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A Note on a Question of I. Laine and C. C. Yang

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Abstract The aim of this paper is to study the order of meromorphic solutions of linear difference equation

$$A_n(z)f(z+n)+\cdots+A_1(z)f(z+1)+A_0(z)f(z)=0,$$

where the coefficients $A_j(z)$ $(j=0,\dots,n)$ are entire functions. We obtain some results by giving some restrictions on coefficients of above equation with no dominating coefficient and partially answer a question of I. Laine and C. C. Yang.

Keywords linear differential equation; linear difference equation; order; accumulation ray; zero sequence

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1. Introduction and main results

Throughout this paper, we will assume that readers are familiar with the fundamental results and standard notations $m(r,f),N(r,f),T(r,f)$, etc. of Nevanlinna's theory of meromorphic functions [1–3]. In particular, we define the order of a meromorphic function $f(z)$ by

$$\sigma(f)=\limsup_{r\to\infty}\frac{\log^+T(r,f)}{\log r},$$

while denote the deficiency of the value a for a nonconstant meromorphic function $f(z)$ by

$$\delta(a,f)=\liminf_{r\to\infty}\frac{m(r,\frac{1}{f-a})}{T(r,f)}=1-\limsup_{r\to\infty}\frac{N(r,\frac{1}{f-a})}{T(r,f)}$$

and denote the Valiron deficiency of the value a for a nonconstant meromorphic function $f(z)$ by

$$\Delta(a,f)=1-\liminf_{r\to\infty}\frac{N(r,\frac{1}{f-a})}{T(r,f)},$$

respectively.

Recently, many interesting results of complex differences and difference equations are obtained [4–8]. Chiang and Feng [7] studied the growth of meromorphic solutions of homogeneous linear difference equation with only one coefficient having the maximal order, they obtained the

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following result.

Theorem 1.1 ([7]) *Let $A_0(z),\dots,A_n(z)$ be entire functions such that there exists an integer l $(0\le l\le n)$, such that*

$$\sigma(A_l)>\max_{\substack{1\le j\le n\\ j\ne l}}\{\sigma(A_j)\}.$$

If $f(z)$ $(\not\equiv 0)$ is a meromorphic solution to

$$A_n(z)f(z+n)+\cdots+A_1(z)f(z+1)+A_0(z)f(z)=0, \tag{1.1}$$

then we have $\sigma(f)\ge\sigma(A_l)+1$.

Laine and Yang [9] obtained that Theorem 1.1 still holds when the dominating coefficient of (1.1) depends on type but not on order. In fact, they proved the following result.

Theorem 1.2 ([9]) *Let $A_0(z),\dots,A_n(z)$ be entire functions of finite order such that among those coefficients having the maximal order $\sigma=\max\{\sigma(A_k),0\le k\le n\}$, exactly one has its type strictly greater than the others. If $f(z)$ $(\not\equiv 0)$ is a meromorphic solution to (1.1), then $\sigma(f)\ge\sigma+1$.*

Remark 1.3 The type of a meromorphic function $f(z)$ of order σ $(0<\sigma<\infty)$, which can be found in [1], is defined by

$$\tau(f)=\limsup_{r\to\infty}\frac{T(r,f)}{r^{\sigma}}.$$

Laine and Yang [9] raised the following question.

Question. Whether all meromorphic solutions $f(z)$ $(\not\equiv 0)$ of Eq. (1.1) satisfy $\sigma(f)\ge 1+\max\{\sigma(A_j):0\le j\le n\}$, if there is no dominating coefficient?

In [10], some examples are given to illustrate that the answer is negative to the question of Laine and Yang in whole. In this paper, we consider this question and obtain some results by giving some restrictions on coefficients of difference Eq. (1.1) with no dominating coefficient.

Before stating our main results, we recall some concepts.

