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On the Point Spectrum and Non-Degenerate Symplectic Structure of Eigenfunction Systems of Off-Diagonal Infinite Dimensional Hamiltonian Operators

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Abstract The point spectrum and non-degenerate symplectic structure of eigenfunction systems of off-diagonal infinite dimensional Hamiltonian operator $H = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ are studied in this article. The necessary and sufficient conditions for the eigenfunction systems of off-diagonal infinite dimensional Hamiltonian operator H to have non-degenerate symplectic structure are given. Further, the necessary and sufficient conditions for point spectrum to be contained in real axis, imaginary axis and other areas are obtained for off-diagonal infinite dimensional Hamiltonian operator H, respectively. As an illustrating example, off-diagonal infinite dimensional Hamiltonian operators derived from the plate bending problem and string vibration problem are used to justify the conclusions.

Keywords point spectrum; non-degenerate symplectic structure; eigenfunction system; off-diagonal infinite dimensional Hamiltonian operator

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1. Introduction

Many partial differential equations arising from mathematical physics and mechanics can be transformed into Hamiltonian systems $u' = Hu$, where u denotes the whole state vector, and $H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}$ is the infinite dimensional Hamiltonian operator. And, as an extension of the traditional separation of variables, the separation of variables based on Hamiltonian systems was proposed in 1991, which provides a new approach to elasticity and related fields [1–3]. After that, many scholars devoted to the work of solving the problems in elasticity by using this new method, and obtained many preferable results [4–6]. The basic theory of this eigenfunction expansion method is the spectral theory and completeness of eigenfunction systems of infinite dimensional Hamiltonian operators [7, 8].

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However, no conclusions have been found to explain the non-degeneracy of symplectic structure of eigenfunction system of infinite dimensional Hamiltonian operators. The expansion of the eigenfunction system of infinite dimensional Hamiltonian operator is premised on the non-degeneracy of symplectic structure, which make sure the existence of coefficients of eigenfunction expansion. Besides, we find that lots of problems in mathematical physics can be solved by the off-diagonal Hamiltonian systems $u' = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right) u$, and the corresponding off-diagonal infinite dimensional Hamiltonian operator $H = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)$ has nice spectral symmetry [9, 10]. Therefore, we research the non-degenerate symplectic structure of eigenfunction systems of off-diagonal infinite dimensional Hamiltonian operators. And further, we characterized the point spectrum of off-diagonal infinite dimensional Hamiltonian operator to be contained in real axis, imaginary axis and in other areas. All the results are not given by anyone else.

2. Preliminaries

Throughout this article, $\mathcal{X}$ is complex Hilbert space, $\mathbb{C}$ and $\mathbb{R}$ are the sets of complex numbers and real numbers, I is identity. The domain, range, null space and point spectrum of operator A are denoted by $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $\sigma_p(A)$. Write $E(\lambda; A)$ for the set composed of all the eigenfunctions related to $\lambda \in \sigma_p(A)$, and let $E(\lambda; A)$ be an empty set if $\lambda \notin \sigma_p(A)$. $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real part and imaginary part of a complex number z, respectively. The inner product on Hilbert space $\mathcal{X} \times \mathcal{X}$ is defined as on the usual product space. We use $(\cdot, \cdot)$ to denote the inner product on corresponding Hilbert space.

Definition 2.1 *Let*

$$H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} : \big(\mathcal{D}(A) \cap \mathcal{D}(C)\big) \times \big(\mathcal{D}(B) \cap \mathcal{D}(A^*)\big) \to \mathcal{X} \times \mathcal{X}$$

be a densely defined closed linear operator. If A is a closed operator, B and C are self-adjoint operators in $\mathcal{X}$, where A^* stands for the adjoint operator of A, then H is called infinite dimensional Hamiltonian operator.

In general case, infinite dimensional Hamiltonian operator H is a symplectic symmetric operator, i.e., H satisfies $J_1 H \subset (J_1 H)^*$, where $J_1 = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. H is called symplectic self-adjoint, if it satisfies $J_1 H = (J_1 H)^*$. H is called off-diagonal infinite dimensional Hamiltonian operator, if $A = 0$.

Definition 2.2 ([11]) *Let $\mathcal{X}$ be a complex Banach space, $T : \mathcal{D}(T) \to \mathcal{X}$ be a closed linear operator in $\mathcal{X}$, then the set*

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is bijective}\}$$

is called the resolvent set of T, and $\mathbb{C} \setminus \rho(T)$ is called the spectrum of T, denoted by $\sigma(T)$.

The spectrum $\sigma(T)$ can be divided into three disjoint parts: point spectrum $\sigma_p(T)$, residual spectrum $\sigma_r(T)$ and continuous spectrum $\sigma_c(T)$, i.e.,

$$\sigma(T) = \sigma_p(T) \cup \sigma_r(T) \cup \sigma_c(T),$$

where

$$\begin{aligned}\sigma_p(T)&=\{\lambda\in\mathbb{C}: T-\lambda \text{ is not injective}\},\\ \sigma_r(T)&=\{\lambda\in\mathbb{C}: T-\lambda \text{ is injective}, \overline{\mathcal{R}(T-\lambda)}\neq\mathcal{X}\},\\ \sigma_c(T)&=\{\lambda\in\mathbb{C}: T-\lambda \text{ is injective}, \overline{\mathcal{R}(T-\lambda)}=\mathcal{X}, \mathcal{R}(T-\lambda)\neq\mathcal{X}\}.\end{aligned}$$

Definition 2.3 ([12]) *Conjugate bi-linear function $\mathcal{B}(\cdot,\cdot):\mathcal{X}\times\mathcal{X}\to\mathbb{C}$ is called a non-degenerate symplectic structure, if following properties hold*

(1) $\mathcal{B}(u,v)=-\overline{\mathcal{B}(v,u)}$ for any $u,v\in\mathcal{X}$.

(2) There exists $u\in\mathcal{X}$, such that $\mathcal{B}(u,v)\neq 0$ for any $v\in\mathcal{X}$.

