$.

Since there is a basis in $\{w_1, h_{j_2} w_1, \ldots, h_{j_q} w_1\}$, without loss of generality, consider $\{2, 3, \ldots, k\}$ $\in \{j_2, j_3, \ldots, j_q\}$, then we have

$$N^H(w_1) = (w_1 w_2 \cdots w_k)^l f,$$

where f is a product of linear forms in $\mathbb{F}[w_1, w_2, \ldots, w_k]$ such that $w_i \nmid f$ for $1 \leqslant i \leqslant k$. □

Choose the triangular basis $\{w_{11}, w_{12}, \dots, w_{1k}, \dots, w_{n1}, w_{n2}, \dots, w_{nk}\}$ for $(V_n \otimes W)^*$ and use the reverse lexicographic order with $w_{11} < w_{12} < \cdots < w_{1k} < \cdots < w_{n1} < w_{n2} < \cdots < w_{nk}$. Note that $w \in < w_{n1}, w_{n2}, \dots, w_{nk} >$ is a distinguished variable in $V_n \otimes W$, which means that w generates $V_n \otimes W$ as an $\mathbb{F}G$-module.

If $k = q = \frac{|H|}{l}$ in Lemma 3.1, then $N^H(w_{n1}) = (w_{n1}w_{n2}\cdots w_{nk})^l$, and $N^{C_p}(N^H(w_{n1})) = N^{C_p}((w_{n1}w_{n2}\cdots w_{nk})^l) \in \mathbb{F}[V_n \otimes W]^G$. Note that $\deg_{w_{ni}}(N^{C_p}(N^H(w_{n1}))) = lp$ for $1 \leqslant i \leqslant k$.

Let $\mathbb{F}[V_n \otimes W]^\sharp$ be the principle ideal of $\mathbb{F}[V_n \otimes W]$ generated by $N^{C_p}((w_{n1}w_{n2}\cdots w_{nk})^l)$. The set $\{N^{C_p}(N^H(w_{n1}))\}$ is a Gröbner basis for the ideal it generates. Then we may divide any given $f \in \mathbb{F}[V_n \otimes W]$ to obtain $f = qN^{C_p}((w_{n1}w_{n2}\cdots w_{nk})^l) + r$ for some q, $r \in \mathbb{F}[V_n \otimes W]$ with $\deg_{w_{ni}}(r) < lp$ for at least one i.

We define $\mathbb{F}[V_n \otimes W]^\flat := \{r \in \mathbb{F}[V_n \otimes W] \mid \deg_{w_{ni}}(r) < lp \text{ for at least one } i\}$. Note that both $\mathbb{F}[V_n \otimes W]^\sharp$ and $\mathbb{F}[V_n \otimes W]^\flat$ are vector spaces and that as vector spaces, $\mathbb{F}[V_n \otimes W] = \mathbb{F}[V_n \otimes W]^\sharp \oplus \mathbb{F}[V_n \otimes W]^\flat$. Moreover, they are G-stable. Therefore, we have the $\mathbb{F}G$-module decomposition

$$\mathbb{F}[V_n \otimes W] = \mathbb{F}[V_n \otimes W]^\sharp \oplus \mathbb{F}[V_n \otimes W]^\flat.$$

Lemma 3.2 *The $\mathbb{F}G$-module $\mathbb{F}[V_n \otimes W]^\flat_d$ can be decomposed as a direct sum of indecomposable projective modules if the following conditions are satisfied:*

(1) $d + n \geqslant lkp + 1$;

(2) $N^H(w_1) = (w_1w_2\cdots w_k)^l$ *for some basis* $\{w_1, w_2, \dots, w_k\}$ *of* W^*.

