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Ball Convergence Theorems for Chebyshev-Halley Method in $\mathbb{B}$-Space

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Abstract A local convergence analysis of Chebyshev-Halley method having third order of convergence for approximating zero of non-linear operator $f(v) = 0$ by using convex majorant function and their condition in $\mathbb{B}$-space (Banach space), is presented in this article. We give the error estimate to show the efficiency of our study. Besides, we established the relation between majorant function and Kantorovich or Smale-type result as special cases of our general theory.

Keywords Chebyshev-Halley method; Majorizing function; Majorant condition; Local convergence; Kantorovich-type condition; Smale-type condition

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1. Introduction

In this article, we are solving the problem of approximation with a local unique solution α of

$$f(v) = 0 \tag{1.1}$$

where, we assume that $f : \Omega \subseteq V_1 \longrightarrow V_2$ is a non-linear operator, on a open convex subset $\Omega \neq 0$ in a $\mathbb{B}$-space V_1 to another $\mathbb{B}$-space V_2. Finding the solution of (1.1) is the best approaching classical problem in scientific, technical and mathematical computational areas as transportation, optimization, dynamics economic system and so on. Generally in these types of problem Eq. (1.1) is represented by integral equation, differential equation, algebraic equation, dynamic system which is mathematically modeled by differential equation, system of linear and non-linear algebraic equation and so on. Along with local as well semi-local convergence the iterative method is one of the most important technique, used to find approximate solution of Eq. (1.1). The semi-local convergence provides information about the initial point, to give conditions ensuring the convergence of iterative method, while in local convergence analysis we find radius of convergence ball and information about the solution. When f is Frèchet differentiable, one can remember that Newton's method [1] which is defined as $v_{n+1} = v_n - f(v_n)f'(v_n)^{-1}$ with initial approximate v_0, is the most widely used iterative method to find approximate solution of (1.1) which is of order two. In order to improve local and semi-local order of convergence of (1.1), some researchers also proposed and analyzed higher order iterative methods. When f is

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twice continuously Frèchet derivative on open convex subset Ω, there are several cubic generalization for Newton's method as Chebyshev's [2], Halley [3], Super-Halley [4], Newton-like [5] and so on. Another classical family of cubic method is Chebyshev-Halley, which was introduced by Gutièrrez and Hernàndez in [6] and the method is defined as, for each $n \geq 0$ with a parameter β,

$$v_{n+1} = v_n - [I + \frac{1}{2}\pounds_f(v_n)\Upsilon_f(v_n)]\Lambda_n f(v_n), \quad n \geq 0 \text{ and } \beta \in [0,1], \tag{1.2}$$

where

$$\Upsilon_f(v_n) = [I - \beta\pounds_f(v_n)]^{-1}, \quad \Lambda_n = f'(v_n)^{-1}$$

and $\pounds_f(v_n) : V_1 \longrightarrow V_2$ is the linear operator, defined by

$$\pounds_f(v_n) = \Lambda_n f''(v_n)\Lambda_n f(v_n), \quad v_n \in V_1.$$

Here, I represents the identity operator, $\Upsilon_n \in \pounds(V_2, V_1)$, where $\pounds(V_2, V_1)$ represents a linearly bounded operator from V_2 into V_1. Thereafter $f'(v_n)$ and $f''(v_n)$ refer to first and second Fréchet-derivative of f, respectively. As special cases, this family includes Chebyshev's method [2, 7]

$$v_{n+1} = v_n - [I + \frac{1}{2}\pounds_f(v_n)]\Lambda_n f(v_n), \quad n \geq 0.$$

Halley method [3, 8]

$$v_{n+1} = v_n - [I + \frac{1}{2}\pounds_f(v_n)(I - \frac{1}{2}\pounds_f(v_n))^{-1}]\Lambda_n f(v_n), \quad n \geq 0$$

and Super-Halley method (Convex acceleration of the Newton method) [4, 9]

$$v_{n+1} = v_n - [I + \frac{1}{2}\pounds_f(v_n)(I - \pounds_f(v_n))^{-1}]\Lambda_n f(v_n), \quad n \geq 0$$

for $\beta = 0$, $\frac{1}{2}$ and 1 in (1.2), respectively, which have also third order of convergence. Many authors have worked for semi-local and local convergence analysis of special cases of Chebyshev-Halley method (Chebyshev's, Halley and Super-Halley method) for solving (1.1) by using Recurrence relation, majorizing sequence, majorant function and so on under mild differentiability condition, majorant condition, weak condition, Hölder condition and so on [2–4, 7–13].

Many authors such as Kumari and Parida [2], Argyros and Ren [8] worked in the direction to show the degree of difficulty for choosing initial points that highlights the importance of study of local convergence. In 2016, Argyros and Magrñàen [14] presented local convergence analysis of Chebyshev-Halley-type method using hypotheses only on the first derivative of the function f and also presented its dynamic. Also, by using hypothesis only up to the first Frèchet derivative of f, Argyros and George [15] presented a local convergence analysis for deformed Euler-Halley method (Chebyshev-Halley method) for solving (1.1) in $\mathbb{B}$-space and gave convergence ball and error estimate. Recently, we provide a local convergence analysis for Chebyshev's method [2] by using new type of majorant condition (1.3) with their properties. Let for $\alpha \in \Omega$, $\Lambda_\alpha = f'(\alpha)^{-1} \in \pounds(V_2, V_1)$. Assume $\delta > 0$ be such that $U(\alpha, \delta) \subseteq \Omega$ and $\phi, \phi_* \colon [0, \delta) \longrightarrow \mathbb{R}$ be functions of class $\mathcal{C}^2[0, \delta)$. Assuming that f'' follows majorant condition [2],

$$\left\|\Lambda_\alpha[f''(z) - f''(v)]\right\| \leq \phi''(\|z - v\| + \|v - \alpha\|) - \phi''(\|v - \alpha\|) \tag{1.3}$$

and

$$\|\Lambda_\alpha[f''(z) - f''(\alpha)]\| \leq \phi''_*(\|z-\alpha\|) - \phi''_*(0), \tag{1.4}$$