Remark 2.4 In Definition 2.3, (1) is the symplectic structure, and (2) is the non-degenerate property. Since $J_1^*=-J_1$, inner product $(J_1\cdot,\cdot)$ satisfies

$$(J_1\xi_1,\xi_2)=\overline{(\xi_2,J_1\xi_1)}=\overline{(-J_1\xi_2,\xi_1)}=-\overline{(J_1\xi_2,\xi_1)},$$

where $\xi_1=(f_1\ \ g_1)^{\mathrm{T}}$, $\xi_2=(f_2\ \ g_2)^{\mathrm{T}}\in\mathcal{X}\times\mathcal{X}$. Thus, inner product $(J_1\cdot,\cdot)$ has the symplectic structure, by Definition 2.3 (1).

3. Theoretical mode

In what follows, we discuss off-diagonal infinite dimensional Hamiltonian operator, which has a nice spectral symmetric property. It is easy to see off-diagonal infinite dimensional Hamiltonian operator $H=\left(\begin{smallmatrix}0 & B\\ C & 0\end{smallmatrix}\right):\mathcal{D}(C)\times\mathcal{D}(B)\to\mathcal{X}\times\mathcal{X}$ satisfies

$$J_1H=(J_1H)^*,\quad J_2H=(J_2H)^*,$$

where $J_2=\left(\begin{smallmatrix}0 & I\\ I & 0\end{smallmatrix}\right)$.

Lemma 3.1 ([7]) *Let $H=\left(\begin{smallmatrix}A & B\\ C & -A^*\end{smallmatrix}\right):\mathcal{D}(H)\subset\mathcal{X}\times\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ be the infinite dimensional Hamiltonian operator. If $\lambda_1,\lambda_2\in\sigma_p(H)$, $\lambda_1\neq-\overline{\lambda}_2$, then $(J_1\xi_1,\xi_2)=0$, where $\xi_1\in E(\lambda_1;H),\xi_2\in E(\lambda_2;H)$.*

Lemma 3.2 ([10]) *Let $H=\left(\begin{smallmatrix}0 & B\\ C & 0\end{smallmatrix}\right):\mathcal{D}(C)\times\mathcal{D}(B)\to\mathcal{X}\times\mathcal{X}$ be the off-diagonal infinite dimensional Hamiltonian operator. Then the following statements hold:*

(1) $0\in\sigma_p(H)\iff 0\in\sigma_p(B)\cup\sigma_p(C)$.

(2) $\sigma_p(H)\setminus\{0\}=\{\lambda\in\mathbb{C}\setminus\{0\}:\lambda^2\in\sigma_p(BC)\}=\{\lambda\in\mathbb{C}\setminus\{0\}:\lambda^2\in\sigma_p(CB)\}$.

Theorem 3.3 *Let $H=\left(\begin{smallmatrix}0 & B\\ C & 0\end{smallmatrix}\right):\mathcal{D}(C)\times\mathcal{D}(B)\to\mathcal{X}\times\mathcal{X}$ be the off-diagonal infinite dimensional Hamiltonian operator. Then*

(1) If $0\in\sigma_p(H)$, then $(J_1\xi_1,\xi_2)\neq 0$ if and only if $\mathcal{N}(B)$ and $\mathcal{N}(C)$ are not orthogonal, where $\xi_1,\xi_2\in E(0;H)$.

(2) If $0\notin\sigma_p(H)$, then the eigenfunction system of H has non-degenerate symplectic structure, if and only if the following statements hold:

(a) $\sigma_p(H)$ is symmetric with respect to the imaginary axis.

(b) $E(\lambda^2; BC)$ *and* $E(\overline{\lambda}^2; CB)$ *are not orthogonal, where* $\lambda \in \mathbb{C} \setminus \{0\}$.

Proof (1) Necessity. Suppose $0 \in \sigma_p(H)$ and $\xi = (f \quad g)^{\mathrm{T}} \in E(0; H)$. It follows from $H\xi = 0$ that $f \in \mathcal{N}(C)$ and $g \in \mathcal{N}(B)$. If $(J_1\xi, \xi) \neq 0$, then we have

$$(J_1\xi, \xi) = \left(\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}\right) = \left(\begin{pmatrix} g \\ -f \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}\right) = (g, f) - (f, g) = -i2\mathrm{Im}((f, g)) \neq 0.$$

Hence, $(f, g) \neq 0$, and hence $\mathcal{N}(B)$ and $\mathcal{N}(C)$ are not orthogonal.

Sufficiency. Suppose $\mathcal{N}(B)$ and $\mathcal{N}(C)$ are not orthogonal. And let $f \in \mathcal{N}(C)$ $(f \neq 0)$. Then there exists $\hat{g} \in \mathcal{N}(B)$ $(\hat{g} \neq 0)$, such that $(f, \hat{g}) \neq 0$. Let $\hat{\xi}_1 = (f \quad \hat{g})^{\mathrm{T}}$, $\hat{\xi}_2 = (f \quad -\hat{g})^{\mathrm{T}}$. Then it is easy to see $H\hat{\xi}_1 = 0$, $H\hat{\xi}_2 = 0$, hence $\hat{\xi}_1, \hat{\xi}_2 \in E(0; H)$. Since

$$(J_1\hat{\xi}_1, \hat{\xi}_1) = \left(\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} f \\ \hat{g} \end{pmatrix}, \begin{pmatrix} f \\ \hat{g} \end{pmatrix}\right) = \left(\begin{pmatrix} \hat{g} \\ -f \end{pmatrix}, \begin{pmatrix} f \\ \hat{g} \end{pmatrix}\right) = (\hat{g}, f) - (f, \hat{g}) = -i2\mathrm{Im}((f, \hat{g}))$$

and

$$(J_1\hat{\xi}_2, \hat{\xi}_1) = \left(\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} f \\ -\hat{g} \end{pmatrix}, \begin{pmatrix} f \\ \hat{g} \end{pmatrix}\right) = \left(\begin{pmatrix} -\hat{g} \\ -f \end{pmatrix}, \begin{pmatrix} f \\ \hat{g} \end{pmatrix}\right) = -(\hat{g}, f) - (f, \hat{g}) = -2\mathrm{Re}((f, \hat{g})),$$

it follows from $(f, \hat{g}) \neq 0$ that $(J_1\hat{\xi}_1, \hat{\xi}_1) \neq 0$ or $(J_1\hat{\xi}_2, \hat{\xi}_1) \neq 0$. Proof is completed.