Proof The proof is by (downward) induction on n. For the case $n = p$. Note that $W = W^{C_p}$. $V_p \otimes W \cong kV_p$ as $\mathbb{F}C_p$-modules. We may choose a basis $\{x_{11}, x_{12}, \dots, x_{1k}, \dots, x_{p1}, x_{p2}, \dots, x_{pk}\}$ of $(V_p \otimes W)^*$ such that $\{x_{1i}, x_{2i}, \dots, x_{pi}\}$ for $1 \leqslant i \leqslant k$ are transitively permuted by the action of C_p simultaneously. Then the monomials of degree d in the variables $\{x_{ji} \mid 1 \leqslant j \leqslant p, 1 \leqslant i \leqslant k\}$ form a vector space basis of $\mathbb{F}[V_p \otimes W]_d$ and this basis of monomials is again permuted by C_p. We may take

$$N^{C_p}((w_{p1}w_{p2}\cdots w_{pk})^l) = N^{C_p}((x_{j_1 1}x_{j_2 2}\cdots x_{j_k k})^l) = (x_{11}x_{21}\cdots x_{p1}\cdots x_{1k}x_{2k}\cdots x_{pk})^l$$

for any $j_1, j_2, \dots, j_k \in \{1, \dots, p\}$. So $\mathbb{F}[V_p \otimes W]^\flat_d$ is spanned by the monomials

$$x_{11}^{e_{11}}x_{21}^{e_{21}}\cdots x_{p1}^{e_{p1}}\cdots x_{1k}^{e_{1k}}x_{2k}^{e_{2k}}\cdots x_{pk}^{e_{pk}},$$

where $e_{ji} < l$ for at least one pair of (j, i). $\mathbb{F}[V_p \otimes W]^\sharp_d$ is spanned by the monomials divisible by $(x_{11}x_{21}\cdots x_{p1}\cdots x_{1k}x_{2k}\cdots x_{pk})^l$. In particular, every monomial, other than the monomial 1, in the basis of $\mathbb{F}[V_p \otimes W]^\flat_d$ lies in a C_p-orbit of order p. Thus for $d \geqslant 1$, we see that $\mathbb{F}[V_p \otimes W]^\flat_d \cong sV_p$ as $\mathbb{F}C_p$-modules for some non-negative integer s. Since indecomposable $\mathbb{F}G$-modules whose dimensions are divided by p are $\{V_p \otimes W | W \in \mathrm{Sim}(H)\}$, by Krull-Schmidt theorem, we have that

$$\mathbb{F}[V_p \otimes W]^\flat_d \cong \bigoplus_{W \in \mathrm{Sim}(H)} s_W V_p \otimes W$$

for some non-negative integers s_W.

For the general case, we suppose that the result holds for $\mathbb{F}[V_{n+1} \otimes W]_d^\flat$ and that $d \geqslant (lkp+1)-n$. Let $\{w_{11}, w_{12}, \ldots, w_{1k}, \ldots, w_{n+1,1}, w_{n+1,2}, \ldots, w_{n+1,k}\}$ be a basis of $(V_n \otimes W)^*$ such that $w \in \langle w_{n+1,1}, w_{n+1,2}, \ldots, w_{n+1,k}\rangle$ the distinguished variable and hence $\langle w_{11}, w_{12}, \ldots, w_{1k}\rangle$ is C_p-fixed. Consider the short exact sequence of $\mathbb{F}C_p$-module

$$0 \to \mathbb{F}[V_{n+1} \otimes W]_d^\flat \xrightarrow{\mu} \mathbb{F}[V_{n+1} \otimes W]_{d+k}^\flat \xrightarrow{\theta} \mathbb{F}[V_n \otimes W]_{d+k}^\flat \to 0,$$

where $\mu(f) = (w_{11}w_{12}\cdots w_{1k})f$ and

$$\theta(w_{ji}) = \begin{cases} 0, & \text{if } j = 1, \\ w_{j-1,i}, & \text{if } 2 \leqslant j \leqslant n+1. \end{cases}$$

