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A Characterization of the Mixed Radial-Angular λ-Central Bounded Mean Oscillation Spaces

Ronghui LIU[1,*], **Suixin HE**[2]

1. *College of Mathematics and Statistics, Northwest Normal University, Gansu* 730070*, P. R. China;*
2. *School of Mathematics and Statistics, Yili Normal University, Xinjiang* 835000*, P. R. China*

Abstract In this paper, we establish a characterization of the mixed radial-angular λ-central bounded mean oscillation spaces via the boundedness of the commutators H_b and its dual H_b^* with a function $b \in \mathrm{CMOL}_{\mathrm{rad}}^{p_2,\lambda} L_{\mathrm{ang}}^{p_1}(\mathbb{R}^n)$.

Keywords Hardy operator; commutator; mixed radial-angular space

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1. Introduction

The classical Hardy operator, as the most fundamental averaging operator, is defined by

$$\mathcal{H}f(x) = \frac{1}{x}\int_0^x f(t)\mathrm{d}t,$$

where the function f is a nonnegative integrable function on $\mathbb{R}^+$ and $x > 0$. Its adjoint operator $\mathcal{H}^*$ is the following

$$\mathcal{H}^* f(x) = \int_x^\infty \frac{f(t)}{t}\mathrm{d}t.$$

The most celebrated integral inequality, due to Hardy [1], can be stated as follows.

Theorem 1.1 ([1]) *If $1 < p < \infty$, then*

$$\|\mathcal{H}f\|_{L^p(\mathbb{R}^+)} \le \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^+)}, \quad \|\mathcal{H}^*f\|_{L^q(\mathbb{R}^+)} \le \frac{p}{p-1}\|f\|_{L^q(\mathbb{R}^+)},$$

where $1/p + 1/q = 1$. Moreover

$$\|\mathcal{H}\|_{L^p(\mathbb{R}^+)\to L^p(\mathbb{R}^+)} = \|\mathcal{H}^*\|_{L^q(\mathbb{R}^+)\to L^q(\mathbb{R}^+)}.$$

Let b be a locally integrable function on $\mathbb{R}^+$. The commutators of Hardy operators are defined by

$$\mathcal{H}_b f = b\mathcal{H}f - \mathcal{H}(fb), \quad \mathcal{H}_b^* f = b\mathcal{H}^* f - \mathcal{H}^*(fb).$$

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* Corresponding author
E-mail address: rhliu@nwnu.edu.cn (Ronghui LIU); hesuixinmath@126.com (Suixin HE)

Subsequently, Long and Wang [2] proved Hardy's integral inequalities for commutators generated by the classical Hardy operators $\mathcal{H}$ and one-side dyadic CMO functions.

Definition 1.2 *Let b be a locally integrable function on $\mathbb{R}^+$, $1<p<\infty$. It is said that b is a one-side dyadic $\mathrm{CMO}^p(\mathbb{R}^+)$ function, if*

$$\|b\|_{\mathrm{CMO}^p(\mathbb{R}^+)}=\sup_{i\in\mathbb{Z}}\Big(\frac{1}{2^i}\int_0^{2^i}|b(t)-b_{(0,2^i]}|^p\mathrm{d}t\Big)^{1/p},$$

where

$$b_{(0,2^i]}=\frac{1}{2^i}\int_0^{2^i}b(t)\mathrm{d}t.$$

Theorem 1.3 ([2]) *Let b be a one-side dyadic $\mathrm{CMO}^p(\mathbb{R}^+)$ function. If $1<p<\infty$ and $1/p+1/p'=1$, then*

$$\|\mathcal{H}_bf\|_{L^p(\mathbb{R}^+)}\lesssim\|b\|_{\mathrm{CMO}^{\max\{p,p'\}}(\mathbb{R}^+)}\|f\|_{L^p(\mathbb{R}^+)}$$

and

$$\|\mathcal{H}_b^*f\|_{L^p(\mathbb{R}^+)}\lesssim\|b\|_{\mathrm{CMO}^{\max\{p,p'\}}(\mathbb{R}^+)}\|f\|_{L^p(\mathbb{R}^+)}.$$

For the multidimensional case $n\geq 2$, generally speaking, there exist two different definitions. One is the rectangle averaging operator, and its norm depends on the dimensions. Another version is the n-dimensional spherical averaging operator, which was introduced by Christ and Grafakos in [3] as follows:

$$\mathrm{H}f(x)=\frac{1}{\nu_n|x|^n}\int_{|t|\leq|x|}f(t)\mathrm{d}t,\quad x\in\mathbb{R}^n\setminus\{0\},$$

where ν_n is the volume of the unit ball in $\mathbb{R}^n$. However, the norm of H is different from the rectangle averaging operator, its norm on $L^p(\mathbb{R}^n)$ was evaluted and found to be equal to that of the 1-dimensional averaging operator. $\|\mathrm{H}\|_{L^p\to L^p}$ $(p>1)$, that is to say, does not depend on the dimension of the space.

Definition 1.4 *Let $1<p<\infty$. A function $f\in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ is said to belong to the spaces $\mathrm{CMO}^p(\mathbb{R}^n)$, if*

$$\|f\|_{\mathrm{CMO}^p(\mathbb{R}^n)}=\sup_{r>0}\Big(\frac{1}{|B(0,r)|}\int_{B(0,r)}|f(x)-f_B|^p\mathrm{d}x\Big)^{1/p}<\infty,$$

where

$$f_B=\frac{1}{|B(0,r)|}\int_{B(0,r)}f(x)\mathrm{d}x.$$

In 2007, Fu et al. [4] proved the following result.

Theorem 1.5 *Let $1<p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$. Then both H_b and H_b^* are bounded from $L^p(\mathbb{R}^n)$ if and only if $b\in\mathrm{CMO}^{\max\{p,p'\}}(\mathbb{R}^n)$.*

In 2000, Alvarez, Guzmán-Partida and Lakey [5] introduced λ-central bounded mean oscillation spaces as follows.