(2) Necessity. Suppose $0 \notin \sigma_p(H)$ and the eigenfunction system of H has non-degenerate symplectic structure. Let $\lambda \in \sigma_p(H)$ $(\lambda \neq 0)$ with $\xi = (f \quad g)^{\mathrm{T}} \in E(\lambda; H)$. Then it is easy to see $f \neq 0$ and $g \neq 0$. In fact, from $H\xi = \lambda\xi$, we obtain

$$\begin{cases} Bg = \lambda f, & (3.1) \\ Cf = \lambda g. & (3.2) \end{cases}$$

In either of the case $f \neq 0, g = 0$ or $f = 0, g \neq 0$, we conclude $\lambda = 0$, by Eqs. (3.1) and (3.2). Conflict. Since the eigenfunction system of H has non-degenerate symplectic structure, according to Lemma 3.1, there exists $\lambda_1 \in \sigma_p(H)$ with $\xi_1 \in E(\lambda_1; H)$, satisfying $\lambda_1 = -\overline{\lambda}$ and $(J_1\xi_1, \xi) \neq 0$. Hence, $\sigma_p(H)$ is symmetric with respect to the imaginary axis. (a) is true.

To prove (b), we prove that, for every $f \in E(\lambda^2; BC)$, there exist an element in $E(\overline{\lambda}^2; CB)$ which is not orthogonal to f. Firstly, by Lemma 3.2 and (3.1)–(3.2), we know $f \neq 0$ and $f \in E(\lambda^2; BC)$, i.e., $BCf = \lambda^2 f$. Hence, if let $\xi = (f \quad \frac{1}{\lambda}Cf)^{\mathrm{T}}$, then it follows from the equality

$$H\xi = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\begin{pmatrix} f \\ \frac{1}{\lambda}Cf \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda}BCf \\ Cf \end{pmatrix} = \lambda\begin{pmatrix} f \\ \frac{1}{\lambda}Cf \end{pmatrix} = \lambda\xi$$

that $\xi = (f \quad \frac{1}{\lambda}Cf)^{\mathrm{T}} \in E(\lambda; H)$. Secondly, for $\xi = (f \quad \frac{1}{\lambda}Cf)^{\mathrm{T}} \in E(\lambda; H)$, since the eigenfunction system of H has non-degenerate symplectic structure, there exists $\xi_0 = (-\frac{1}{\overline{\lambda}}Bg_0 \quad g_0)^{\mathrm{T}} \in E(-\overline{\lambda}; H)$, such that

$$(J_1\xi, \xi_0) \neq 0,$$

and hence $(f, g_0) \neq 0$. In fact, we choose $g_0 \in E(\overline{\lambda}^2; CB)$, then $CBg_0 = \overline{\lambda}^2 g_0$. Let $\xi_0 =$

$(-\frac{1}{\lambda}Bg_0 \quad g_0)^{\mathrm{T}}$. Then

$$H\xi_0 = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\lambda}Bg_0 \\ g_0 \end{pmatrix} = \begin{pmatrix} Bg_0 \\ -\frac{1}{\lambda}CBg_0 \end{pmatrix} = \begin{pmatrix} Bg_0 \\ -\overline{\lambda}g_0 \end{pmatrix} = -\overline{\lambda} \begin{pmatrix} -\frac{1}{\lambda}Bg_0 \\ g_0 \end{pmatrix} = -\overline{\lambda}\xi_0, \tag{3.3}$$

and hence $\xi_0 \in E(-\overline{\lambda}; H)$. Since

$$\begin{aligned}(J_1\xi, \xi_0) &= \left(\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} f \\ \frac{1}{\lambda}Cf \end{pmatrix}, \begin{pmatrix} -\frac{1}{\lambda}Bg_0 \\ g_0 \end{pmatrix}\right) = \left(\begin{pmatrix} \frac{1}{\lambda}Cf \\ -f \end{pmatrix}, \begin{pmatrix} -\frac{1}{\lambda}Bg_0 \\ g_0 \end{pmatrix}\right) \\ &= (\frac{1}{\lambda}Cf, -\frac{1}{\lambda}Bg_0) - (f, g_0) = -\frac{1}{\lambda^2}(BCf, g_0) - (f, g_0) \\ &= -\frac{1}{\lambda^2}(\lambda^2 f, g_0) - (f, g_0) = -2(f, g_0), \end{aligned} \tag{3.4}$$

it follows from $(J_1\xi, \xi_0) \neq 0$ that $(f, g_0) \neq 0$. (b) is true.

Sufficiency. Suppose the assertions (a) and (b) hold. Let $\lambda \in \sigma_p(H)$ with eigenfunction $\xi = (f \quad g)^{\mathrm{T}} \in E(\lambda; H)$, the assertion (a) implies $-\overline{\lambda} \in \sigma_p(H)$. By Lemma 3.2 (2), $\xi = (f \quad g)^{\mathrm{T}} \in E(\lambda; H)$ implies $f \in E(\lambda^2; BC)$. From assertion (b), there exists $g_0 \in E(\overline{\lambda}; CB)$, such that $(f, g_0) \neq 0$. We make $\xi_0 = (-\frac{1}{\lambda}Bg_0 \quad g_0)^{\mathrm{T}}$, then it follows from Eq. (3.3) that $\xi_0 \in E(-\overline{\lambda}; H)$, and from Eq. (3.4) and $(f, g_0) \neq 0$ follows $(J_1\xi, \xi_0) \neq 0$. Thus, the eigenfunction system of H has non-degenerate symplectic structure. □

Lemma 3.4 ([10]) *Let $H = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right) : \mathcal{D}(C) \times \mathcal{D}(B) \to \mathcal{X} \times \mathcal{X}$ be the off-diagonal infinite dimensional Hamiltonian operator. Then the following statements hold:*

(1) $\lambda \in \sigma_p(H)$, if and only if $-\lambda \in \sigma_p(H)$.

(2) $\sigma_p(H) \cup \sigma_r(H)$ is symmetric with respect to the real axis, and $\sigma_r(H)$ is not symmetric with respect to the real axis, if $\sigma_r(H) \neq \varnothing$.

(3) $\sigma_p(H) \cup \sigma_r(H)$ is symmetric with respect to the imaginary axis, and $\sigma_r(H)$ is not symmetric with respect to the imaginary axis, if $\sigma_r(H) \neq \varnothing$.