$\forall\, z, v \in U(\alpha, \delta)$, where $\|z - v\| + \|v - \alpha\| < \delta$, and the following conditions hold

(1) $\phi''(0) \geq 0$, $\phi''_*(0) > 0$, $\phi'_*(0) = -1$;

(2) ϕ''_* is convex in $[0, \delta)$, ϕ'' and ϕ''_* are strictly increasing in $[0, \delta)$;

(3) ϕ'_* has zeroes in $(0, \delta)$ and assume that δ_0 be smallest zero of ϕ'_* in $(0, \delta)$;

(4) $\|\Lambda_\alpha f''(\alpha)\| \leq \phi''_*(0)$,

and we prove that Chebyshev method is cubically convergent.

In this article, we present the local convergence analysis of Chebyshev-Halley method for finding approximate zero of (1.1) by using the majorant conditions (1.3) and (1.4). Here, we show that the method is cubically convergent under properties (1)–(4) and also provide the error estimate. Moreover, we also establish the relation between majorant condition and Kantorovich-type or Smale-type assumption as special cases. Here, the majorant conditions (1.3) and (1.4) are generalization of Kantorovich and Smale-type assumption.

The organization of our article is as follows: In Section 1, we present majorant functions with their condition and properties. In Section 2, we establish relation between the majorant function and non-linear operator and then we define some functions for our convenience, which are used in the article, whereas in Subsection 2.1, we present the main theorem where we gave local convergence theorem for Chebyshev-Halley method and show that the method is cubically convergent. At last we specialize the majorant function ϕ and ϕ_* to obtain Kantorovich and Smale-type result and to show the efficiency of our method and we give some numerical examples in Section 3.

2. Local convergence analysis

Here, we present the local convergence analysis of Chebyshev-Halley method. Let for $\delta > 0$, $U(\alpha, \delta) = \{z \in V_1 : \|z - \alpha\| < \delta\}$ be the open ball and $\bar{U}(\alpha, \delta)$ be its closure. For the study of the local convergence analysis of Chebyshev-Halley method, we introduce some functions j_1, j_2, j_3, Q and W on $[0, \delta]$. Here, we define

$$j_1(\varsigma) = \beta\phi''_*(\varsigma)(2 + \phi'_*(\varsigma))\varsigma - (\phi'_*(\varsigma))^2, \quad \beta \in [0,1] \tag{2.1}$$

and

$$Q(\varsigma) = -\frac{1}{2}\frac{j_2(\varsigma)}{\phi'_*(\varsigma)}\Big[\frac{1}{\varsigma} - \frac{\phi''_*(\varsigma)^2}{\phi'_*(\varsigma)} + \frac{1}{4}\frac{\phi'_*(\varsigma)^3}{\phi''_*(\varsigma)^2}\varsigma + \beta j_3(\varsigma)\frac{\phi''_*(\varsigma)}{\varsigma}\Big], \quad \beta \in [0,1]$$

where

$$j_2(\varsigma) = \frac{1}{1 - j_3} \quad \text{and} \quad j_3(\varsigma) = \frac{\beta\phi''_*(\varsigma)(2 + \phi'_*(\varsigma))\varsigma}{(\phi'_*(\varsigma))^2}.$$

Assume that $Q(\varsigma) > 0$, for $\beta \in [0,1]$ and $\varsigma \in [0, \delta_1)$. Note that, j_1 is a continuous function on $[0, \delta]$ by the condition (1)–(4) with

$$j_1(0) = -(\phi'_*(0))^2 = -1 < 0$$

and

$$j_1(\varsigma) = \beta\phi_*''(\varsigma)(2+\phi_*'(\varsigma))\varsigma - (\phi_*'(\varsigma))^2 = 2\beta\phi_*''(\varsigma)\varsigma > 0$$

for $\beta \neq 0$. Then, intermediate value theorem implies that j_1 has at least one zero in $[0,\delta]$ and we denote δ_1 as its minimal zero. Hence $j_1(\delta_1) = 0$, which implies that $j_3(\delta_1) = 1$ for $\beta \neq 0$. By [2] and definition of ϕ_*', we conclude that ϕ_*' is strictly increasing and convex in $[0,\delta)$ and $\phi_*'(\varsigma) \in (0,1)$ for $\varsigma \in (0,\delta_0)$. Also, we have

$$j_3(0) = 0 \text{ and } j_3(\delta_1) = 1 > 0$$

thus, j_3 is increasing function and $j_3(\varsigma) \in [0,1)$ for $\varsigma \in [0,\delta_1)$. Now, for j_2, we have $j_2(0) = 1$ and it is increasing in $\varsigma \in [0,\delta_1)$. Then, we can conclude that for $\varsigma \in [0,\delta_1)$, $j_2(\varsigma) > 0$. If

$$W(\varsigma) = Q(\varsigma)\varsigma^2 - 1 \tag{2.2}$$

has at least zero in $(0,\ \delta_1)$, let δ_* be the minimal zero in $(0,\delta_1)$. Here, $W(0) < 0$ and $Q(\varsigma)$ is strictly increasing. Then, we conclude that $W(\varsigma) < 0$ for $\varsigma \in [0,\delta_*)$.