Definition 1.6 *Let $1 < q < \infty$ and $-1/q < \lambda < 1/n$. A function $f \in L^q_{\text{loc}}(\mathbb{R}^n)$ is said to belong to the λ-central bounded mean oscillation space $\text{CMO}^{q,\lambda}(\mathbb{R}^n)$ if*

$$\|f\|_{\text{CMO}^{q,\lambda}(\mathbb{R}^n)} = \sup_{r>0} \Big(\frac{1}{|B(0,r)|^{1+\lambda q}} \int_{B(0,r)} |f(x) - f_B|^q \text{d}x \Big)^{1/q} < \infty.$$

Remark 1.7 (1) If $\lambda = 0$, then λ-central bounded mean oscillation spaces reduce to the spaces $\text{CMO}^p(\mathbb{R}^n)$, that is

$$\text{CMO}^{p,0}(\mathbb{R}^n) = \text{CMO}^p(\mathbb{R}^n), \quad 1 < p < \infty.$$

(2) If $1 < p_2 \le \tilde{p_2} < \infty$ and $-1/p_2 < \lambda < 1/n$, then we also have

$$\text{CMO}^{\tilde{p_2},\lambda}(\mathbb{R}^n) \subset \text{CMO}^{p_2,\lambda}(\mathbb{R}^n).$$

(3) If $-1/\tilde{p} < \lambda_1 \le \lambda_2 < 1/n$, then

$$\text{CMO}^{\tilde{p},\lambda_1}(\mathbb{R}^n) \subset \text{CMO}^{\tilde{p},\lambda_2}(\mathbb{R}^n).$$

In 2013, Zhao et al. [6] obtained the following conclusion.

Theorem 1.8 *Let $1 < p, q < \infty$, $0 \le \lambda < 1/n$, $\frac{1}{p} - \frac{1}{q} = \lambda$. Then both H_b and H^*_b are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in \text{CMO}^{\max\{q,p'\},\lambda}(\mathbb{R}^n)$.*

On the other hand, the mixed radial-angular spaces $L^p_{\text{rad}} L^{\tilde{p}}_{\text{ang}}(\mathbb{R}^n)$, as a formal extension of the Lebesgue spaces $L^p(\mathbb{R}^n)$, were introduced to the study of regularity and some important estimates, such as angular regularity and Strichartz estimates [7–11]. The boundedness for certain classical operators in harmonic analysis on mixed radial-angular spaces $L^p_{\text{rad}} L^{\tilde{p}}_{\text{ang}}(\mathbb{R}^n)$ was established successively in [7–9, 12–15].

Inspired by the results above, it is natural to introduce the mixed radial-angular CMO spaces and λ-central bounded mean oscillation spaces, etc. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$, $n \ge 2$, with normalized Lebesgue measure $\text{d}\sigma = \text{d}\sigma(\cdot)$. For any $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, applying the spherical coordinate formula, we write

$$\begin{aligned} \|f\|_{L^p(\mathbb{R}^n)} &= \Big(\int_0^\infty \int_{S^{n-1}} |f(r\theta)|^p \text{d}\sigma(\theta) r^{n-1} \text{d}r \Big)^{1/p} \\ &= \Big(\int_0^\infty \|f(r\cdot)\|^p_{L^p(S^{n-1})} r^{n-1} \text{d}r \Big)^{1/p}. \end{aligned}$$

Therefore, from the perspective of radial and angular integrability, Lebesgue norms can be interpreted as certain special norms with the same integrability in the radial and angular directions. Inspired by this version, we naturally consider the case of Lebesgue norms with different integrability in the radial and angular directions, namely,

$$\|f\|_{L^p_{\text{rad}} L^{\tilde{p}}_{\text{ang}}(\mathbb{R}^n)} := \left(\int_0^\infty \|f(r\cdot)\|^p_{L^{\tilde{p}}(S^{n-1})} r^{n-1} \text{d}r \right)^{1/p}, \quad 1 \le p, \ \tilde{p} \le \infty,$$

when $p = \infty$ or $\tilde{p} = \infty$, we just need to make the usual modifications in the above definition, but we do not use these cases in current work.

Similarly, for the norm $\|f\|_{\text{CMO}^p(\mathbb{R}^n)}$, we can also rewrite as follows:

$$\|f\|_{\text{CMO}^p(\mathbb{R}^n)} = \sup_{r>0} \Big(\frac{1}{\nu_n r^n} \int_0^r \int_{S^{n-1}} |f(\rho\theta) - f_B|^p \text{d}\sigma(\theta) \rho^{n-1} \text{d}\rho \Big)^{1/p},$$

and along this line of distinguishing between the radial and angular directions integrabilities, the author et al. in [16] introduces the mixed radial-angular homogeneous CMO spaces and the mixed radial-angular λ-central bounded mean oscillation spaces, respectively.

Definition 1.9 *Let $1 < p_1,\ p_2 < \infty$. A function $f \in L^{p_2}_{\mathrm{rad,loc}} L^{p_1}_{\mathrm{ang}}((0,\infty)\times S^{n-1})$ is said to belong to the mixed radial-angular homogeneous* CMO *spaces* $\mathrm{CMOL}^{p_2}_{\mathrm{rad}} L^{p_1}_{\mathrm{ang}}(\mathbb{R}^n)$, *if*

$$\|f\|_{\mathrm{CMOL}^{p_2}_{\mathrm{rad}} L^{p_1}_{\mathrm{ang}}(\mathbb{R}^n)} = \sup_{r>0} \Big(\frac{1}{\nu_n r^n} \int_0^r \Big(\int_{S^{n-1}} |f(\rho\theta) - f_B|^{p_1} \mathrm{d}\sigma(\theta) \Big)^{p_2/p_1} \rho^{n-1} \mathrm{d}\rho \Big)^{1/p_2},$$

where $f \in L^{p_2}_{\mathrm{rad,loc}} L^{p_1}_{\mathrm{ang}}((0,\infty)\times S^{n-1})$ means that the function f defined on $(0, r_0)\times S^{n-1} \subset (0,\infty)\times S^{n-1}$ satisfies $\|f\|_{L^{p_2}_{\mathrm{rad}} L^{p_1}_{\mathrm{ang}}((0,r_0)\times S^{n-1})} < \infty$ for any $r_0 \in (0,\infty)$.

Remark 1.10 (1) If $1 < p_1 = p_2 = p < \infty$, then

$$\mathrm{CMOL}^{p_2}_{\mathrm{rad}} L^{p_1}_{\mathrm{ang}}(\mathbb{R}^n) = \mathrm{CMO}^p(\mathbb{R}^n).$$

(2) If $1 < p_1 \le \tilde{p_1} < \infty$, then by the Hölder inequality, we have

$$\mathrm{CMOL}^{p_2}_{\mathrm{rad}} L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n) \subset \mathrm{CMOL}^{p_2}_{\mathrm{rad}} L^{p_1}_{\mathrm{ang}}(\mathbb{R}^n).$$

(3) If $1 < p_2 \le \tilde{p_2} < \infty$, then by the Hölder inequality, we also have

$$\mathrm{CMOL}^{\tilde{p_2}}_{\mathrm{rad}} L^{p_1}_{\mathrm{ang}}(\mathbb{R}^n) \subset \mathrm{CMOL}^{p_2}_{\mathrm{rad}} L^{p_1}_{\mathrm{ang}}(\mathbb{R}^n).$$

The mixed radial-angular homogeneous λ-central bounded mean oscillation spaces are defined as follows.