Now $\mathbb{F}[V_{n+1} \otimes W]_d^\flat$ and $\mathbb{F}[V_{n+1} \otimes W]_{d+k}^\flat$ are both free by the induction hypothesis, and hence $\mathbb{F}[V_{n+1} \otimes W]_d^\flat$ is injective by Alperin [7, Theorem 4, p.41]. Thus the above sequence splits and $\mathbb{F}[V_n \otimes W]_{d+k}^\flat$ is projective. By the fact that projective C_p-module is free, we can write that $\mathbb{F}[V_n \otimes W]_{d+k}^\flat \cong sV_p$. Similar to the case $n = p$, we have that

$$\mathbb{F}[V_n \otimes W]_{d+k}^\flat \cong \bigoplus_{W \in \mathrm{Sim}(H)} s'_W V_p \otimes W$$

for some non-negative integers s'_W. □

Theorem 3.3 *Let d be a non-negative integer and write $d = q(lkp) + r$ where $0 \leqslant r \leqslant lkp - 1$. Then*

$$\mathbb{F}[V_n \otimes W]_d \cong \mathbb{F}[V_n \otimes W]_r \oplus \Big(\bigoplus_{W \in \mathrm{Sim}(H)} s_W V_p \otimes W \Big)$$

as $\mathbb{F}G$-modules for some non-negative integers s_W.

Proof The proof is by induction on q. If $q = 0$, it is trivially true. For $d \geqslant lkp$, we have a G-equivariant isomorphism

$$\mu : \mathbb{F}[V_n \otimes W]_{d-lkp} \to \mathbb{F}[V_n \otimes W]_d^\sharp,$$

where $\mu(f) = N^{C_p}(N^H(w))f$ for a distinguished variable w. By Lemma 3.2, we have that

$$\begin{aligned}
\mathbb{F}[V_n \otimes W]_d &\cong \mathbb{F}[V_n \otimes W]_d^\sharp \oplus \mathbb{F}[V_n \otimes W]_d^\flat \\
&\cong \mathbb{F}[V_n \otimes W]_{d-lkp} \oplus \Big(\bigoplus_{W \in \mathrm{Sim}(H)} s'_W V_p \otimes W \Big) \\
&\cong (\mathbb{F}[V_n \otimes W]_r \oplus \Big(\bigoplus_{W \in \mathrm{Sim}(H)} t'_W V_p \otimes W) \Big) \oplus \Big(\bigoplus_{W \in \mathrm{Sim}(H)} s'_W V_p \otimes W \Big) \\
&\cong \mathbb{F}[V_n \otimes W]_r \oplus \Big(\bigoplus_{W \in \mathrm{Sim}(H)} s_W V_p \otimes W \Big),
\end{aligned}$$

where the third isomorphism follows from the induction hypothesis. Furthermore, we may compute that

$$\sum_{W \in \mathrm{Sim}(H)} s_W \dim_{\mathbb{F}} W = \frac{\binom{kn+d-1}{d} - \binom{kn+r-1}{r}}{p}. \quad \square$$

3.2. The $\mathbb{F}G$-module structure of $\mathbb{F}[V]$

We now consider the decomposable $\mathbb{F}G$-module V which is reduced, i.e., there is not trivial $\mathbb{F}G$-module contained in V. Let V be a reduced finite dimensional representation of G and decompose V into a direct sum of indecomposable summands:

$$V = (V_{n_1} \otimes W_1) \oplus (V_{n_2} \otimes W_2) \oplus \cdots \oplus (V_{n_m} \otimes W_m),$$

where W_i's are simple H-modules satisfying that there is $w_{i1} \in W_i^*$ such that

$$N^H(w_{i1}) = (w_{i1} w_{i2} \cdots w_{ik_i})^{l_i}$$

with $k_i = \dim_{\mathbb{F}}(W_i)$.

Consider the twisted derivation $\triangle : \mathbb{F}[V] \to \mathbb{F}[V]$ defined as $\triangle = \sigma - Id$, where $C_p = \langle \sigma \rangle$.