Definition 1.4 ([11]) *Let $g(z)$ be a meromorphic function in the complex plane $\mathbb{C}$, and let $\arg z=\theta\in\mathbb{R}$ be a ray from the origin. We denote the exponent of convergence of the zero sequence of $g(z)$ at the ray $\arg z$ by $\lambda_\theta(g)=\lim_{\varepsilon\to 0^+}\lambda_{\theta,\varepsilon}(g)$, where*

$$\lambda_{\theta,\varepsilon}=\limsup_{r\to\infty}\frac{\log^+ n_{\theta-\varepsilon,\theta+\varepsilon}(r,0,g)}{\log r},$$

here $n_{\alpha,\beta}(r,0,g)$ is the number of zeros of g counting multiplicity in $\{z:\alpha<\arg z<\beta\}\cap\{z:|z|<r\}$.

We call the ray $\arg z=\theta$ which has the property $\lambda_\theta(g)=\sigma(g)$ an accumulation ray of the zero sequence of g.

It follows from Definition 1.4 and the following Lemma 2.1 that the number of accumulation rays of the zero sequence of every non-trivial solution of differential equation

$$\omega''(z)+P(z)\omega(z)=0, \tag{1.2}$$

where $P(z) = a_k z^k + \cdots + a_0$, $a_k \neq 0$, is no more than $k+2$, and the set of the accumulation rays of the zero sequence of every non-trivial solution of (1.2) is a subset of $\{\theta_j : 0 \leq j \leq k+1\}$, where $\theta_j = \frac{2j\pi - \arg a_k}{k+2}$, $j = 0, 1, \ldots, k+1$.

Definition 1.5 ([12]) *Let $\omega(z)$ be a non-trivial solution of (1.2), where $P(z) = a_k z^k + \cdots + a_0, a_k \neq 0$. We denote by $p(\omega)$ the number of ray $\arg z = \theta_j$ which are not accumulation rays of the zero sequence of $\omega(z)$, where $\theta_j = \frac{2j\pi - \arg a_k}{k+2}$, $j = 0, 1, \ldots, k+1$.*

Remark 1.6 By Definition 1.5 and the following Lemma 2.1, we can see that $p(\omega)$ must be an even number.

Now we state our main results.

Theorem 1.7 *Suppose that $A_0(z), \ldots, A_n(z)$ are entire functions satisfying the following conditions:*

(i) $A_0(z), A_1(z)$ are two linearly independent solutions of (1.2), where $P(z) = a_k z^k + \cdots + a_0, a_k \neq 0$ and the number of accumulation rays of the zero sequence of $A_1(z)$ is less than $k+2$;

(ii) $\max\{\sigma(A_2), \ldots, \sigma(A_n)\} = \rho < \frac{k+2}{2}$.

If $f(z)$ is a nontrivial meromorphic solution of (1.1), then $\sigma(f) \geq 1 + \max\{\sigma(A_j) : 0 \leq j \leq n\} = 1 + \frac{k+2}{2}$.

Corollary 1.8 *Suppose that $A_0(z), \ldots, A_n(z)$ are entire functions satisfying the following conditions:*

(i) $A_0(z), A_1(z)$ are two linearly independent solutions of (1.2), where $P(z) = a_k z^k + \cdots + a_0, a_k \neq 0$ and $\delta(0, A_1(z)) > 0$;

(ii) $\max\{\sigma(A_2), \ldots, \sigma(A_n)\} = \rho < \frac{k+2}{2}$.

If $f(z)$ is a nontrivial meromorphic solution of (1.1), then $\sigma(f) \geq 1 + \max\{\sigma(A_j) : 0 \leq j \leq n\} = 1 + \frac{k+2}{2}$.

The next result shows that two coefficients of (1.1) are nontrivial solutions of (1.3) and (1.4), respectively,

$$\omega''(z) + P_1(z)\omega(z) = 0, \tag{1.3}$$

$$\omega''(z) + P_2(z)\omega(z) = 0, \tag{1.4}$$

where $P_1(z) = a_k z^k + \cdots + a_0, a_k \neq 0, P_2(z) = b_m z^m + \cdots + b_0, b_m \neq 0$.