Definition 1.11 *Let $1 < p_1,\ p_2 < \infty$ and $-1/p_2 < \lambda < 1/n$. A function $f \in L^{p_2}_{\mathrm{rad,loc}} L^{p_1}_{\mathrm{ang}}((0,\infty)\times S^{n-1})$ is said to belong to the mixed radial-angular homogeneous λ-central bounded mean oscillation spaces* $\mathrm{CMOL}^{p_2,\lambda}_{\mathrm{rad}} L^{p_1}_{\mathrm{ang}}(\mathbb{R}^n)$, *if*

$$\begin{aligned}&\|f\|_{\mathrm{CMOL}^{p_2,\lambda}_{\mathrm{rad}} L^{p_1}_{\mathrm{ang}}(\mathbb{R}^n)} \\ &= \sup_{r>0} \Big(\frac{1}{\nu_n r^{n+n\lambda p_2}} \int_0^r \Big(\int_{S^{n-1}} |f(\rho\theta) - f_B|^{p_1} \mathrm{d}\sigma(\theta) \Big)^{p_2/p_1} \rho^{n-1} \mathrm{d}\rho \Big)^{1/p_2} < \infty.\end{aligned}$$

Remark 1.12 (1) If $1 < p_1 = p_2 = p < \infty$ and $-1/p_2 < \lambda < 1/n$, then

$$\mathrm{CMOL}^{p_2,\lambda}_{\mathrm{rad}} L^{p_1}_{\mathrm{ang}}(\mathbb{R}^n) = \mathrm{CMO}^{p,\lambda}(\mathbb{R}^n).$$

(2) If $1 < p_1 \le \tilde{p_1} < \infty$, and $-1/p_2 < \lambda < 1/n$, then we have

$$\mathrm{CMOL}^{p_2,\lambda}_{\mathrm{rad}} L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n) \subset \mathrm{CMOL}^{p_2,\lambda}_{\mathrm{rad}} L^{p_1}_{\mathrm{ang}}(\mathbb{R}^n).$$

(3) If $1 < p_2 \le \tilde{p_2} < \infty$ and $-1/p_2 < \lambda < 1/n$, then we also have

$$\mathrm{CMOL}^{\tilde{p_2},\lambda}_{\mathrm{rad}} L^{p_1}_{\mathrm{ang}}(\mathbb{R}^n) \subset \mathrm{CMOL}^{p_2,\lambda}_{\mathrm{rad}} L^{p_1}_{\mathrm{ang}}(\mathbb{R}^n).$$

(4) If $-1/\tilde{p} < \lambda_1 \le \lambda_2 < 1/n$, then

$$\mathrm{CMOL}^{\tilde{p},\lambda_1}_{\mathrm{rad}} L^{p_1}_{\mathrm{ang}}(\mathbb{R}^n) \subset \mathrm{CMOL}^{\tilde{p},\lambda_2}_{\mathrm{rad}} L^{p_1}_{\mathrm{ang}}(\mathbb{R}^n).$$

A question that arises naturally is whether or not the commutators H_b and H_b^* are bounded on the mixed radial-angular $L^{p_1}_{\rm rad}L^{\tilde{p}_1}_{\rm ang}(\mathbb{R}^n)$? Furthermore, if the answer is positive, what kind of function space does the symbol b belong to ?

The main purpose in this paper is to solve the above problems, and our result can be formulated as follows.

Theorem 1.13 *Suppose that $1 < p_1, \tilde{p_1}, \tilde{p}, p < \infty$, $0 \le \lambda < 1/n$ and $\lambda = \frac{1}{p_1} - \frac{1}{p}$, then both the commutators H_b and H_b^* are bounded from $L^{p_1}_{\rm rad}L^{\tilde{p}_1}_{\rm ang}(\mathbb{R}^n)$ to $L^{p}_{\rm rad}L^{\tilde{p}}_{\rm ang}(\mathbb{R}^n)$ if and only if*

$$b \in \mathrm{CMOL}^{\max\{p,p_1'\},\lambda}_{\rm rad} L^{\max\{\tilde{p},\tilde{p_1}'\}}_{\rm ang}(\mathbb{R}^n).$$

Let $\lambda = 0$ in Theorem 1.13, we have

Corollary 1.14 *Suppose that $1 < p, \tilde{p}, \tilde{p}_1 < \infty$, then both the commutators H_b and H_b^* are bounded from $L^{p}_{\rm rad}L^{\tilde{p}_1}_{\rm ang}(\mathbb{R}^n)$ to $L^{p}_{\rm rad}L^{\tilde{p}}_{\rm ang}(\mathbb{R}^n)$ if and only if $b \in \mathrm{CMOL}^{\max\{p,p'\}}_{\rm rad} L^{\max\{\tilde{p},\tilde{p_1}'\}}_{\rm ang}(\mathbb{R}^n)$.*

The rest of this paper is organized as follows. In Section 2, we will prove Theorem 1.13. We would like to remark that some ideas for our arguments are taken from [6, 16–18].

As a rule, $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $C_k = B_k \backslash B_{k-1}$ and $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, where χ_E is the characteristic function of set E, we use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \le CB$. The notation $f \approx g$ means that there exist constants $C_1, C_2 > 0$ such that $C_1 g \le f \le C_2 g$.

2. Proofs of main results

In this section, we provide the proof of Theorem 1.13.