Remark 3.5 Let $H = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right) : \mathcal{D}(C) \times \mathcal{D}(B) \to \mathcal{X} \times \mathcal{X}$ be the off-diagonal infinite dimensional Hamiltonian operator. Then $\sigma_p(H)$ is symmetric with respect to the imaginary axis, if and only if $\sigma_p(H)$ is symmetric with respect to the real axis.

Proof The conclusion holds immediately, by Lemma 3.4. □

Theorem 3.6 *Let $H = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right) : \mathcal{D}(C) \times \mathcal{D}(B) \to \mathcal{X} \times \mathcal{X}$ be the off-diagonal infinite dimensional Hamiltonian operator. Then the eigenfunction system of H has non-degenerate symplectic structure, if and only if the following assertion (1) holds, and one of (2) and (3) is satisfied meanwhile, where*

(1) $E(\lambda^2; BC)$ and $E(\overline{\lambda}^2; CB)$ are not orthogonal, where $\lambda \in \mathbb{C} \setminus \{0\}$.

(2) $\sigma_p(H)$ is symmetric with respect to the real axis or imaginary axis.

(3) $\sigma_r(H) = \varnothing$.

Proof The colclusions hold immediately by Theorem 3.3 and Remark 3.5. □

Corollary 3.7 *Let $H = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right) : \mathcal{D}(C) \times \mathcal{D}(B) \to \mathcal{X} \times \mathcal{X}$ be off-diagonal infinite dimensional Hamiltonian operator. If $B = \pm C$, then the eigenfunction system of H has non-degenerate symplectic structure, if and only if one of the following assertions hold:*

(1) $\sigma_p(H)$ is symmetric with respect to the imaginary axis or real axis.

(2) $\sigma_r(H) = \varnothing$.

Proof If $B = \pm C$ and $0 \in \sigma_p(H)$, then it follows from B, C are self-adjoint operators that $\mathcal{N}(B)$ and $\mathcal{N}(C)$ are not orthogonal. If $0 \notin \sigma_p(H)$, it is clear that $E(\lambda^2; BC)$ and $E(\overline{\lambda}^2; CB)$ are not orthogonal. Besides, for off-diagonal infinite dimensional Hamiltonian operator H, $\sigma_p(H)$ is symmetric with respect to the imaginary axis is equivalent to $\sigma_r(H) = \varnothing$, by Lemma 3.4. Thus, H has non-degenerate symplectic structure, if and only if either (1) or (2) holds. □

We found in elasticity that the operator B in many off-diagonal infinite dimensional Hamiltonian operator $H = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)$, is usually an identity operator. To this case, we draw the following corollary.

Corollary 3.8 *Let $H = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right) : \mathcal{D}(C) \times \mathcal{D}(B) \to \mathcal{X} \times \mathcal{X}$ be the off-diagonal infinite dimensional Hamiltonian operator. If either B or C is identity operator, then the following assertions hold:*

(1) If $\sigma_p(H) \subset i\mathbb{R}$, then the eigenfunction system of H has non-degenerate symplectic structure, here $i\mathbb{R} = \{i\lambda : \lambda \in \mathbb{R}\}$.

(2) If $\sigma_p(H) \subset \mathbb{R}$, then the eigenfunction system of H has non-degenerate symplectic structure.

(3) If $\sigma_p(H) \subset \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, and $\sigma_p(H)$ is symmetric with respect to real axis or imaginary axis, then the eigenfunction system of H has non-degenerate symplectic structure.

Proof By Theorem 3.6, the conclusions hold clearly. □

In what follows, the conditions under which the point spectrum is distributed in real axis, imaginary axis or other regions are discussed, respectively.

Lemma 3.9 *Let $H = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right) : \mathcal{D}(C) \times \mathcal{D}(B) \to \mathcal{X} \times \mathcal{X}$ be the off-diagonal infinite dimensional Hamiltonian operator. If $\lambda_1, \lambda_2 \in \sigma_p(H)$ and $\lambda_1 \neq \overline{\lambda}_2$, then $(J_2\xi_1, \xi_2) = 0$, where $\xi_1 \in E(\lambda_1; H), \xi_2 \in E(\lambda_2; H)$ and $J_2 = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$.*

Proof Since $\lambda_1, \lambda_2 \in \sigma_p(H)$, and $\xi_1 \in E(\lambda_1; H)$, $\xi_2 \in E(\lambda_2; H)$, making inner product on two sides of $H\xi_1 = \lambda_1\xi_1$ with $J_2\xi_2$, we have

$$(J_2\xi_2, H\xi_1) = (J_2\xi_2, \lambda_1\xi_1) = \overline{\lambda}_1(J_2\xi_2, \xi_1) = \overline{\lambda}_1(\xi_2, J_2\xi_1).$$

Making inner product on two sides of $H\xi_2 = \lambda_2\xi_2$ with $J_2\xi_1$, we have

$$(J_2\xi_1, H\xi_2) = (J_2\xi_1, \lambda_2\xi_2) = \overline{\lambda}_2(J_2\xi_1, \xi_2).$$

Since

$$(J_2\xi_2, H\xi_1) = (\xi_2, J_2H\xi_1) = (J_2H\xi_2, \xi_1) = (H\xi_2, J_2\xi_1) = \overline{(J_2\xi_1, H\xi_2)},$$

we have that

$$\overline{\lambda}_1(\xi_2, J_2\xi_1) = \overline{\overline{\lambda}_2(J_2\xi_1, \xi_2)},$$

i.e.,

$$(\lambda_1 - \overline{\lambda}_2)(J_2\xi_1, \xi_2) = 0.$$

Since $\lambda_1 \neq \overline{\lambda}_2$, we have $(J_2\xi_1, \xi_2) = 0$. □

Remark 3.10 Let $H = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right) : \mathcal{D}(C) \times \mathcal{D}(B) \to \mathcal{X} \times \mathcal{X}$ be the off-diagonal infinite dimensional Hamiltonian operator. If $(J_2\xi, \xi) \neq 0$ holds for some $\lambda \in \sigma_p(H)$ with $\xi \in E(\lambda; H)$, then $\lambda \in \mathbb{R}$.