Lemma 2.1 *Suppose $\|v-\alpha\| \le \varsigma < \delta_0$. If ϕ_* is a function of class C^2 on $[0,\delta_0)$ and majorize f at α, then $\Lambda = f'(v)^{-1}$ exists in $\pounds(V_2,V_1)$, and*

(i)

$$\|\Lambda f'(\alpha)\| \le -\frac{1}{\phi_*'(\|v-\alpha\|)} \le -\frac{1}{\phi_*'(\varsigma)}. \tag{2.3}$$

(ii)

$$\|\Lambda_\alpha f''(v)\| \le \phi_*''(\|v-\alpha\|) \le \phi_*''(\varsigma). \tag{2.4}$$

(iii) In $\pounds(V_2,V_1)$, $I-\beta\pounds_f(v)$ is invertible and

$$\|(I-\beta\pounds_f(v))^{-1}\| \le \frac{1}{1-j_3(\|v-\alpha\|)} \le \frac{1}{1-j_3(\varsigma)} = j_2(\varsigma), \quad \beta\in[0,1]. \tag{2.5}$$

Proof Let $v \in \bar{U}(\alpha,\delta_0)$ and $\varsigma \in [0,\delta_0)$. By Taylor series we have

$$f'(v) = f'(\alpha) + \int_0^1 [f''(\alpha+\vartheta(v-\alpha)) - f''(\alpha)](v-\alpha)\mathrm{d}\vartheta + f''(\alpha)(v-\alpha)$$

or

$$\Lambda_\alpha[f'(v)-f'(\alpha)] = \int_0^1 \Lambda_\alpha[f''(\alpha+\vartheta(v-\alpha)) - f''(\alpha)](v-\alpha)\mathrm{d}\vartheta + \Lambda_\alpha f''(\alpha)(v-\alpha),$$

where, $\Lambda_\alpha = f'(\alpha)^{-1}$. Applying the conditions (1), (4), Eq. (1.4), and from [2], $\phi_*'(\varsigma) \in (0,1)$ on the above equation, we get

$$\begin{aligned}\|\Lambda_\alpha(f'(v)-f'(\alpha))\| \le & \int_0^1 \|\Lambda_\alpha[f''(\alpha+\vartheta(v-\alpha)) - f''(\alpha)]\|\|(v-\alpha)\|\mathrm{d}\vartheta + \\ & \|\Lambda_\alpha f''(\alpha)\|\|(v-\alpha)\| \\ \le & \int_0^1 [\phi_*''(\vartheta\|v-\alpha\|) - \phi_*''(0)]\|v-\alpha\|\mathrm{d}\vartheta + \phi_*''(0)\|v-\alpha\| \\ \le & \phi_*'(\|v-\alpha\|) - \phi_*'(0) \le \phi_*'(\varsigma) - \phi_*'(0) < 1.\end{aligned}$$

Then, by $\mathbb{B}$-lemma on invertible operator [16,17], $\Lambda \in \pounds(V_2, V_1)$ and

$$\|\Lambda f'(\alpha)\| \leq \frac{1}{1-(\phi_*'(\|v-\alpha\|)-\phi_*'(0))} \leq -\frac{1}{\phi_*'(\|v-\alpha\|)} \leq -\frac{1}{\phi_*'(\varsigma)},$$

which proves Eq. (2.3). Next using Eq. (1.4), (4) and since ϕ_*'' is strictly increasing, we obtain

$$\begin{aligned}\|\Lambda_\alpha f''(v)\| &\leq \|\Lambda_\alpha[f''(v)-f''(\alpha)]\| + \|\Lambda_\alpha f''(\alpha)\| \\ &\leq \phi_*''(\|v-\alpha\|) - \phi_*''(0) + \phi_*''(0) \\ &\leq \phi_*''(\|v-\alpha\|) \leq \phi_*''(\varsigma),\end{aligned}$$

which proves Eq. (2.4).

Now, by using (2.3), (2.4) and the definition of $\pounds_f(v)$, for $v \in \bar{U}(\alpha, \delta^*)$ we have

$$\begin{aligned}\|\beta\pounds_f(v)\| \leq &|\beta|\|\Lambda f'(\alpha)\|\|\Lambda_\alpha f''(v)\|\|\Lambda f'(\alpha)\|\times \\ &\Big\|\int_0^1 \Lambda_\alpha f'(\alpha+\vartheta(v-\alpha))\mathrm{d}\vartheta(v-\alpha)\Big\|, \quad \beta\in[0,1] \\ \leq&\beta\|\Lambda f'(\alpha)\|^2\|\Lambda_\alpha f''(v)\|\times \\ &\Big\|\Big\{\int_0^1 \Lambda_\alpha[f'(\alpha+\vartheta(v-\alpha))-f'(\alpha)]\mathrm{d}\vartheta + I\Big\}(v-\alpha)\Big\| \\ \leq&\beta\frac{\phi_*''(\|v-\alpha\|)}{\phi_*'(\|v-\alpha\|)^2}\big(2+\phi_*'(\|v-\alpha\|)\big)(\|v-\alpha\|) \\ \leq& j_3(\|v-\alpha\|) \leq j_3(\varsigma) < 1.\end{aligned}$$