Theorem 1.9 *Suppose that $A_0(z), \ldots, A_n(z)$ are entire functions satisfying*

$$\max\{\sigma(A_2), \ldots, \sigma(A_n)\} = \rho < \max\{\sigma(A_j) : j = 0, 1\},$$

and $A_1(z), A_0(z)$ are nontrivial solutions of (1.3) and (1.4), respectively. Suppose that $A_0(z)$ and $A_1(z)$ satisfy one of the following conditions:

(i) $k \neq m$;

(ii) $k = m, \arg a_k \neq \arg b_m$, the number of accumulation rays of the zero sequence of $A_1(z)$ is less than $m+2$;

(iii) $k = m$ and $a_k = cb_m$, where $c > 0$ and $c \neq 1$.

If $f(z)$ is a nontrivial meromorphic solution of (1.1), *then*

$$\sigma(f) \geq 1 + \max\{\sigma(A_j) : 0 \leq j \leq n\}.$$

2. Auxiliary lemmas

First we state some notations. Let $\alpha < \beta$ be such that $\beta - \alpha < 2\pi$, and let $r > 0$. Denote

$$S(\alpha, \beta) = \{z : \alpha < \arg z < \beta\},$$
$$S(\alpha, \beta, r) = \{z : \alpha < \arg z < \beta\} \cap \{z : |z| < r\}.$$

For simplicity, set $S = S(\alpha, \beta)$. Let $\overline{S}$ denote the closure of S. Suppose that A is an entire function of order $\sigma \in (0, +\infty)$. We say that A blows up exponentially in $\overline{S}$ if for any $\theta \in (\alpha, \beta)$ the relation

$$\lim_{r\to\infty} \frac{\log\log |A(re^{i\theta})|}{\log r} = \sigma$$

holds. We also say that A decays to zero exponentially in $\overline{S}$ if for any $\theta \in (\alpha, \beta)$ the relation

$$\lim_{r\to\infty} \frac{\log\log |A(re^{i\theta})|^{-1}}{\log r} = \sigma$$

holds.

The following Lemma 2.1, originally due to Hille [13], can also be found in [14] and [15]. It is about the asymptotic properties of solutions of (1.2) and plays an important role in proving Theorem 1.7. The method used in proving the lemma is typically referred to as the method of asymptotic integration.

Lemma 2.1 *Let $A(z)$ be a nontrivial solution of $\omega'' + P(z)\omega = 0$, where $P(z) = a_n z^n + \cdots + a_0, a_n \neq 0$. Set $\theta_j = \frac{2j\pi - \arg a_n}{n+2}$ and $S_j = S(\theta_j, \theta_{j+1})$, where $j = 0, 1, 2, \ldots, n+1$ and $\theta_{n+2} = \theta_0 + 2\pi$. Then A has the following properties.*

(i) In each sector S_j, A either blows up or decays to zero exponentially.

(ii) If for some j, A decays to zero in S_j, then it must blow up in S_{j-1} and S_{j+1}. However, it is possible for A to blow up in many adjacent sectors.

(iii) If A decays to zero in S_j, then A has at most finitely many zeros in any closed sub-sector within $S_{j-1} \cup \overline{S_j} \cup S_{j+1}$.

(iv) If A blows up in S_{j-1} and S_j, then for each $\varepsilon > 0, A$ has infinitely many zeros in each sector $\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon)$ and, furthermore, as $r \to \infty$,

$$n(\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, A) = (1 + o(1)) \frac{2\sqrt{|a_n|}}{\pi(n+2)} r^{\frac{n+2}{2}},$$

where $n(\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, A)$ is the number of zeros of A in the region $\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r)$ counting multiplicity.

The following lemma on difference analogue of logarithmic derivative is due to Chiang [7].

Lemma 2.2 ([7]) *Let η_1, η_2 be two arbitrary complex numbers, and let $f(z)$ be a meromorphic* *function of finite order σ. Let $\varepsilon > 0$ be a given real constant. Then the following statements hold.*

(i) There exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero such that if $z = re^{i\varphi_0}$ satisfying $\varphi_0 \in [0, 2\pi) \setminus E_1$, then there is a constant $R_0 = R_0(\varphi_0) > 1$ such that for z satisfying $\arg z = \varphi_0$ and $|z| \geq R_0$, we have

$$\exp\{-r^{\sigma-1+\varepsilon}\} \leq |\frac{f(z+\eta_1)}{f(z+\eta_2)}| \leq \exp\{r^{\sigma-1+\varepsilon}\}. \tag{2.1}$$

(ii) There exists a set $E_2 \subset (1, +\infty)$ with finite logarithmic measure such that for all $|z| = r \notin E_2 \cup [0, 1]$, we have (2.1).