Proof The conclusion is clear, by the converse negative proposition of Lemma 3.9. □

Lemma 3.11 *Let $H = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right) : \mathcal{D}(C) \times \mathcal{D}(B) \to \mathcal{X} \times \mathcal{X}$ be the off-diagonal infinite dimensional Hamiltonian operator. Then the following statements hold:*

(1) If $\sigma_p(H) \subset i\mathbb{R} \setminus \{0\}$, $E(\lambda^2; BC)$ and $E(\lambda^2; CB)$ are not orthogonal, then $(f, g) \in i\mathbb{R} \setminus \{0\}$ for each $\lambda \in \sigma_p(H)$ with $\xi = (f \;\; g)^{\mathrm{T}} \in E(\lambda; H)$.

(2) If $\sigma_p(H) \subset \mathbb{R} \setminus \{0\}$, $E(\lambda^2; BC)$ and $E(\lambda^2; CB)$ are not orthogonal, then $(f, g) \in \mathbb{R} \setminus \{0\}$ for each $\lambda \in \sigma_p(H)$ with $\xi = (f \;\; g)^{\mathrm{T}} \in E(\lambda; H)$.

(3) If $\sigma_p(H) \subset \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, then $(f, g) = 0$ for each $\lambda \in \sigma_p(H)$ with $\xi = (f \;\; g)^{\mathrm{T}} \in E(\lambda; H)$.

Proof (1) If $\sigma_p(H) \subset i\mathbb{R} \setminus \{0\}$, for each $\lambda \in \sigma_p(H)$ with $\xi = (f \;\; g)^{\mathrm{T}} \in E(\lambda; H)$, we have

$$(J_2\xi, \xi) = 2\mathrm{Re}((f, g)) = 0,$$

by Lemma 3.9. Since $E(\lambda^2; BC)$ and $E(\lambda^2; CB)$ are not orthogonal, the eigenfunction system of H has non-degenerate symplectic structure, by Theorem 3.6. It follows that

$$(J_1\xi, \xi) = -i2\mathrm{Im}((f, g)) \neq 0,$$

and hence, $(f, g) \in i\mathbb{R} \setminus \{0\}$.

(2) If $\sigma_p(H) \subset \mathbb{R} \setminus \{0\}$, for each $\lambda \in \sigma_p(H)$ with $\xi = (f \;\; g)^{\mathrm{T}} \in E(\lambda; H)$, we have

$$(J_1\xi, \xi) = -i2\mathrm{Im}((f, g)) = 0,$$

by Lemma 3.1. Since $E(\lambda^2; BC)$ and $E(\lambda^2; CB)$ are not orthogonal, the eigenfunction system of H has non-degenerate symplectic structure, by Theorem 3.6. Since $\lambda \in \mathbb{R} \setminus \{0\}$, $\xi = (f \;\; g)^{\mathrm{T}} \in E(\lambda; H)$ is equivalent to $f \in E(\lambda^2; BC)$ and $g \in E(\lambda^2; CB)$. From Eq. (3.3), we know $\xi_0 = (-\frac{1}{\lambda}Bg \;\; g)^{\mathrm{T}} \in E(-\lambda; H)$, and the non-degenerate symplectic structure of eigenfunction system of H implies

$$(J_1\xi, \xi_0) = -2(f, g) \neq 0,$$

by Eq. (3.4). Thus, $(f, g) \in \mathbb{R} \setminus \{0\}$.

(3) If $\sigma_p(H) \subset \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, for each $\lambda \in \sigma_p(H)$ with $\xi \in E(\lambda; H)$, both $(J_2\xi, \xi) = 0$ and $(J_1\xi, \xi) = 0$ hold, by Lemmas 3.1 and 3.9. Thus $(f, g) = 0$. □

Lemma 3.12 *Let $H = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right) : \mathcal{D}(C) \times \mathcal{D}(B) \to \mathcal{X} \times \mathcal{X}$ be the off-diagonal infinite dimensional Hamiltonian operator. If either $B|_{E_2(\lambda;H)}$ or $C|_{E_1(\lambda;H)}$ is semi-definite, then $\sigma_p(H) \subset \big(\mathbb{R} \cup$*

$i\mathbb{R}$), *where* $B|_{E_2(\lambda;H)}$ *and* $C|_{E_1(\lambda;H)}$ *are the restriction of operators* B, C *on* $E_2(\lambda;H), E_1(\lambda;H)$, *respectively, and* $E(\lambda;H) = (E_1(\lambda;H) \quad E_2(\lambda;H))^{\mathrm{T}}$.

Proof The conclusion holds immediately by the proof in [9]. □

Theorem 3.13 *Let* $H = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} : \mathcal{D}(C)\times\mathcal{D}(B) \to \mathcal{X}\times\mathcal{X}$ *be the off-diagonal infinite dimensional Hamiltonian operator. If* $E(\lambda^2; BC)$ *and* $E(\overline{\lambda}^2; CB)$ *are not orthogonal, then the following statements hold:*

(1) $\sigma_p(H) \subset \mathbb{R}\setminus\{0\}$, *if and only if* $(Bg, g) = (Cf, f) \neq 0$ *and* $(f, g) \in \mathbb{R}\setminus\{0\}$ *for* $\lambda \in \sigma_p(H)$ *with* $\xi = (f \quad g)^{\mathrm{T}} \in E(\lambda;H)$.

(2) $\sigma_p(H) \subset i\mathbb{R}\setminus\{0\}$, *if and only if* $(Bg, g) = -(Cf, f) \neq 0$ *and* $(f, g) \in i\mathbb{R}\setminus\{0\}$ *for* $\lambda \in \sigma_p(H)$ *with* $\xi = (f \quad g)^{\mathrm{T}} \in E(\lambda;H)$.

(3) $\sigma_p(H) \subset \big(\mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R})\big)\cup\{0\}$, *if and only if* $(Bg, g) = (Cf, f) = 0$ *for* $\lambda \in \sigma_p(H)$ *with* $\xi = (f \quad g)^{\mathrm{T}} \in E(\lambda;H)$.

Proof We prove (1) and (3), the proof of (2) is analogous with (1).