Thus, by $\mathbb{B}$-Lemma on invertible operator, it follows that $(I-\beta\pounds_f(v))^{-1} \in \pounds(V_2, V_1)$ and

$$\begin{aligned}\|\Upsilon_f(v_n)\| = \|(I-\beta\pounds_f(v))^{-1}\| &\leq \frac{1}{1-\|\pounds_f(v)\|} \\ &\leq \frac{1}{1-j_3(\|v-\alpha\|)} = j_2(\|v-\alpha\|) \leq j_2(\varsigma)\end{aligned}$$

which proves Eq. (2.5). □

2.1. Main result

Here, the main local convergence results for the Chebyshev-Halley method has been given.

Theorem 2.2 *Let $f : \Omega \subseteq V_1 \longrightarrow V_2$ be a twice continuously Fréchet differentiable nonlinear operator where V_1, V_2 are $\mathbb{B}$-space and Ω is a non-empty open convex subset in $\mathbb{B}$-space, $\Lambda_\alpha = f'(\alpha)^{-1} \in \pounds(V_2, V_1)$ exist and the condition (1)–(4) hold. With initial approximation $v_0 \in U(\alpha, \delta^*)$, the sequence $\{v_n\}$ obtained by Chebyshev-Halley method is well defined, contained in $U(\alpha, \delta^*)$ for all $n \geq 0$ and converges to a unique zero α of operator f in $U(\alpha, \delta_0)$. Additionally, the following error estimation also holds*

$$\|v_{n+1}-\alpha\| \leq Q(\delta^*)\|v_n-\alpha\|^3. \tag{2.6}$$

In general, we can conclude that Chebyshev-Halley method converging to α is cubic.

Proof We used $f(\alpha)=0$ and from Eq. (1.2) to get

$$\begin{aligned} v_{n+1}-\alpha &= v_n-\big[I+\frac{1}{2}\pounds_f(v_n)\varUpsilon_f(v_n)\big]\varLambda_n f(v_n)-\alpha \\ &=(v_n-\alpha)-\varUpsilon_f(v_n)\big[I+\big(\frac{1}{2}-\beta\big)\pounds_f(v_n)\big]\varLambda_n f(v_n) \\ &=\varUpsilon_f(v_n)\big[(v_n-\alpha)(I-\beta\pounds_f(v_n))-\big\{I+\big(\frac{1}{2}-\beta\big)\pounds_f(v_n)\big\}\varLambda_n f(v_n)\big] \\ &=\varUpsilon_f(v_n)\big[A_f(v_n)+B_f(v_n)\big]. \end{aligned}$$

By Taylor's expansion, we have

$$\begin{aligned} A_f(v_n)=&v_n-\alpha-\varLambda_n f(v_n)-\frac{1}{2}\pounds_f(v_n)\varLambda_n f(v_n) \\ =&\varLambda_n\big[f(\alpha)-f(v_n)-(v_n-\alpha)f'(v_n)-\frac{(v_n-\alpha)^2}{2}f''(v_n)\big]+ \\ &\frac{1}{2}\varLambda_n f''(v_n)(v_n-\alpha+\varLambda_n f(v_n))(v_n-\alpha-\varLambda_n f(v_n)) \\ =&\varLambda_n\int_0^1(1-\vartheta)[f''(v_n+\vartheta(\alpha-v_n))-f''(v_n)](\alpha-v_n)^2\mathrm{d}\vartheta+ \\ &\frac{1}{2}\varLambda_n f''(v_n)\varLambda_n\int_0^1(1-\vartheta)f''(v_n+\vartheta(\alpha-v_n))(\alpha-v_n)^2\mathrm{d}\vartheta\times \\ &\Big[2I+\int_0^1(1-\vartheta)\varLambda_n f''(v_n+\vartheta(\alpha-v_n))(\alpha-v_n)\mathrm{d}\vartheta\Big](\alpha-v_n) \end{aligned}$$

and

$$\begin{aligned} B_f(v_n)&=-\beta\pounds_f(v_n)(v_n-\alpha)+\beta\pounds_f(v_n)\varLambda_n f(v_n) \\ &=-\beta\pounds_f(v_n)\varLambda_n[(v_n-\alpha)f'(v_n)-f(v_n)] \\ &=-\beta\pounds_f(v_n)\varLambda_n\int_0^1(1-\vartheta)f''(v_n+\vartheta(\alpha-v_n))\mathrm{d}\vartheta(\alpha-v_n)^2. \end{aligned}$$