Proof For convenience, we write $f(x)\chi_i = f_i(x)$. Then

$$\begin{aligned}
&\|\mathrm{H}_b(f)\|^p_{L^p_{\rm rad}L^{\tilde{p}}_{\rm ang}(\mathbb{R}^n)} = \int_0^\infty \Big(\int_{S^{n-1}} |\mathrm{H}_b(f)(r\theta_1)|^{\tilde{p}} \mathrm{d}\sigma(\theta_1)\Big)^{p/\tilde{p}} r^{n-1}\mathrm{d}r \\
&= \sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^k} \Big(\int_{S^{n-1}} \Big| \frac{1}{(\nu_n r^n)} \int_{|y|<r} f(y)(b(r\theta_1) - b(y))\mathrm{d}y \Big|^{\tilde{p}} \mathrm{d}\sigma(\theta_1)\Big)^{p/\tilde{p}} r^{n-1}\mathrm{d}r \\
&\lesssim \sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^k} \Big(\int_{S^{n-1}} \Big(\sum_{i=-\infty}^{k} \int_{C_i} \Big| f(y)(b(r\theta_1) - b(y))\Big| \mathrm{d}y\Big)^{\tilde{p}} \mathrm{d}\sigma(\theta_1)\Big)^{p/\tilde{p}} 2^{-knp} r^{n-1}\mathrm{d}r.
\end{aligned}$$

We first estimate the inner integral as follows:

$$\begin{aligned}
&\int_{C_i} \Big| f(y)(b(r\theta_1) - b(y))\Big| \mathrm{d}y \le \int_{2^{i-1}}^{2^i} \int_{S^{n-1}_y} |f(\rho\theta_2)(b(r\theta_1) - b(\rho\theta_2))| \mathrm{d}\sigma(\theta_2)\rho^{n-1}\mathrm{d}\rho \\
&\le \int_{2^{i-1}}^{2^i} \int_{S^{n-1}_y} |f(\rho\theta_2)(b(r\theta_1) - b_{B_k})| \mathrm{d}\sigma(\theta_2)\rho^{n-1}\mathrm{d}\rho + \\
&\quad \int_{2^{i-1}}^{2^i} \int_{S^{n-1}_y} |f(\rho\theta_2)(b(\rho\theta_2) - b_{B_k})| \mathrm{d}\sigma(\theta_2)\rho^{n-1}\mathrm{d}\rho \\
&=: I_1 + I_2. \qquad (2.1)
\end{aligned}$$

For I_1, by Hölder's inequality, $\frac{1}{\tilde{p}_1}+\frac{1}{\tilde{p}_1'}=1$, we get

$$\begin{aligned} I_1 &\lesssim |b(r\theta_1)-b_{B_k}|\int_{2^{i-1}}^{2^i}\Big(\int_{S_y^{n-1}}|f(\rho\theta_2)|^{\tilde{p}_1}\mathrm{d}\sigma(\theta_2)\Big)^{1/\tilde{p}_1}\rho^{n-1}\mathrm{d}\rho \\ &\lesssim |b(r\theta_1)-b_{B_k}|2^{in/p_1'}\|f\chi_i\|_{L_{\rm rad}^{p_1}L_{\rm ang}^{\tilde{p}_1}(\mathbb{R}^n)}. \end{aligned} \tag{2.2}$$

For I_2, a standard calculation tells us that

$$\begin{aligned} &|b(\rho\theta_2)-b_{B_k}|\le |b(\rho\theta_2)-b_{B_i}|+\sum_{j=i}^{k}|b_{B_j}-b_{B_{j-1}}| \\ &\le |b(\rho\theta_2)-b_{B_i}|+\sum_{j=i}^{k}\frac{1}{|B_j|}\int_{B_j}|b(y)-b_{B_{j+1}}|\mathrm{d}y \\ &\le |b(\rho\theta_2)-b_{B_i}|+\sum_{j=i}^{k}\frac{1}{|B_j|}\int_{B_{j+1}}|b(y)-b_{B_{j+1}}|\mathrm{d}y \\ &\le |b(\rho\theta_2)-b_{B_i}|+\sum_{j=i}^{k}\frac{1}{|B_j|}\int_0^{2^{j+1}}\int_{S^{n-1}}|b(\rho_1\theta_2)-b_{B_{j+1}}|\mathrm{d}\sigma(\theta_2)\rho_1^{n-1}\mathrm{d}\rho_1 \\ &\le |b(\rho\theta_2)-b_{B_i}|+\sum_{j=i}^{k}\frac{1}{|B_j|}\int_0^{2^{j+1}}\int_{S^{n-1}}|b(\rho_1\theta_2)-b_{B_{j+1}}|\mathrm{d}\sigma(\theta_2)\rho_1^{n-1}\mathrm{d}\rho_1 \\ &\lesssim |b(\rho\theta_2)-b_{B_i}|+\sum_{j=i}^{k}\frac{1}{|B_j|}\int_0^{2^{j+1}}\Big(\int_{S^{n-1}}|b(\rho_1\theta_2)-b_{B_{j+1}}|^{\tilde{p}}\mathrm{d}\sigma(\theta_2)\Big)^{1/\tilde{p}}\rho_1^{n-1}\mathrm{d}\rho_1 \\ &\lesssim |b(\rho\theta_2)-b_{B_i}|+\sum_{j=i}^{k}\frac{2^{jn/p'}}{|B_j|}\Big(\int_0^{2^{j+1}}\Big(\int_{S^{n-1}}|b(\rho_1\theta_2)-b_{B_{j+1}}|^{\tilde{p}}\mathrm{d}\sigma(\theta_2)\Big)^{p/\tilde{p}}\rho_1^{n-1}\mathrm{d}\rho\Big)^{1/p} \\ &\lesssim |b(\rho\theta_2)-b_{B_i}|+\sum_{j=i}^{k}2^{(j+1)n\lambda}\|b\|_{L_{\rm rad}^{p,\lambda}L_{\rm ang}^{\tilde{p}}(\mathbb{R}^n)} \\ &\lesssim |b(\rho\theta_2)-b_{B_i}|+(1-2^{n\lambda})^{-1}(2^{in\lambda}-2^{kn\lambda})\|b\|_{{\rm CMOL}_{\rm rad}^{p,\lambda}L_{\rm ang}^{\tilde{p}}(\mathbb{R}^n)}. \end{aligned}$$