(1) Suppose $\lambda \in \sigma_p(H) \subset \mathbb{R}\setminus\{0\}$ and $\xi = (f \quad g)^{\mathrm{T}} \in E(\lambda;H)$. Since $f \neq 0$ and $g \neq 0$ by Eqs. (3.1)–(3.2), making inner product on two sides of (3.1) with g and (3.2) with f, respectively, gives

$$\begin{cases} (Bg, g) = \lambda(f, g), & (3.5)\\ (Cf, f) = \lambda(g, f). & (3.6)\end{cases}$$

Since $\lambda \in \mathbb{R}\setminus\{0\}$, by calculations (3.5) + (3.6) and (3.5) − (3.6), we have

$$\begin{cases} (Bg, g) + (Cf, f) = 2\lambda \mathrm{Re}((f, g)), & (3.7)\\ (Bg, g) - (Cf, f) = 2\lambda i \mathrm{Im}((f, g)). & (3.8)\end{cases}$$

By Lemma 3.11, we know $(f, g) \in \mathbb{R}\setminus\{0\}$ and it follows from (3.7) and (3.8)

$$(Bg, g) = (Cf, f) \neq 0.$$

Conversely, suppose $(Bg, g) = (Cf, f) \neq 0$ and $(f, g) \in \mathbb{R}\setminus\{0\}$, since B and C are self adjoint operators, (Bg, g) and (Cf, f) are real numbers. From Eqs. (3.7) and (3.8), it follows that $\lambda \in \mathbb{R}\setminus\{0\}$.

(3) We prove in two cases.

Case I. Consider $0 \in \sigma_p(H)$ with $\xi = (f \quad g)^{\mathrm{T}} \in E(0;H)$. By Eqs. (3.5) and (3.6), we have

$$(Bg, g) = (Cf, f) = 0.$$

Conversely, if $(Bg, g) = (Cf, f) = 0$ for $\lambda \in \sigma_p(H)$ with $\xi = (f \quad g)^{\mathrm{T}} \in E(\lambda;H)$, then $\lambda = 0$. If otherwise $\lambda \neq 0$, since $B|_{E_2(\lambda;H)}$ and $C|_{E_1(\lambda;H)}$ are semi-definite, by Lemma 3.12, we have $\sigma_p(H) \subset (\mathbb{R}\cup i\mathbb{R})\setminus\{0\}$, and further by Lemma 3.11, we have $(f, g) \neq 0$. By Eqs. (3.5) and (3.6), we have $(Bg, g) \neq 0$ or $(Cf, f) \neq 0$. Conflict.

Case II. Consider $\sigma_p(H) \subset \mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R})$. For $\lambda \in \sigma_p(H) \subset \mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R})$ and $\xi = (f \quad g)^{\mathrm{T}} \in$

$E(\lambda;H)$, by calculation (3.5)+(3.6) and (3.5)−(3.6),

$$\begin{cases} (Bg,g)+(f,Cf)=2\mathrm{Re}(\lambda)(f,g), & (3.9)\\ (Bg,g)-(f,Cf)=2i\mathrm{Im}(\lambda)(f,g). & (3.10)\end{cases}$$

By Lemma 3.11, we know that $(f,g)=0$. Hence, by Eqs. (3.9) and (3.10),

$$(Bg,g)=(Cf,f)=0.$$

Conversely, suppose $(Bg,g)=(Cf,f)=0$ for each $\lambda\in\sigma_p(H)$, $\lambda\neq 0$ and $\xi=(f\ \ g)^{\mathrm{T}}\in E(\lambda;H)$. We prove by contraction. If $\sigma_p(H)\subset i\mathbb{R}\setminus\{0\}$, by Lemma 3.11, we have $(f,g)\in i\mathbb{R}\setminus\{0\}$. Then, Eq. (3.10) is not equal. Conflict. If $\sigma_p(H)\subset\mathbb{R}\setminus\{0\}$, by Lemma 3.11, we have $(f,g)\in\mathbb{R}\setminus\{0\}$. Then, Eq. (3.9) is not equal. Conflict. Therefore, $\sigma_p(H)\subset\mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R})$. □

Corollary 3.14 *Let $H=\left(\begin{smallmatrix}0&B\\C&0\end{smallmatrix}\right):\mathcal{D}(C)\times\mathcal{D}(B)\to\mathcal{X}\times\mathcal{X}$ be the off-diagonal infinite dimensional Hamiltonian operator. Suppose $B|_{E_2(\lambda;H)}$ or $C|_{E_1(\lambda;H)}$ is semi-definite operator, $E(\lambda^2;BC)$ and $E(\overline{\lambda}^2;CB)$ are not orthogonal, then the following statements hold:*

(1) $\sigma_p(H)\subset\mathbb{R}$, if and only if either $B|_{E_2(\lambda;H)}\geq 0$, $C|_{E_1(\lambda;H)}\geq 0$ hold or $-B|_{E_2(\lambda;H)}\geq 0$, $-C|_{E_1(\lambda;H)}\geq 0$ hold. Here, $B|_{E_2(\lambda;H)}\geq 0$ means $B|_{E_2(\lambda;H)}$ is a positive operator, i.e., $(B|_{E_2(\lambda;H)}g,g)\geq 0$ for all $g\in E_2(\lambda;H)$.

(2) $\sigma_p(H)\subset i\mathbb{R}$, if and only if either $B|_{E_2(\lambda;H)}\geq 0$, $-C|_{E_1(\lambda;H)}\geq 0$ hold or $-B|_{E_2(\lambda;H)}\geq 0$, $C|_{E_1(\lambda;H)}\geq 0$ hold.

(3) $\sigma_p(H)\subset(\mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R}))\cup\{0\}$, if and only if $B|_{E_2(\lambda;H)}$ and $C|_{E_1(\lambda;H)}$ are conservative operators, i.e., $(B|_{E_2(\lambda;H)}g,g)=(C|_{E_1(\lambda;H)}f,f)=0$ for $f\in E_1(\lambda;H), g\in E_2(\lambda;H)$.