By using (1.3), (1.4), (2.3)–(2.5), we obtain in turn that

$$\begin{aligned} \|A_f(v_n)\|\le&\|\varLambda_n f'(\alpha)\|\times\Big\|\int_0^1(1-\vartheta)\varLambda_\alpha[f''(v_n^\vartheta)-f''(v_n)](\alpha-v_n)^2\mathrm{d}\vartheta\Big\|+ \\ &\frac{1}{2}\|\varLambda_n f'(\alpha)\|\|\varLambda_\alpha f''(v_n)\|\|\varLambda_n f'(\alpha)\|\times\Big\|\int_0^1(1-\vartheta)\varLambda_\alpha f''(v_n^\vartheta)(\alpha-v_n)^2\mathrm{d}\vartheta\Big\|\times \\ &\Big[2+\|\varLambda_n f'(\alpha)\|\Big\|\int_0^1(1-\vartheta)f''(v_n^\vartheta)(\alpha-v_n)\mathrm{d}\vartheta\Big\|\Big]\|\alpha-v_n\| \\ \le&-\frac{1}{\phi_*'(\|v_n-\alpha\|)}\times \\ &\int_0^1(1-\vartheta)\big[\phi''(\vartheta\|v_n-\alpha\|+\|v_n-\alpha\|)-\phi''(\|v_n-\alpha\|)\big]\mathrm{d}\vartheta\|\alpha-v_n\|^2+ \\ &\frac{1}{2}\frac{\phi_*''(\|v_n-\alpha\|)}{\phi_*'(\|v_n-\alpha\|)^2}\int_0^1(1-\vartheta)\phi_*''(\vartheta\|v_n-\alpha\|)\mathrm{d}\vartheta\|\alpha-v_n\|^2\times \\ &\Big[2-\frac{1}{\phi_*'(\|v_n-\alpha\|)}\int_0^1(1-\vartheta)\phi_*''(\vartheta\|v_n-\alpha\|)\mathrm{d}\vartheta\|\alpha-v_n\|\Big]\|\alpha-v_n\|, \end{aligned} \tag{2.7}$$

where, $v_n^{\vartheta} = v_n + \vartheta(\alpha - v_n)$ and

$$\begin{aligned}\|B_f(v_n)\| &\le |\beta| \|\pounds_f(v_n)\| \|\Lambda_n f'(\alpha)\| \Big\| \int_0^1 (1-\vartheta)\Lambda_\alpha f''(v_n^{\vartheta}) \Big\| \mathrm{d}\vartheta \|\alpha - v_n\|^2 \\ &\le -\beta \frac{j_3(\|v_n - \alpha\|)}{\phi_*'(\|v_n - \alpha\|)} \int_0^1 (1-\vartheta)\phi_*''(\vartheta\|v_n - \alpha\|)\mathrm{d}\vartheta \|\alpha - v_n\|^2. \end{aligned} \tag{2.8}$$

Therefore,

$$\|v_{n+1} - \alpha\| \le \big\|\Upsilon_f(v_n)\big\| \big[\|A_f(v_n)\| + \|B_f(v_n)\|\big]. \tag{2.9}$$

Now, by Lemma 2.1 and the convexity of ϕ'' (see [1]), we get

$$\phi''(\vartheta\|v_n - \alpha\| + \|v_n - \alpha\|) - \phi''(\|v_n - \alpha\|) \le [\phi''(\vartheta\delta^* + \delta) - \phi''(\delta^*)] \frac{\vartheta\|\alpha - v_n\|}{\vartheta\delta^*}. \tag{2.10}$$

Therefore, by using Lemma 2.1, Eqs. (2.4), (2.5), (2.7), (2.8) and (2.10) in (2.9), we get

$$\begin{aligned}\|v_{n+1} - \alpha\| \le & -\frac{1}{2}\frac{j_2(\|v_n - \alpha\|)\|v_n - \alpha\|^3}{\delta^*\phi_*'(\|v_n - \alpha\|)} + \frac{1}{2} j_2(\|v_n - \alpha\|)\frac{\phi_*''(\|v_n - \alpha\|)^2}{\phi_*'(\|v_n - \alpha\|)^2}\|v_n - \alpha\|^3 - \\ & \frac{1}{8} j_2(\|v_n - \alpha\|)\frac{\phi_*''(\|v_n - \alpha\|)^3}{\phi_*'(\|v_n - \alpha\|)^3}\delta^*\|v_n - \alpha\|^3 - \\ & \frac{\beta}{2} j_2(\|v_n - \alpha\|) j_3(\|v_n - \alpha\|)\frac{\phi_*''(\|v_n - \alpha\|)}{\delta^*\phi_*'(\|v_n - \alpha\|)}\|v_n - \alpha\|^3 \\ \le & -\frac{1}{2}\frac{j_2(\|v_n - \alpha\|}{\phi_*'(\|v_n - \alpha\|)}\Big[\frac{1}{\delta^*} - \frac{\phi_*''(\|v_n - \alpha\|)^2}{\phi_*'(\|v_n - \alpha\|)} + \frac{1}{4}\frac{\phi_*'(\|v_n - \alpha\|)^3}{\phi_*'(\|v_n - \alpha\|)^2}\delta^* + \\ & \beta j_3(\|v_n - \alpha\|)\frac{\phi_*''(\|v_n - \alpha\|)}{\delta^*}\Big]\|v_n - \alpha\|^3 \\ \le & -\frac{1}{2}\frac{j_2(\delta^*)}{\phi_*'(\delta^*)}\Big[\frac{1}{\delta^*} - \frac{\phi_*''(\delta^*)^2}{\phi_*'(\delta^*)} + \frac{1}{4}\frac{\phi_*'(\delta^*)^3}{\phi_*'(\delta^*)^2}\delta^* + \beta j_3(\delta^*)\frac{\phi_*''(\delta^*)}{\delta^*}\Big]\|v_n - \alpha\|^3 \\ \le & Q(\delta^*)\|v_n - \alpha\|^3 \le \frac{\|v_n - \alpha\|^3}{(\delta^*)^2} \le \|v_n - \alpha\| < \delta^* \end{aligned}$$

which shows existence of (2.6) and also we can conclude that, $v_{n+1} \in U(\alpha, \delta^*)$, and $\lim_{n\to\infty} v_n = \alpha$.