By Hölder's inequality again, $\frac{1}{p_1}+\frac{1}{p_1'}=1$ and $\frac{1}{p_2}+\frac{1}{p_2'}=1$, we have

$$\begin{aligned} I_2 &\lesssim \int_{2^{i-1}}^{2^i}\int_{S_y^{n-1}}|f(\rho\theta_2)(b(\rho\theta_2)-b_{B_i})|\mathrm{d}\sigma(\theta_2)\rho^{n-1}\mathrm{d}\rho+ \\ &\quad 2^{in/p_1'}(1-2^{n\lambda})^{-1}(2^{in\lambda}-2^{kn\lambda})\|b\|_{{\rm CMOL}_{\rm rad}^{p,\lambda}L_{\rm ang}^{\tilde{p}}(\mathbb{R}^n)}\|f\|_{L_{\rm rad}^{p_1}L_{\rm ang}^{\tilde{p}_1}(\mathbb{R}^n)} \\ &=\int_{2^{i-1}}^{2^i}\Big(\int_{S_y^{n-1}}|f(\rho\theta_2)|^{\tilde{p}_1}\mathrm{d}\sigma(\theta_2)\Big)^{1/\tilde{p}_1}\Big(\int_{S_y^{n-1}}|b(\rho\theta_2)-b_{B_i}|^{\tilde{p}_1'}\mathrm{d}\sigma(\theta_2)\Big)^{1/\tilde{p}_1'}\rho^{n-1}\mathrm{d}\rho \\ &\lesssim 2^{in/p_1'+in\lambda}\|b\|_{{\rm CMOL}_{\rm rad}^{p_1',\lambda}L_{\rm ang}^{\tilde{p}_1'}(\mathbb{R}^n)}\|f\chi_i\|_{L_{\rm rad}^{p_1}L_{\rm ang}^{\tilde{p}_1}(\mathbb{R}^n)}+ \\ &\quad 2^{in/p_1'}(1-2^{n\lambda})^{-1}(2^{in\lambda}-2^{kn\lambda})\|b\|_{{\rm CMOL}_{\rm rad}^{p,\lambda}L_{\rm ang}^{\tilde{p}}(\mathbb{R}^n)}\|f\|_{L_{\rm rad}^{p_1}L_{\rm ang}^{\tilde{p}_1}(\mathbb{R}^n)}. \end{aligned} \tag{2.3}$$

Combining the estimates (2.1)–(2.3), together with the Hölder inequality, leads to that

$$\|{\rm H}_b(f)\|_{L_{\rm rad}^{p}L_{\rm ang}^{\tilde{p}}(\mathbb{R}^n)}^p$$

$$
\begin{aligned}
&\lesssim \sum_{k=-\infty}^{\infty}\int_{2^{k-1}}^{2^k}\Big(\int_{S^{n-1}}\Big(\sum_{i=-\infty}^{k}\int_{C_i}\Big|f(y)(b(r\theta_1)-b(y))\Big|\mathrm{d}y\Big)^{\tilde{p}}\mathrm{d}\sigma(\theta_1)\Big)^{p/\tilde{p}}2^{-knp}r^{n-1}\mathrm{d}r\\
&=\sum_{k=-\infty}^{\infty}\int_{2^{k-1}}^{2^k}\Big(\int_{S_x^{n-1}}|b(r\theta_1)-b_{B_k}|^{\tilde{p}}\Big|\sum_{i=-\infty}^{k}2^{in/p_1'}\|f\chi_i\|_{L^{p_1}_{\mathrm{rad}}L^{\tilde{p}_1}_{\mathrm{ang}}(\mathbb{R}^n)}\Big|^{\tilde{p}}\times\\
&\quad \mathrm{d}\sigma(\theta_1)\Big)^{p/\tilde{p}}2^{-knp}r^{n-1}\mathrm{d}r+\\
&\quad \|b\|^p_{\mathrm{CMOL}^{p_1',\lambda}_{\mathrm{rad}}L^{\tilde{p_1}'}_{\mathrm{ang}}(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\int_{2^{k-1}}^{2^k}\Big(\int_{S_x^{n-1}}\Big|\sum_{i=-\infty}^{k}2^{in/p_1'+in\lambda}\|f\chi_i\|_{L^{p_1}_{\mathrm{rad}}L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)}\Big|^{\tilde{p}}\times\\
&\quad \mathrm{d}\sigma(\theta_1)\Big)^{p/\tilde{p}}2^{-knp}r^{n-1}\mathrm{d}r+\\
&\quad \|b\|^p_{\mathrm{CMOL}^{p,\lambda}_{\mathrm{rad}}L^{\tilde{p}}_{\mathrm{ang}}(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\int_{2^{k-1}}^{2^k}\Big(\int_{S_x^{n-1}}\Big|\sum_{i=-\infty}^{k}2^{in/p_1'}\frac{2^{in\lambda}-2^{kn\lambda}}{1-2^{n\lambda}}\|f\chi_i\|_{L^{p_1}_{\mathrm{rad}}L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)}\Big|^{\tilde{p}}\times\\
&\quad \mathrm{d}\sigma(\theta_1)\Big)^{p/\tilde{p}}2^{-knp}r^{n-1}\mathrm{d}r\\
&\lesssim \sum_{k=-\infty}^{\infty}\int_{2^{k-1}}^{2^k}\Big(\int_{S_x^{n-1}}|b(r\theta_1)-b_{B_k}|^{\tilde{p}}\mathrm{d}\sigma(\theta_1)\Big)^{p/\tilde{p}}2^{-knp}r^{n-1}\mathrm{d}r\times\\
&\quad \Big(\sum_{i=-\infty}^{k}2^{in/p_1'}\|f\chi_i\|_{L^{p_1}_{\mathrm{rad}}L^{\tilde{p}_1}_{\mathrm{ang}}(\mathbb{R}^n)}\Big)^p+\\
&\quad \|b\|^p_{\mathrm{CMOL}^{p_1',\lambda}_{\mathrm{rad}}L^{\tilde{p_1}'}_{\mathrm{ang}}(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\int_{2^{k-1}}^{2^k}\Big(\sum_{i=-\infty}^{k}2^{in/p_1'+in\lambda}\|f\chi_i\|_{L^{p_1}_{\mathrm{rad}}L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)}\Big)^p2^{-knp}r^{n-1}\mathrm{d}r+\\
&\quad \|b\|^p_{\mathrm{CMOL}^{p,\lambda}_{\mathrm{rad}}L^{\tilde{p_1}'}_{\mathrm{ang}}(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\Big(\sum_{i=-\infty}^{k}2^{in/p_1'}\frac{2^{in\lambda}-2^{kn\lambda}}{1-2^{n\lambda}}\|f\chi_i\|_{L^{p_1}_{\mathrm{rad}}L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)}\Big)^p2^{-knp}r^{n-1}\mathrm{d}r\\
&\lesssim \|b\|^p_{\mathrm{CMOL}^{p,\lambda}_{\mathrm{rad}}L^{\tilde{p}}_{\mathrm{ang}}(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}2^{kn+kn\lambda p}\Big(\sum_{i=-\infty}^{k}2^{in/p_1'}\|f\chi_i\|_{L^{p_1}_{\mathrm{rad}}L^{\tilde{p}_1}_{\mathrm{ang}}(\mathbb{R}^n)}\Big)^p2^{-knp}+\\
&\quad \|b\|^p_{\mathrm{CMOL}^{p_1',\lambda}_{\mathrm{rad}}L^{\tilde{p_1}'}_{\mathrm{ang}}(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\Big(\sum_{i=-\infty}^{k}2^{in/p_1'+in\lambda}\|f\chi_i\|_{L^{p_1}_{\mathrm{rad}}L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)}\Big)^p2^{kn-knp}+\\
&\quad \|b\|^p_{\mathrm{CMOL}^{p,\lambda}_{\mathrm{rad}}L^{\tilde{p}}_{\mathrm{ang}}(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\Big(\sum_{i=-\infty}^{k}2^{in/p_1'}\frac{2^{in\lambda}-2^{kn\lambda}}{1-2^{n\lambda}}\|f\chi_i\|_{L^{p_1}_{\mathrm{rad}}L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)}\Big)^p2^{kn-knp}\\
&\lesssim \|b\|^p_{\mathrm{CMOL}^{\max\{p,p_1'\},\lambda}_{\mathrm{rad}}L^{\max\{\tilde{p},\tilde{p_1}'\}}_{\mathrm{ang}}(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\|f\chi_k\|^p_{L^{p_1}_{\mathrm{rad}}L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)}+\\
&\quad \|b\|^p_{\mathrm{CMOL}^{\max\{p,p_1'\},\lambda}_{\mathrm{rad}}L^{\max\{\tilde{p},\tilde{p_1}'\}}_{\mathrm{ang}}(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}2^{kn-knp}\Big(\sum_{i=-\infty}^{k}2^{in/p_1'}\frac{2^{in\lambda}-2^{kn\lambda}}{1-2^{n\lambda}}\|f\chi_i\|_{L^{p_1}_{\mathrm{rad}}L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)}\Big)^p\\
&=: J_1+J_2.
\end{aligned}
$$