Let $z_{i1}, z_{i2}, \ldots, z_{ik_i}$ be distinguished variables in $V_{n_i} \otimes W_i$ such that $\{z_{ji1} = \triangle^j(z_{i1}), z_{ji2} = \triangle^j(z_{i2}), \ldots, z_{jik_i} = \triangle^j(z_{ik_i}) | 0 \leqslant j \leqslant n_i - 1, 1 \leqslant i \leqslant m\}$ is a vector space basis for V^*. Let $N_i = N^{C_p}((z_{i1} z_{i2} \cdots z_{ik_i})^{l_i})$ be the norm polynomial for $V_{n_i} \otimes W_i$.

Given proper order of the variables of V^*, we have that the set $\{N_1, N_2, \ldots, N_m\}$ is a Gröbner basis for the ideal they generate [8, Theorem 1.6.7]. For more background of Gröbner basis, textbook [8] and [9] are suggested. Then for any $f \in \mathbb{F}[V]$, there is a decomposition

$$f = f_1 N_1 + f_2 N_2 + \cdots + f_m N_m + r,$$

where $\deg_{z_{iq}}(f_j) < l_i p$ for at least one $q \in \{1, 2, \ldots, k_i\}$, all $i < j$, and $\deg_{z_{iq}}(r) < l_i p$ for at least one $q \in \{1, 2, \ldots, k_i\}$, all $i \in \{1, 2, \ldots, m\}$. Note that the decomposition depends upon the choice and order of distinguished variables but is otherwise unique.

Let $\mathbb{F}[V]^\sharp$ denote the ideal of $\mathbb{F}[V]$ generated by the norms $N_1, N_2, \ldots, N_m$ and $\mathbb{F}[V]^\flat := \{r \in \mathbb{F}[V] \mid \deg_{z_{iq}} r < l_i p$ for at least one $q \in \{1, 2, \ldots, k_i\}$, all $i \in \{1, 2, \ldots, m\}\}$. Since $\mathbb{F}[V]^\sharp$ and $\mathbb{F}[V]^\flat$ are $\mathbb{F}G$-modules, we have the $\mathbb{F}G$-module decomposition

$$\mathbb{F}[V] = \mathbb{F}[V]^\sharp \oplus \mathbb{F}[V]^\flat.$$

The decomposition $V = (V_{n_1} \otimes W_1) \oplus (V_{n_2} \otimes W_2) \oplus \cdots \oplus (V_{n_m} \otimes W_m)$ induces an isomorphism $\mathbb{F}[V] \cong \mathbb{F}[V_{n_1} \otimes W_1] \otimes \mathbb{F}[V_{n_2} \otimes W_2] \otimes \cdots \otimes \mathbb{F}[V_{n_m} \otimes W_m]$. This isomorphism in turn yields an $\mathbb{N}^m$ multi-grading on $\mathbb{F}[V]$:

$$\mathbb{F}[V]_{(d_1, d_2, \ldots, d_m)} = \mathbb{F}[V_{n_1} \otimes W_1]_{d_1} \otimes \mathbb{F}[V_{n_2} \otimes W_2]_{d_2} \otimes \cdots \otimes \mathbb{F}[V_{n_m} \otimes W_m]_{d_m}.$$

Before proving the periodicity theorem, we shall show the decomposition of the tensor products of indecomposable modules with projective indecomposable modules.

Lemma 3.4 *Let W, W' be simple H-modules. Then*

$$(V_n \otimes W) \otimes (V_p \otimes W') \cong \bigoplus_{W \in \mathrm{Sim}(H)} s_W (n V_p \otimes W)$$

as $\mathbb{F}G$-modules for some non-negative integers s_W.

Proof We proceed by induction on n. For $n = 1$, by Maschke's Theorem, we have

$$\begin{aligned}(V_1 \otimes W) \otimes (V_p \otimes W') &\cong V_p \otimes (W \otimes W') \\ &\cong V_p \otimes (\bigoplus_{W\in \mathrm{Sim}(H)} s_W W) \cong \bigoplus_{W\in \mathrm{Sim}(H)} s_W V_p \otimes W\end{aligned}$$

for some non-negative integers s_W.