For uniqueness part, let $\sigma \in \bar{U}(\alpha, \delta_0)$ be another zero of the operator f. Define the linear operator, $\pounds = \int_0^1 f'(\sigma + \vartheta(\alpha - \sigma))\mathrm{d}\vartheta$. Then, as in result [2], $\phi_*'(\varsigma) \in (0, 1)$, we get

$$\begin{aligned} & \|\Lambda_\alpha\| \Big\| \int_0^1 f'(\sigma + \psi(\alpha - \sigma))\mathrm{d}\psi - f'(\alpha) \Big\| \\ & \le \int_0^1 \big[\phi_*''(\vartheta\|\sigma - \alpha\|) - \phi_*''(0)\big]\|\sigma - \alpha\|\mathrm{d}\vartheta + \phi_*''(0)\|\sigma - \alpha\| \\ & \le \phi_*'(\|\sigma - \alpha\|) - \phi_*'(0) < 1. \end{aligned}$$

Then, $\pounds$ is invertible by $\mathbb{B}$-Lemma. Thus from the identity $0 = f(\alpha) - f(\sigma) = \pounds(\alpha - \sigma)$, we get $\alpha = \sigma$. Hence the theorem is proved. □

Moreover, in the case when the function ϕ_* is replaced by ϕ in majorant conditions (1)–(4) and in the definition of j_1, j_2, j_3, Q and j_4, let these replacements be denoted by k_1, k_2, k_3, Q^* and k_4, respectively. Furthermore δ_0, δ_1 and δ_* are denoted by ς_0, ς_1 and ς_*, respectively.

Theorem 2.3 *Under the condition (1)–(4) with majorant function ϕ replacing ϕ_*, with initial point $v_0 \in U(\alpha, \varsigma_*)$, the sequence $\{v_n\}$ obtained by Chebyshev-Halley method is well defined, contained in $U(\alpha, \varsigma_*)$ for all $n \geq 0$, and converges to a unique solution α of operator f from (1.1) in $U(\alpha, \varsigma_0)$. Additionally, the error estimate also holds:*

$$\|v_{n+1} - \alpha\| \leq Q^*(\varsigma_*)\|v_n - \alpha\|. \tag{2.11}$$

Again we conclude that the Chebyshev-Halley method has third order of convergence to α.

Remark 2.4 As ϕ and ϕ_* are the majorant and centered majorant condition in $U(\alpha, \delta)$ by (1.3) and (1.4), respectively, we can conclude that

$$\phi_*''(\varsigma) \leq \phi''(\varsigma), \quad \varsigma \in [0, \delta]$$

and $\dfrac{\phi''}{\phi_*''}$ can be arbitrarily large [1].

Remark 2.5 Define an autonomous differential equation f of the form [1]

$$f'(v) = \mathcal{K}(f(v)),$$

where, $\mathcal{K}$ is a differentiable operator set up as $\mathcal{K} : V_2 \longrightarrow V_1$. Then the result obtained by Theorem 2.2 can be used to find the solution α of an autonomous differential operator f. Since,

$$f''(\alpha) = f'(\alpha)\mathcal{K}'(f(\alpha)) = \mathcal{K}(f(\alpha))\mathcal{K}'(f(\alpha)) = \mathcal{K}(0)\mathcal{K}'(0), \tag{2.12}$$

we can use the result without knowing the solution α.

3. Special cases with examples

Here, we establish a relation based on Kantorovich and Smale type assumptions under the majorant condition. We consider two special cases.

3.1. Kantorovich-type assumption

We define majorizing functions

$$\phi(\varsigma) = \frac{\mu}{6}\varsigma^3 + \frac{\nu}{2}\varsigma^2 - \varsigma + a_0 \tag{3.1}$$

and

$$\phi_*(\varsigma) = \frac{\eta}{6}\varsigma^3 + \frac{\nu}{2}\varsigma^2 - \varsigma + a_1 \tag{3.2}$$

for positive μ, ν, η, a_0 and a_1. We choose these cubic polynomials ϕ and ϕ_* in Eqs. (1.3) and (1.4) satisfying (1)–(4). With the choice of $\phi(\varsigma)$ and $\phi_*(\varsigma)$ from Eqs. (3.1) and (3.2), the majorant conditions (1.3) and (1.4) reduce to the general and center Lipschitz condition, respectively

$$\big\|\Lambda_\alpha[f''(z) - f''(v)]\big\| \leq \mu\|z - v\|,$$

$$\|\Lambda_\alpha[f''(v) - f''(\alpha)]\| \leq \eta\|v - \alpha\|,$$

$\forall\, z, v \in U(\alpha, \delta)$. Functions ϕ, ϕ_* satisfy the condition (1)–(4), with $\|\Lambda_\alpha f''(\alpha)]\| \leq \nu$. Now, from Condition (3), we have

$$\delta_0 = \frac{-\nu + \sqrt{\nu^2 + 2\mu}}{\mu} \quad \text{and} \quad \varsigma_0 = \frac{-\nu + \sqrt{\nu^2 + 2\eta}}{\eta}.$$