Proof By Theorem 3.13, the assertions hold immediately. □

Corollary 3.15 *Let $H=\left(\begin{smallmatrix}0&B\\C&0\end{smallmatrix}\right):\mathcal{D}(C)\times\mathcal{D}(B)\to\mathcal{X}\times\mathcal{X}$ be the off-diagonal infinite dimensional Hamiltonian operator. If either B or C is identity operator, then the following assertions hold:*

(1) $\sigma_p(H)\subset\mathbb{R}\setminus\{0\}$, if and only if $(Bg,g)=(Cf,f)\neq 0$ and $(f,g)\in\mathbb{R}\setminus\{0\}$, for $\lambda\in\sigma_p(H)$ with $\xi=(f\ \ g)^{\mathrm{T}}\in E(\lambda;H)$.

(2) $\sigma_p(H)\subset i\mathbb{R}\setminus\{0\}$, if and only if $(Bg,g)=-(Cf,f)\neq 0$ and $(f,g)\in i\mathbb{R}\setminus\{0\}$ for $\lambda\in\sigma_p(H)$ with $\xi=(f\ \ g)^{\mathrm{T}}\in E(\lambda;H)$.

(3) $\sigma_p(H)\subset(\mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R}))\cup\{0\}$ if and only if $(Bg,g)=(Cf,f)=0$ for $\lambda\in\sigma_p(H)$ with $\xi=(f\ \ g)^{\mathrm{T}}\in E(\lambda;H)$.

Proof By Theorem 3.13, the conclusions hold clearly. □

4. Applications

In this section, two examples are presented to illustrate the previous results. We always assume that $\mathcal{X}=L^2[0,1]$.

Example 4.1 The free vibration of rectangular thin plates with two opposite edges simply supported is given by

$$D(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})^2 W-\rho\omega^2 W=0, \tag{4.1}$$

where $W(x,y)$ is vibration mode, ρ is mass density, D is bending stiffness and ω is natural frequency. The boundary conditions for simply supported edges $y=0$ and $y=1$ are

$$W=0,\ \frac{\partial^2 W}{\partial y^2}=0,\ \text{for } y=0 \text{ and } y=1. \tag{4.2}$$

Introduce the bending moment functions

$$M_x=-D(\frac{\partial^2 W}{\partial x^2}+\nu\frac{\partial^2 W}{\partial y^2}),\quad M_y=-D(\frac{\partial^2 W}{\partial y^2}+\nu\frac{\partial^2 W}{\partial x^2}),$$

then we have

$$M_x+M_y=-D(1+\nu)\nabla^2 W.$$

Let $M=-\frac{M_x+M_y}{D(1+\nu)}=\nabla^2 W$, $\theta=\frac{\partial W}{\partial x}$, $\varphi=\frac{\partial M}{\partial x}$, $\frac{\partial\theta}{\partial x}=\frac{\partial^2 W}{\partial x^2}=M-\frac{\partial^2 W}{\partial y^2}$, $\frac{\partial\varphi}{\partial x}=\frac{\partial^2 M}{\partial x^2}=-\frac{\partial^2 M}{\partial y^2}+\frac{\rho\omega^2}{D}W$. Then, the Eq. (4.1) can be transformed into the Hamiltonian system

$$\frac{\partial}{\partial x}\begin{pmatrix} W\\ M\\ \varphi\\ \theta\end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & I\\ 0 & 0 & I & 0\\ \frac{\rho\omega^2}{D} & -\frac{\partial^2}{\partial y^2} & 0 & 0\\ -\frac{\partial^2}{\partial y^2} & I & 0 & 0\end{pmatrix}\begin{pmatrix} W\\ M\\ \varphi\\ \theta\end{pmatrix}.$$

Write

$$\mathbf{H}=\begin{pmatrix} 0 & B\\ C & 0\end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & I\\ 0 & 0 & I & 0\\ \frac{\rho\omega^2}{D} & -\frac{\partial^2}{\partial y^2} & 0 & 0\\ -\frac{\partial^2}{\partial y^2} & I & 0 & 0\end{pmatrix}.$$

According to the boundary conditions (4.2), we have

$$\mathcal{D}(\mathbf{H})=\left\{(W\quad M\quad \varphi\quad \theta)^{\mathrm{T}}\in\mathcal{X}^4\ \middle|\ \begin{array}{l} W(0)=W(1)=M(0)=M(1)=0 \text{ and } W',M'\text{are}\\ \text{absolutely continuous, } W'',M''\in L^2[0,1]\end{array}\right\}.$$

Direct calculations show that

$$\sigma_p(\mathbf{H})=\left\{\lambda_k=(\mathrm{sgn}k)\sqrt{(k\pi)^2+\sqrt{\frac{\rho\omega^2}{D}}}, k=\pm1,\pm2,\ldots\right\},$$

where $\mathrm{sgn}k$ is the usual sign function, and the eigenfunctions corresponding to eigenvalues $\lambda_k, (k=\pm1,\pm2,\ldots)$ are

$$\xi_k=\Big(\sqrt{\frac{D}{\rho\omega^2}}\sin k\pi y\quad \sin k\pi y\quad \lambda_k\sin k\pi y\quad \lambda_k\sqrt{\frac{D}{\rho\omega^2}}\sin k\pi y\Big)^{\mathrm{T}}.$$

Denote $\xi_k = (\xi_k^{(1)} \ \ \xi_k^{(2)} \ \ \xi_k^{(3)} \ \ \xi_k^{(4)})^{\mathrm{T}}$ $(k = \pm 1, \pm 2, \ldots)$, then

$$\begin{aligned}
(J_1\xi_k, \xi_{-k}) =& \left(\begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_k^{(1)} \\ \xi_k^{(2)} \\ \xi_k^{(3)} \\ \xi_k^{(4)} \end{pmatrix}, \begin{pmatrix} \xi_{-k}^{(1)} \\ \xi_{-k}^{(2)} \\ \xi_{-k}^{(3)} \\ \xi_{-k}^{(4)} \end{pmatrix} \right) \\
=&(\xi_k^{(3)}, \xi_{-k}^{(1)}) + (\xi_k^{(4)}, \xi_{-k}^{(2)}) - (\xi_k^{(1)}, \xi_{-k}^{(3)}) - (\xi_k^{(2)}, \xi_{-k}^{(4)}) \\
=& \int_0^1 \lambda_k \sin k\pi y \cdot \sqrt{\frac{D}{\rho\omega^2}} \sin(-k)\pi y \mathrm{d}y + \int_0^1 \lambda_k \sqrt{\frac{D}{\rho\omega^2}} \sin k\pi y \cdot \sin(-k)\pi y \mathrm{d}y - \\
& \int_0^1 \sqrt{\frac{D}{\rho\omega^2}} \sin k\pi y \cdot \lambda_{-k} \sin(-k)\pi y \mathrm{d}y - \int_0^1 \sin k\pi y \cdot \lambda_{-k} \sqrt{\frac{D}{\rho\omega^2}} \sin(-k)\pi y \mathrm{d}y \\
=&(\lambda_{-k} - \lambda_k) \cdot \sqrt{\frac{D}{\rho\omega^2}} = -2\sqrt{\frac{k^2\pi^2 D}{\rho\omega^2} + \sqrt{\frac{D}{\rho\omega^2}}} \neq 0.
\end{aligned}$$