For J_1, it follows from $p_1 < p$ that

$$
\begin{aligned}
J_1 &= \|b\|^p_{\mathrm{CMOL}_{\mathrm{rad}}^{\max\{p,p_1'\},\lambda} L_{\mathrm{ang}}^{\max\{\tilde{p},\tilde{p_1}'\}}(\mathbb{R}^n)} \sum_{k=-\infty}^{\infty} \|f\chi_i\|^p_{L^{p_1}_{\mathrm{rad}} L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)} \\
&\le \|b\|^p_{\mathrm{CMOL}_{\mathrm{rad}}^{\max\{p,p_1'\},\lambda} L_{\mathrm{ang}}^{\max\{\tilde{p},\tilde{p_1}'\}}(\mathbb{R}^n)} \sum_{k=-\infty}^{\infty} \Big(\int_{2^{k-1}}^{2^k} \Big(\int_{S^{n-1}} |f(r\theta)|^{\tilde{p}_1} \mathrm{d}\sigma(\theta)\Big)^{p_1/\tilde{p}_1} r^{n-1}\mathrm{d}r\Big)^{p/p_1} \\
&\le \|b\|^p_{\mathrm{CMOL}_{\mathrm{rad}}^{\max\{p,p_1'\},\lambda} L_{\mathrm{ang}}^{\max\{\tilde{p},\tilde{p_1}'\}}(\mathbb{R}^n)} \Big(\sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^k} \Big(\int_{S^{n-1}} |f(r\theta)|^{\tilde{p}_1} \mathrm{d}\sigma(\theta)\Big)^{p_1/\tilde{p}_1} r^{n-1}\mathrm{d}r\Big)^{p/p_1} \\
&= \|b\|^p_{\mathrm{CMOL}_{\mathrm{rad}}^{\max\{p,p_1'\},\lambda} L_{\mathrm{ang}}^{\max\{\tilde{p},\tilde{p_1}'\}}(\mathbb{R}^n)} \|f\|^p_{L^{p_1}_{\mathrm{rad}} L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)}.
\end{aligned}
$$

For J_2, by adopting similar strategy, we get

$$
\begin{aligned}
J_2 = {} & \|b\|^p_{\mathrm{CMOL}_{\mathrm{rad}}^{\max\{p,p_1'\},\lambda} L_{\mathrm{ang}}^{\max\{\tilde{p},\tilde{p_1}'\}}(\mathbb{R}^n)} \sum_{k=-\infty}^{\infty} 2^{kn-knp}\Big(\sum_{i=-\infty}^{k} 2^{in/p_1'} \frac{2^{in\lambda}-2^{kn\lambda}}{1-2^{n\lambda}} \|f\chi_i\|_{L^{p_1}_{\mathrm{rad}} L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)}\Big)^p \\
= {} & \|b\|^p_{\mathrm{CMOL}_{\mathrm{rad}}^{\max\{p,p_1'\},\lambda} L_{\mathrm{ang}}^{\max\{\tilde{p},\tilde{p_1}'\}}(\mathbb{R}^n)} \sum_{k=-\infty}^{\infty} \times \\
& \Big(\sum_{i=-\infty}^{k} 2^{in/p_1'} \frac{2^{in\lambda}-2^{kn\lambda}}{1-2^{n\lambda}} 2^{kn/p-kn} \|f\chi_i\|_{L^{p_1}_{\mathrm{rad}} L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)}\Big)^p \\
\le {} & \|b\|^p_{\mathrm{CMOL}_{\mathrm{rad}}^{\max\{p,p_1'\},\lambda} L_{\mathrm{ang}}^{\max\{\tilde{p},\tilde{p_1}'\}}(\mathbb{R}^n)} \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{k} \Big(2^{in/p_1'} \frac{2^{in\lambda}-2^{kn\lambda}}{1-2^{n\lambda}} 2^{kn/p-kn}\Big)^{p/2} \times \\
& \Big(\sum_{i=-\infty}^{k} \big(2^{in/p_1'} \frac{2^{in\lambda}-2^{kn\lambda}}{1-2^{n\lambda}} 2^{kn/p-kn} \|f\chi_i\|_{L^{p_1}_{\mathrm{rad}} L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)}\big)^{p'/2}\Big)^{p/p'} \|f\chi_i\|^p_{L^{p_1}_{\mathrm{rad}} L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)} \\
\lesssim {} & \|b\|^p_{\mathrm{CMOL}_{\mathrm{rad}}^{\max\{p,p_1'\},\lambda} L_{\mathrm{ang}}^{\max\{\tilde{p},\tilde{p_1}'\}}(\mathbb{R}^n)} \sum_{i=-\infty}^{\infty} \|f\chi_i\|^p_{L^{p_1}_{\mathrm{rad}} L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)} \\
= {} & \|b\|^p_{\mathrm{CMOL}_{\mathrm{rad}}^{\max\{p,p_1'\},\lambda} L_{\mathrm{ang}}^{\max\{\tilde{p},\tilde{p_1}'\}}(\mathbb{R}^n)} \|f\|^p_{L^{p_1}_{\mathrm{rad}} L^{\tilde{p_1}}_{\mathrm{ang}}(\mathbb{R}^n)}.
\end{aligned}
$$