3.2. Smale-type assumption

Denote $\gamma_* = \sup_{n>1} \| \frac{\Lambda_\alpha f^{(n)}(\alpha)}{n!} \|^{\frac{1}{n-1}}$ and suppose that f satisfies

$$\|\Lambda_\alpha f^{(n)}(\alpha)\| \leq n! \gamma_*^{n-1}, \quad n \geq 2.$$

Define two majorant functions ϕ and ϕ^* by

$$\phi(\varsigma) = \frac{\gamma_0 \varsigma^2}{1 - \gamma_0 \varsigma} - \varsigma + b_0, \quad b_0 > 0,\; 0 \leq \varsigma < \frac{1}{\gamma_0}$$

and

$$\phi_*(\varsigma) = \frac{\gamma_1 \varsigma^2}{1 - \gamma_1 \varsigma} - \varsigma + a_1, \quad b_1 > 0,\; 0 \leq \varsigma < \frac{1}{\gamma_1}.$$

Here, the majorant functions ϕ and ϕ^* satisfy all conditions (1)–(4), Eqs (1.3), and (1.4) reduce to

$$\|[\Lambda_\alpha[f''(z) - f''(v)]\| \leq 2\gamma_0 \Big[\frac{1}{(1 - \gamma_0\|z - v\| - \gamma_0\|z - \alpha\|)^3} - \frac{1}{(1 - \gamma_0\|v - \alpha\|)^3}\Big]$$

and

$$\|\Lambda_\alpha[f''(v) - f''(\sigma)]\| \leq 2\gamma_1 \Big[\frac{1}{(1 - \gamma_1\|v - \alpha\|)^3} - 1\Big].$$

By the majorant condition (3), we get

$$\delta_0 = \big(1 - \frac{1}{\sqrt{2}}\big)\frac{1}{\gamma_1} \quad \text{and} \quad \varsigma_0 = \big(1 - \frac{1}{\sqrt{2}}\big)\frac{1}{\gamma_0}.$$

Remark 3.1 We can see that $\gamma_1 \leq \gamma_0$, and $\gamma_1 = \gamma_0 = \gamma_*$ certainly can be taken as a choice for γ_1 and γ_0.

Example 3.2 Returning to Remark 2.5, let $\Omega = V_1 = V_2 = \mathbb{R}$ and define autonomous differentiable operator f as $f(v) = e^v - 1$. Then, we can choose differentiable operator

$$\mathcal{K}(v) = v + 1$$

on $\bar{B}(0, 1)$, with the max norm $\|\cdot\| = \|\cdot\|_\infty$ and norm $\|\cdot\|$ reduces to usual modulus $|\cdot|$. Here, $\alpha = 0$ is a zero of function f. Also, $f'(\alpha) = \mathcal{K}(f(\alpha)) = e^\alpha = e^0 = 1$. Then, $\|\Lambda_\alpha\| = 1$ and by Eq. (2.12), $f''(\alpha) = 1$. Note that,

$$\big\|\Lambda_\alpha\big[f''(z) - f''(v)\big]\big\| \leq e\|z - v\|,$$

$$\|\Lambda_\alpha[f''(v) - f''(\alpha)]\| \leq (e - 1)\|v - \alpha\|$$

and

$$\|\Lambda_\alpha f''(\alpha)\| \leq 1.$$

Thus, we get $\mu = e$, $\nu = 1$, and $\eta = e - 1$. Here, $\eta < \mu$ then by Remark 2.4, Eq. (2.1), $\phi_*''(\varsigma) < \phi''(\varsigma)$ and hence, we get $\delta_0 = 0.643849667898780$ and $\varsigma_0 = 0.565444814154378$. For β=0, $\frac{1}{2}$ and 1 in Eqs. (2.1) and (2.2), we get δ_1, ς_1, δ_*, ς_* and summarized in Table 1.

β	δ_1	ς_1	δ_*	ς_*
0	0.643849667898780	0.565444814154378	0.351105548877659	0.314916447380077
$\frac{1}{2}$	0.320364522820381	0.287887897567324	0.262755241747792	0.238898271178141
1	0.253825706795416	0.230120489416514	0.212739022947239	0.194933484896947

Table 1 The value of δ_1, ς_1, δ_* and ς_*

Let us take the initial point $v_0 = 0.05 \in U(0,1)$. We can generate sequence $\{v_n\}$ by using Chebyshev-Halley method. If we take β=0, $\frac{1}{2}$ and 1, the Chebyshev-Halley method becomes Chebyshev, Halley and Super-Halley method, respectively. In Tables 2–4, we summarize the comparison of error estimates given in Eqs. (2.6) and (2.11) for β=0, $\frac{1}{2}$ and 1. In these Tables we show that the error bonds on the distance $\|v_n - \alpha\|$, which is obtained by using center and general majorant function ϕ_* and ϕ rather only ϕ.

n	R.H.S of (2.6)	R.H.S of (2.11)
0	0.001013992229226	0.001260431725833
1	8.457236862715380e-09	2.019143993470636e-08
2	4.906932637824294e-24	8.300632514651191e-23
3	9.584178682871008e-70	5.766906029467023e-66
4	7.141498009277176e-207	1.933917472982467e-195
5	0	0