Lemma 2.3 ([12]) *Let f and g be linearly independent solutions of (1.2), where $P(z) = a_n z^n + \cdots + a_0$ is a polynomial of degree n with $a_n \neq 0$. Set $h = g/f$ and $\alpha = \frac{n+2}{2}$. Then the following hold:*

(1) $T(r, h) = (1 + o(1))\frac{2\sqrt{|a_n|}}{\pi\alpha} r^\alpha$ as $r \to \infty$.

(2) There exists at most $n + 2$ distinct values $b_1, b_2, \ldots, b_m$ in the extended complex plane with the following properties:

(i) For each $k, \delta(b_k, h) = \Delta(b_k, h)$, and $\Delta(b_k, h)$ is a positive integral multiple of $\frac{1}{\alpha}$.

(ii) $\sum_{k=1}^{m} \delta(b_k, h) = 2$.

(iii) If $b \neq b_k$ for $k = 1, \ldots, m$, then as $r \to \infty, N(r, \frac{1}{h-b}) = (1 + o(1))T(r, h)$.

Lemma 2.4 ([12]) *Let $f \not\equiv 0$ be any solution of (1.2), where $P(z) = a_n z^n + \cdots + a_0$ is a polynomial of degree n with $a_n \neq 0$, and set $\alpha = \frac{n+2}{2}$. Then $p(f)$ is an even number and as $r \to \infty$, the following three formulas hold:*

$$n(r, \frac{1}{f}) = (1 + o(1))\frac{2\alpha - p(f)}{\pi\alpha}\sqrt{|a_n|}r^\alpha,$$
$$N(r, \frac{1}{f}) = (1 + o(1))\frac{2\alpha - p(f)}{\pi\alpha^2}\sqrt{|a_n|}r^\alpha,$$
$$T(r, f) = (1 + o(1))\frac{4\alpha - p(f)}{2\pi\alpha^2}\sqrt{|a_n|}r^\alpha.$$

Hence $\delta(0, f) = \Delta(0, f) = \frac{p(f)}{4\alpha - p(f)}$.

The following result gives the estimation of the maximum modulus of nontrivial solution of $\omega'' + P(z)\omega = 0$, where $P(z) = a_n z^n + \cdots + a_0$.

Lemma 2.5 ([12]) *Let A be defined as in Lemma 2.1. Then the following equality holds:*

$$\log M(r, A) = (1 + o(1))\frac{2\sqrt{|a_n|}}{n+2} r^{\frac{n+2}{2}}, \text{ as } r \to \infty.$$

3. Proofs of Theorems

Theorem 1.7 can be verified by following the proof of Theorem 1.8 in [16] with suitable modifications.

Proof of Theorem 1.7 Suppose that $f(z)$ is a nontrivial meromorphic solution of (1.1) with $\sigma(f)=\sigma<+\infty$. By (1.1), we obtain

$$-A_0(z)=A_n(z)\frac{f(z+n)}{f(z)}+\cdots+A_1(z)\frac{f(z+1)}{f(z)}. \tag{3.1}$$

Using Lemma 2.2, we get that for any $\varepsilon>0$, there exists a set $E\subset[0,2\pi)$ that has linear measure zero such that if $z=re^{i\varphi_0}$ satisfying $\varphi_0\in[0,2\pi)\setminus E$, there is a constant $R_0=R_0(\varphi_0)>1$ such that for z satisfying $\arg z=\varphi_0$ and $|z|\geq R_0$, we have

$$|\frac{f(z+l)}{f(z)}|\leq\exp\{r^{\sigma-1+\varepsilon}\},\quad l=1,\ldots,n. \tag{3.2}$$

From condition (ii) of Theorem 1.7 and the definition of order, we see that for any $\varepsilon>0$,

$$|A_j(z)|\leq\exp\{r^{\rho+\varepsilon}\},\quad j=2,\ldots,n, \tag{3.3}$$

for sufficient large $r=|z|$.