For the general case, we assume the induction hypothesis $(V_{n-1} \otimes W) \otimes (V_p \otimes W') \cong \bigoplus_{W\in \mathrm{Sim}(H)} s_W((n-1)V_p \otimes W)$ and consider the short exact sequence:

$$0 \to V_{n-1} \otimes W \to V_n \otimes W \to V_1 \otimes W \to 0.$$

Since $V_p \otimes W'$ is projective, hence flat, tensoring with $V_p \otimes W'$ yields a new short exact sequence:

$$0 \to (V_{n-1} \otimes W) \otimes (V_p \otimes W') \to (V_n \otimes W) \otimes (V_p \otimes W') \to (V_1 \otimes W) \otimes (V_p \otimes W') \to 0.$$

Since the last term is projective, this sequence splits and thus

$$\begin{aligned}(V_n \otimes W) \otimes (V_p \otimes W') &\cong ((V_1 \otimes W) \otimes (V_p \otimes W')) \oplus ((V_{n-1} \otimes W) \otimes (V_p \otimes W')) \\ &\cong \Big(\bigoplus_{W\in \mathrm{Sim}(H)} s_W V_p \otimes W\Big) \oplus \Big(\bigoplus_{W\in \mathrm{Sim}(H)} s_W((n-1)V_p \otimes W)\Big) \\ &\cong \bigoplus_{W\in \mathrm{Sim}(H)} s_W(nV_p \otimes W).\end{aligned}$$

Note that s_W is dependent on the tensor product of simple H-module. □

Theorem 3.5 *Let $V = (V_{n_1} \otimes W_1) \oplus (V_{n_2} \otimes W_2) \oplus \cdots \oplus (V_{n_m} \otimes W_m)$. Let $d_1, d_2, \ldots, d_m$ be non-negative integers and write $d_i = q_i(lk_ip) + r_i$, where $0 \leqslant r_i \leqslant lk_ip - 1$ for $i = 1, 2, \ldots, m$. Then*

$$\mathbb{F}[V]_{(d_1,d_2,\ldots,d_m)} \cong \mathbb{F}[V]_{(r_1,r_2,\ldots,r_m)} \oplus \Big(\bigoplus_{W\in \mathrm{Sim}(H)} s_W V_p \otimes W\Big),$$

as $\mathbb{F}G$-modules for some non-negative integers s_W.

Proof We will induct on the number of direct summands. Write $V = (V_{n_1} \otimes W_1) \oplus T$ for $T = (V_{n_2} \otimes W_2) \oplus \cdots \oplus (V_{n_m} \otimes W_m)$. By induction, we have

$$\begin{aligned}\mathbb{F}[V]_{(d_1,d_2,\ldots,d_m)} &\cong \mathbb{F}[V_{n_1} \otimes W_1]_{d_1} \otimes \mathbb{F}[T]_{(d_2,\ldots,d_m)} \\ &\cong \Big(\mathbb{F}[V_{n_1} \otimes W_1]_{r_1} \oplus \Big(\bigoplus_{W\in \mathrm{Sim}(H)} s'_W V_p \otimes W\Big)\Big) \otimes \\ &\quad \Big(\mathbb{F}[T]_{(r_2,\ldots,r_m)} \oplus \Big(\bigoplus_{W\in \mathrm{Sim}(H)} s''_W V_p \otimes W\Big)\Big) \\ &\cong \mathbb{F}[V]_{(r_1,r_2,\ldots,r_m)} \oplus \Big(\bigoplus_{W\in \mathrm{Sim}(H)} s_W V_p \otimes W\Big).\end{aligned}$$

And we may compute that

$$\frac{\prod_{i=1}^{m} \binom{k_in_i+d_i-1}{d_i} - \prod_{i=1}^{m} \binom{k_in_i+r_i-1}{r_i}}{p} = \sum_{W\in \mathrm{Sim}(H)} s_W \dim_{\mathbb{F}} W. \quad \square$$

Acknowledgements The authors would like to express their sincere gratitude to the referees and editors for a very careful reading of this paper and for all their insightful comments, which led to a number of improvements.

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