We now consider the converse in this position. Suppose that

$$b \in \mathrm{CMOL}_{\mathrm{rad}}^{\max\{p,p_1'\},\lambda} L_{\mathrm{ang}}^{\max\{\tilde{p},\tilde{p_1}'\}}(\mathbb{R}^n).$$

For convenience, denote $s = \max\{p, p_1'\}$ and $t = \max\{\tilde{p}, \tilde{p_1}'\}$, then

$$
\begin{aligned}
& \Big(\frac{1}{\nu_n r^{n+n\lambda s}} \int_0^r \Big(\int_{S^{n-1}} |b(\rho\theta)-b_B|^t \mathrm{d}\sigma(\theta)\Big)^{s/t} \rho^{n-1}\mathrm{d}\rho\Big)^{1/s} \\
& \quad = \Big(\frac{1}{\nu_n r^{n+n\lambda s}} \int_0^r \Big(\int_{S^{n-1}} \Big(\frac{1}{\nu_n r^n} \int_0^r \int_{S^{n-1}} \Big|b(\rho\theta)-b(\rho_1\theta_1)\Big| \mathrm{d}\sigma(\theta_1)\rho_1^{n-1}\mathrm{d}\rho_1\Big)^t \times \\
& \quad \mathrm{d}\sigma(\theta)\Big)^{s/t} \rho^{n-1}\mathrm{d}\rho\Big)^{1/s} \\
& \quad = \Big(\frac{1}{{\nu_n}^{1+s} r^{n+n\lambda s+ns}} \int_0^r \Big(\int_{S^{n-1}} \Big(\int_0^r \int_{S^{n-1}} \Big|b(\rho\theta)-b(\rho_1\theta_1)\Big| \mathrm{d}\sigma(\theta_1)\rho_1^{n-1}\mathrm{d}\rho_1\Big)^t \times \\
& \quad \mathrm{d}\sigma(\theta)\Big)^{s/t} \rho^{n-1}\mathrm{d}\rho\Big)^{1/s}
\end{aligned}
$$

$$\leq \Big(\frac{1}{{\nu_n}^{1+s} r^{n+n\lambda s+ns}} \int_0^r \Big(\int_{S^{n-1}} \Big(\int_0^\rho \int_{S^{n-1}} \Big| b(\rho\theta) - b(\rho_1\theta_1) \Big| \mathrm{d}\sigma(\theta_1) \times$$
$$\chi_{\{\rho_1<r\}}(\rho_1)\rho_1^{n-1}\mathrm{d}\rho_1\Big)^t \mathrm{d}\sigma(\theta)\Big)^{s/t} \rho^{n-1}\mathrm{d}\rho\Big)^{1/s} +$$
$$\Big(\frac{1}{{\nu_n}^{1+s} r^{n+n\lambda s+ns}} \int_0^r \Big(\int_{S^{n-1}} \Big(\int_\rho^\infty \int_{S^{n-1}} \Big| b(\rho\theta) - b(\rho_1\theta_1) \Big| \mathrm{d}\sigma(\theta_1) \times$$
$$\chi_{\{\rho_1<r\}}(\rho_1)\rho_1^{n-1}\mathrm{d}\rho_1\Big)^t \mathrm{d}\sigma(\theta)\Big)^{s/t} \rho^{n-1}\mathrm{d}\rho\Big)^{1/s}$$
$$=: K_1 + K_2.$$

For K_1, a simple calculation deduces that

$$K_1 = \Big(\frac{1}{{\nu_n}^{1+s} r^{n+n\lambda s+ns}} \int_0^r \Big(\int_{S^{n-1}} \rho_1^{sn} \Big(\frac{1}{\rho_1^n} \int_0^\rho \int_{S^{n-1}} \Big| b(\rho\theta) - b(\rho_1\theta_1) \Big| \mathrm{d}\sigma(\theta_1) \times$$
$$\chi_{\{\rho_1<r\}}(\rho_1)\rho_1^{n-1}\mathrm{d}\rho_1\Big)^t \mathrm{d}\sigma(\theta)\Big)^{s/t} \rho^{n-1}\mathrm{d}\rho\Big)^{1/s}$$
$$= \Big(\frac{r^{sn}}{{\nu_n}^{1+s} r^{n+n\lambda s+ns}} \int_0^r \Big(\int_{S^{n-1}} \Big(\mathrm{H}_b(\chi_{B(0,r)})(\rho\theta)\Big)^t \mathrm{d}\sigma(\theta)\Big)^{s/t} \rho^{n-1}\mathrm{d}\rho\Big)^{1/s}.$$

For K_2, we also have

$$K_2 = \Big(\frac{1}{{\nu_n}^{1+s} r^{n+n\lambda s+ns}} \int_0^r \Big(\int_{S^{n-1}} \Big(\int_0^\rho \int_{S^{n-1}} \frac{\Big| b(\rho\theta) - b(\rho_1\theta_1) \Big|}{\rho^n} \mathrm{d}\sigma(\theta_1) \times$$
$$\chi_{\{\rho_1<r\}}(\rho_1)\rho^n\rho_1^{n-1}\mathrm{d}\rho_1\Big)^t \mathrm{d}\sigma(\theta)\Big)^{s/t} \rho^{n-1}\mathrm{d}\rho\Big)^{1/s}$$
$$= \Big(\frac{r^{sn}}{{\nu_n}^{1+s} r^{n+n\lambda s+ns}} \int_0^r \Big(\int_{S^{n-1}} \Big(\mathrm{H}_b^*(\chi_{B(0,r)})(\rho\theta)\Big)^t \mathrm{d}\sigma(\theta)\Big)^{s/t} \rho^{n-1}\mathrm{d}\rho\Big)^{1/s}.$$

Next, we will divide four cases to obtain the estimates of K_1 and K_2: $s = p$, $t = \tilde{p}$; $s = p$, $t = \tilde{p}_1{}'$; $s = p_1'$, $t = \tilde{p}$ and $s = p_1'$, $t = \tilde{p}_1{}'$. In fact, we only need to consider the following two cases.