Using Lemma 2.1, set $\theta_j=\frac{2j\pi-\arg a_k}{k+2}$ and $S_j=\{z:\theta_j<\arg z<\theta_{j+1}\}$, $j=0,1,\ldots,k+1, \theta_{k+2}=\theta_0+2\pi$. Since the number of accumulation rays of the zero sequence of $A_1(z)$ is less than $k+2$, we can get that $p(A_1)\geq 2$. It follows from Lemma 2.1 that there exists at least one sector of the $k+2$ sectors S_j, such that $A_1(z)$ decays to zero exponentially, without loss of generality, say $S_0=\{z:\theta_0<\arg z<\theta_1\}$. That is, for any $\theta\in(\theta_0,\theta_1)$,

$$\lim_{r\to\infty}\frac{\log\log|A_1(re^{i\theta})|^{-1}}{\log r}=\frac{k+2}{2}. \tag{3.4}$$

Next we claim that $A_0(z)$ must blow up exponentially in S_0. Suppose on the contrary that A_0 decays to zero exponentially in S_0. It follows from Lemma 2.3 that as $r\to\infty$

$$N(r,\frac{1}{h-b})=(1+o(1))T(r,h)=(1+o(1))\frac{2\sqrt{|a_k|}}{\pi\alpha}r^{\alpha} \tag{3.5}$$

holds for any $b\in\mathbb{C}$, with at most finitely many exceptions, where $h=\frac{A_1}{A_0},\alpha=\frac{k+2}{2}$. Set $\omega=A_1-bA_0$. Then ω is a solution of (1.2). Recall that both $A_0(z)$ and $A_1(z)$ decay to zero exponentially in S_0 simultaneously. From this fact and Lemma 2.1, we can get ω also decays to zero exponentially in S_0. Thus $p(\omega)\geq 2$. However, it follows from Lemma 2.4 that

$$N(r,\frac{1}{h-b})=N(r,\frac{1}{\omega})=(1+o(1))\frac{2\alpha-p(\omega)}{\pi\alpha^2}\sqrt{|a_k|}r^{\alpha} \tag{3.6}$$

as $r\to\infty$. From (3.5) and (3.6), we see that $p(\omega)=0$. This is absurd. Hence $A_0(z)$ blows up exponentially in S_0. This is for any $\theta\in(\theta_0,\theta_1)$,

$$\lim_{r\to\infty}\frac{\log\log|A_0(re^{i\theta})|}{\log r}=\frac{k+2}{2}. \tag{3.7}$$

From (3.1)–(3.4) and (3.7), we have for any $\varepsilon>0,\theta\in(\theta_0,\theta_1)\setminus E$,

$$\begin{aligned}\exp\{r^{\frac{k+2}{2}-\varepsilon}\}&\leq|A_0(re^{i\theta})|\leq|A_n(z)||\frac{f(z+n)}{f(z)}|+\cdots|A_2(z)||\frac{f(z+2)}{f(z)}|+|A_1(z)||\frac{f(z+1)}{f(z)}|\\&\leq(n-1)\exp\{r^{\rho+\varepsilon}\}\exp\{r^{\sigma-1+\varepsilon}\}+o(1)\exp\{r^{\sigma-1+\varepsilon}\}\\&\leq n\exp\{r^{\rho+\varepsilon}\}\exp\{r^{\sigma-1+\varepsilon}\}.\end{aligned} \tag{3.8}$$

From (3.8), we get that $\sigma \geq 1+\frac{k+2}{2}$. Hence, the conclusion of Theorem 1.7 holds. This completes the proof. □

Proof of Corollary 1.8 Since $A_1(z)$ is a solution of (1.2), from Lemma 2.4, we know that $\delta(0, A_1) = \frac{p(A_1)}{4\alpha - p(A_1)}$, where $\alpha = \frac{k+2}{2}$. It follows from $\delta(0, A_1) > 0$ that $p(A_1) > 0$, which means that the number of accumulation rays of the zero sequence of A_1 is less than $k+2$. Therefore, we get the conclusion of Corollary 1.8 from Theorem 1.7. □

Proof of Theorem 1.9 (i) It follows from Bank and Laine's result [17, Theorem 1] that $\sigma(A_0) = \frac{m+2}{2}$ and $\sigma(A_1) = \frac{k+2}{2}$. Without loss of generality, we assume that $k < m$. Therefore, we have $\max\{\sigma(A_1), \sigma(A_2), \ldots, \sigma(A_n)\} < \sigma(A_0) = \frac{m+2}{2}$. Hence the conclusion follows from Theorem 1.1.