Thus, the eigenfunction system of **H** has non-degenerate symplectic structure. The conclusion of Corollary 3.8 is true. Besides,

$$\begin{aligned}
(Bg, g) &= \left(\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \xi_k^{(3)} \\ \xi_k^{(4)} \end{pmatrix}, \begin{pmatrix} \xi_k^{(3)} \\ \xi_k^{(4)} \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \lambda_k \sin k\pi y \\ \lambda_k \sqrt{\frac{D}{\rho\omega^2}} \sin k\pi y \end{pmatrix}, \begin{pmatrix} \lambda_k \sin k\pi y \\ \lambda_k \sqrt{\frac{D}{\rho\omega^2}} \sin k\pi y \end{pmatrix} \right) \\
&= -1 - k^2\pi^2 \sqrt{\frac{D}{\rho\omega^2}}, \\
(Cf, f) &= \left(\begin{pmatrix} \frac{\rho\omega^2}{D} & -\frac{\partial^2}{\partial y^2} \\ -\frac{\partial^2}{\partial y^2} & I \end{pmatrix} \begin{pmatrix} \xi_k^{(1)} \\ \xi_k^{(2)} \end{pmatrix}, \begin{pmatrix} \xi_k^{(1)} \\ \xi_k^{(2)} \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} \frac{\rho\omega^2}{D} & -\frac{\partial^2}{\partial y^2} \\ -\frac{\partial^2}{\partial y^2} & I \end{pmatrix} \begin{pmatrix} \sqrt{\frac{D}{\rho\omega^2}} \sin k\pi y \\ \sin k\pi y \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{D}{\rho\omega^2}} \sin k\pi y \\ \sin k\pi y \end{pmatrix} \right) \\
&= -1 - k^2\pi^2 \sqrt{\frac{D}{\rho\omega^2}}.
\end{aligned}$$

Therefore, $(Bg, g) = (Cf, f) \neq 0$. Theorem 3.13 is valid.

Example 4.2 Consider the string vibration problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, & t > 0, \ 0 < x < 1, \\ u(t, 0) = u(t, 1) = 0, & t \geq 0, \\ u(0, x) = \varphi(x), u_t(0, x) = \phi(x), & 0 \leq x \leq 1. \end{cases} \tag{4.3}$$

Let $v = \frac{\partial u}{\partial t}$. Then Eq. (4.3) can be transformed into Hamiltonian system

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & I \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

Write

$$\mathbf{H} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$$

and

$$\mathcal{D}(\mathbf{H}) = \big\{(u \quad v)^{\mathrm{T}} \in \mathcal{X}^2 | u(0) = u(1) = 0, u' \text{ are absolutely continuous, } u', u'' \in L^2[0,1]\big\}.$$

Direct calculations show that

$$\sigma_p(\mathbf{H}) = \{\lambda_k = ik\pi, k = \pm 1, \pm 2, \ldots\},$$

and the eigenfunctions corresponding with eigenvalues λ_k $(k = \pm 1, \pm 2, \ldots)$ are

$$\xi_k = (\sin k\pi x \quad ik\pi \sin k\pi x)^{\mathrm{T}}.$$

Denote $\xi_k = (\xi_k^{(1)} \quad \xi_k^{(2)})^{\mathrm{T}}$ $(k = \pm 1, \pm 2, \ldots)$, then

$$\begin{aligned}
(J_1\xi_k, \xi_k) &= \left(\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \sin k\pi x \\ ik\pi \sin k\pi x \end{pmatrix}, \begin{pmatrix} \sin k\pi x \\ ik\pi \sin k\pi x \end{pmatrix}\right) \\
&= (ik\pi \sin k\pi x, \sin k\pi x) - (\sin k\pi x, ik\pi \sin k\pi x) \\
&= ik\pi \neq 0.
\end{aligned}$$

Thus, the eigenfunction system of $\mathbf{H}$ has non-degenerate symplectic structure. The conclusion of Corollary 3.8 is true. Besides,

$$\begin{aligned}
(Bg, g) &= (B\xi_k^{(2)}, \xi_k^{(2)}) = (ik\pi \sin k\pi x, ik\pi \sin k\pi x) = \frac{k^2\pi^2}{2}, \\
(Cf, f) &= (C\xi_k^{(1)}, \xi_k^{(1)}) = (-k^2\pi^2 \sin k\pi x, \sin k\pi x) = -\frac{k^2\pi^2}{2}.
\end{aligned}$$

Therefore, $(Bg, g) = -(Cf, f) \neq 0$. Theorem 3.13 is valid.

5. Conclusions

This paper is devoted to the research of the point spectrum and non-degenerate symplectic structure of eigenfunction systems of off-diagonal infinite dimensional Hamiltonian operators, which have a wide applications in physics and mechanics. As the theoretical basis of symplectic Fourier series expansion method, the distribution of point spectrum and non-degenerate symplectic structure of eigenfunction systems of infinite dimensional Hamiltonian operators are particularly important. However, the non-degenerate property of symplectic structure has not been proved. Besides, in many practical problems, the point spectrum of off-diagonal infinite dimensional Hamiltonian operators are either real numbers or pure imaginary numbers. Hence, for off-diagonal infinite dimensional Hamiltonian operators, we gave the necessary and sufficient conditions for the point spectrum to be contained in the real axis or imaginary axis. The conclusions of this paper have never been made. The results obtained in this paper are based on the special structure of infinite dimensional Hamiltonian operators, may not be valid for other operator matrices.

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