Case 1. $s = p$, $t = \tilde{p}$, since the commutators H_b and H_b^* are bounded from $L^{p_1}_{\rm rad}L^{\tilde{p}_1}_{\rm ang}(\mathbb{R}^n)$ to $L^{p}_{\rm rad}L^{\tilde{p}}_{\rm ang}(\mathbb{R}^n)$, we get

$$K_1 = \Big(\frac{r^{pn}}{{\nu_n}^{1+p} r^{n+n\lambda p+np}} \int_0^r \Big(\int_{S^{n-1}} \Big(\mathrm{H}_b(\chi_{B(0,r)})(\rho\theta)\Big)^{\tilde{p}} \mathrm{d}\sigma(\theta)\Big)^{p/\tilde{p}} \rho^{n-1}\mathrm{d}\rho\Big)^{1/p}$$
$$\lesssim \frac{r^n}{r^{n/p+n\lambda+n}} \|\chi_{B(0,r)}\|L^{p_1}_{\rm rad}L^{\tilde{p}_1}_{\rm ang}(\mathbb{R}^n) \lesssim 1.$$

Similarly, we can also obtain $K_2 \lesssim 1$.

Case 2. $s = p_1'$, $t = \tilde{p}$, since the dual space of $L^{p}_{\rm rad}L^{\tilde{p}}_{\rm ang}(\mathbb{R}^n)$ is $L^{p'}_{\rm rad}L^{\tilde{p}'}_{\rm ang}(\mathbb{R}^n)$ (see [19]), we know that the commutators H_b and H_b^* are bounded from $L^{p'}_{\rm rad}L^{\tilde{p}'}_{\rm ang}(\mathbb{R}^n)$ to $L^{p_1'}_{\rm rad}L^{\tilde{p}_1'}_{\rm ang}(\mathbb{R}^n)$. Therefore, we derive

$$K_1 = \Big(\frac{r^{p_1'n}}{{\nu_n}^{1+p_1'} r^{n+n\lambda p_1'+np_1'}} \int_0^r \Big(\int_{S^{n-1}} \Big(\mathrm{H}_b(\chi_{B(0,r)})(\rho\theta)\Big)^{\tilde{p}} \mathrm{d}\sigma(\theta)\Big)^{p_1'/\tilde{p}} \rho^{n-1}\mathrm{d}\rho\Big)^{1/p_1'}$$
$$\lesssim \frac{r^n}{r^{n/p_1'+n\lambda+n}} \|\chi_{B(0,r)}\|L^{p'}_{\rm rad}L^{\tilde{p}'}_{\rm ang}(\mathbb{R}^n) \lesssim 1.$$

Therefore, the proof of Theorem 1.13 is completed. □

References

[1] G. H. HARDY. *Note on a theorem of Hilbert*. Math. Z., 1920, **6**(3-4): 314–317.

[2] Shunchao LONG, Jian WANG. *Commutators of Hardy operators*. J. Math. Anal. Appl., 2002, **274**(2): 626–644.

[3] M. CHRIST, L. GRAFAKOS. *Best constants for two nonconvolution inequalities*. Proc. Amer. Math. Soc., 1995, **123**(6): 1687–1693.

[4] Zunwei FU, Zongguang LIU, Shanzhen LU, et al. *Characterization for commutators of n-dimensional fractional Hardy operators*. Sci. China Ser. A, 2007, **50**(10): 1418–1426.

[5] J. ALVAREZ, J. LAKEY, M. GUZMÁN-PARTIDA. *Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measures*. Collect. Math., 2000, **51**(1): 1–47.

[6] Fayou ZHAO, Shanzhen LU. *A characterization of λ-central BMO spaces*. Front. Math. China, 2013, **8**(1): 229–238.

[7] F. CACCIAFESTA, R. LUCÀ, *Singular integrals with angular regularity*. Proc. Amer. Math. Soc., 2016, **144**: 3413–3418.

[8] P. D'ANCONA, R. LUCÀ. *Stein-Weiss and Caffarelli-Kohn-Nirenberg inequalities with higher angular integrability*. J. Math. Anal. Appl., 2012, **388**(2): 1061–1079.

[9] P. D'ANCONA, R. LUCÀ. *On the regularity set and angular integrability for the Navier-Stokes equation*. Arch. Rational Mech. Anal., 2016, **221**(3): 1255–1284.

[10] J. STERBENZ. *Angular regularity and Strichartz estimates for the wave equation*. Int. Math. Res. Not., 2005, **4**: 187–231.

[11] T. TAO. *Spherically averaged endpoint Strichartz estimates for the two-dimensional Schrödinger equation*. Comm. Partial Differential Equations, 2000, **25**(7-8): 1471–1485.

[12] Ronghui LIU, Feng LIU, Huoxiong WU. *Mixed radial-angular integrability for rough singular integrals and maximal operators*. Proc. Amer. Math. Soc., 2020, **148**(9): 3943–3956.

[13] Ronghui LIU, Feng LIU, Huoxiong WU. *On the mixed radial-angular integrability of Marcinkiewicz integrals with rough kernels*. Acta Math. Sci. Ser. B (Engl. Ed.)., 2021, **41**(1): 241–256.

[14] Ronghui LIU, Huoxiong WU. *Rough singular integrals and maximal operator with radial-angular integrability*. Proc. Amer. Math. Soc., 2022, **150**(3): 1141–1151.

[15] Ronghui LIU, Huoxiong WU. *Mixed radial-angular integrability for rough maximal singular integrals and Marcinkiewicz integrals with mixed homogeneity*. Math. Nachr., 2023, **296**(7): 2942–2957.

[16] Ronghui LIU, Shuangping TAO. *Mixed radial-angular integrability for Hardy type operators*. Bull. Korean Math. Soc., 2023, **60**(5): 1409–1425.

[17] Ronghui LIU, Shuangping TAO, Huoxiong WU. *Mixed radial-angular integrabilities for commutators of fractional Hardy operators*. Bull. Sci. Math., 2023, (in press).

[18] Ronghui LIU, Shuangping TAO, Huoxiong WU. *Characterizations of the mixed radial-angular central Campanato space via the commutators of Hardy type*. Forum Math., 2023, **35**(5): 1327–1346.

[19] A. BENEDEK, R. PANZONE. *The space L^p, with mixed norm*. Duke Math. J., 1961, **28**: 301–324.