(ii) The condition implies that the set of accumulation rays of the zero sequence of $A_0(z)$ and $A_1(z)$ are not the same. From Lemma 2.1, there exists a sector $S(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$, such that for any $\theta \in (\alpha, \beta)$, (3.4) and (3.7) hold. Thus using a similar reasoning as in the proof of Theorem 1.7, we can get the conclusion.

(iii) Without loss of generality, we assume that $0 < c < 1$. Thus $|a_k| < |b_m|$. Suppose that $\sigma(f) = \sigma < 1+\frac{k+2}{2}$. By Lemma 2.2, there exists a set $E_2 \subset (1, \infty)$ with finite logarithmic measure such that for all $|z| = r \notin E_2 \cup [0, 1]$, (3.2) holds. From the condition $\max\{\sigma(A_2), \ldots, \sigma(A_n)\} = \rho < \frac{k+2}{2}$, we get that for sufficient small $\varepsilon > 0$, (3.3) holds for sufficient large $r = |z|$. Since $A_0(z)$ and $A_1(z)$ are nontrivial solutions of (1.3) and (1.4), respectively, by Lemma 2.5, as $r \to \infty$, the following equalities hold:

$$\log M(r, A_1) = (1+o(1))\frac{2\sqrt{|a_k|}}{k+2} r^{\frac{k+2}{2}} \tag{3.9}$$

and

$$\log M(r, A_0) = (1+o(1))\frac{2\sqrt{|b_m|}}{k+2} r^{\frac{k+2}{2}}. \tag{3.10}$$

Now we choose a sequence of points $\{z_l\}$ tending to infinity satisfying $|z_l| = r_l \in (1, +\infty) \setminus E_2$, such that

$$|A_0(z_l)| = M(r_l, A_0). \tag{3.11}$$

Combining (3.1)–(3.3), (3.9)–(3.11), as $l \to \infty$, we get

$$\begin{aligned}
(1+o(1))\frac{2\sqrt{|b_m|}}{k+2} r_l^{\frac{k+2}{2}} &= \log M(r_l, A_0) = \log |A_0(z_l)| \\
&\leq \log\{(n-1)\exp\{r_l^{\rho+\varepsilon}\}\exp\{r_l^{\sigma-1+\varepsilon}\} + |A_1(z_l)|\exp\{r_l^{\sigma-1+\varepsilon}\}\} \\
&\leq \log(n-1) + r_l^{\rho+\varepsilon} + 2r_l^{\sigma-1+\varepsilon} + \log|A_1(z_l)| + O(1) \\
&\leq r_l^{\rho+\varepsilon} + 2r_l^{\sigma-1+\varepsilon} + (1+o(1))\frac{2\sqrt{|a_k|}}{k+2} r_l^{\frac{k+2}{2}} + O(1) \\
&= (1+o(1))\frac{2\sqrt{|a_k|}}{k+2} r_l^{\frac{k+2}{2}}\Big[1 + \frac{r_l^{\rho+\varepsilon} + 2r_l^{\sigma-1+\varepsilon} + O(1)}{(1+o(1))\frac{2\sqrt{|a_k|}}{k+2} r_l^{\frac{k+2}{2}}}\Big] \\
&= (1+o(1))\frac{2\sqrt{|a_k|}}{k+2} r_l^{\frac{k+2}{2}}.
\end{aligned}$$

This implies that $|b_m| \leq |a_k|$, which contradicts the condition $|b_m| > |a_k|$. Hence $\sigma(f) \geq 1 + \max\{\sigma(A_j) : 0 \leq j \leq n\}$. This completes the proof of Theorem 1.9. □

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