Table 2 Comparison of the error estimates for $\delta_* = 0.351105548877659$ and $\varsigma_* = 0.314916447380077$

n	R.H.S of (2.6)	R.H.S of (2.11)
0	0.001810536296938	0.002190201132329
1	8.596445817925019e-08	1.840882137192507e-07
2	9.201402179839152e-21	1.093079971965235e-19
3	1.128390089848572e-59	2.288388713356089e-56
4	2.081014051364180e-176	2.099729633345768e-166
5	0	0

Table 3 Comparison of the error estimates for $\delta_* = 0.262755241747792$ and $\varsigma_* = 0.238898271178141$

n	R.H.S of (2.6)	R.H.S of (2.11)
0	0.002761949237911	0.003289554751472
1	4.655354488315909e-07	9.367818415031833e-07
2	2.229276556580038e-18	2.163428256185775e-17
3	2.447921722443905e-52	2.664738515581832e-49
4	3.241143487581930e-154	4.979559243914742e-145
5	0	0

Table 4 Comparison of the error estimates for $\delta_* = 0.212739022947239$ and $\varsigma_* = 0.194933484896947$

Example 3.3 We assume a nonlinear Hammerstein integral equation of the second kind which is defined as

$$v(t)=\vartheta(t)+\lambda\int_a^b \mathcal{P}(t,\psi)v(\psi)^3\mathrm{d}\psi,\quad t\in[a,b],\ \lambda\in\mathbb{R}, \tag{3.3}$$

where, $\vartheta(t)>0$ for $t\in[a,b]$, $\vartheta(t),v(\psi)\in[a,b]$ and

$$\mathcal{P}(t,\psi)=\begin{cases}\frac{(b-t)(\psi-a)}{b-a}, & \psi\le t,\\ \frac{(b-\psi)(t-a)}{b-a}, & t\le\psi,\end{cases}$$

is the Green's kernel of the integral equation, which is non-negative and continuous in $[a,b]\times[a,b]$.

Now, we take $[a,b]$ as $[0,1]$ in Eq. (3.3). Let $V_1{=}V_2{=}\mathcal{C}[0,1]$, equipped with max norm and let $\Omega{=}\bar{U}(0{,}1)$. For solving Eq. (3.3), it is equivalent to the solution of equation $f(v)=0$, where $f:\Omega\subseteq\mathcal{C}[0,1]\longrightarrow\mathcal{C}[0,1]$ is defined as

$$f(v)(t)=v(t)-\vartheta(t)-\lambda\int_0^1 \mathcal{P}(t,\psi)v(\psi)^3\mathrm{d}\psi,\quad t\in[0,1],\ \lambda\in\mathbb{R}$$

where

$$\mathcal{P}(t,\psi)=\begin{cases}(1-t)\psi, & \psi\le t,\\ (1-\psi)t, & t\le\psi,\end{cases}$$

defined on $[0,1]\times[0,1]$. Now, here we choose $\vartheta(t)=1$ and $\lambda=\frac{2}{5}$ then,

$$f(v)(t)=v(t)-1-\frac{2}{5}\int_0^1 \mathcal{P}(t,\psi)v(\psi)^3\mathrm{d}\psi,\quad v\in\mathcal{C}[0,1],\ t\in[0,1]. \tag{3.4}$$

Now, the first and second Fréchet derivative are

$$f'(v)u(t)=u(t)-\frac{6}{5}\int_0^1 \mathcal{P}(t,\psi)v(\psi)^2u(\psi)\mathrm{d}\psi,\quad v\in\mathcal{C}[0,1],\ t\in[0,1]$$

and

$$f''(v)(uw)(t)=-\frac{12}{5}\int_0^1 \mathcal{P}(t,\psi)v(\psi)u(\psi)w(\psi)\mathrm{d}\psi,\quad v\in\mathcal{C}[0,1],\ t\in[0,1],$$

respectively. Here, $\alpha(t)=\alpha=0$ is the zero of function (3.4). So $f'(\alpha(t))=I$, then $\|f'(\alpha)^{-1}\|=\|\Lambda_\alpha\|=1$. Now,

$$\|f''(z)-f''(v)\|\le\frac{12}{5}\Big\|\int_0^1 \mathcal{P}(t,\psi)\mathrm{d}\psi\Big\|\|z-v\|.$$

Since,

$$\Big\|\int_0^1 \mathcal{P}(t,\psi)\mathrm{d}\psi\Big\|=\max_{t\in[0,1]}\Big|\int_0^t(1-t)\psi\mathrm{d}\psi+\int_t^1 t(1-\psi)\mathrm{d}\psi\Big|$$
$$=\max_{t\in[0,1]}\Big|-\frac{1}{2}\big(t-\frac{1}{2}\big)^2+\frac{1}{8}\Big|=\frac{1}{8},$$

we have

$$\big\|\Lambda_\alpha[f''(z)-f''(v)]\big\|\le\frac{3}{10}\|z-v\|.$$

Then, by the special case 3.1, we get $\mu{=}\eta{=}\frac{3}{10}$ and $\nu=0$. Thus, by Remark 2.4 and Eq. (2.1), we see that $\phi''_*(\varsigma)=\phi''(\varsigma)$ for Kantorovich-type assumption 3.1. Then, we get $\delta_0=\varsigma_0=2.581988897471611$. For $\beta{=}0$, $\frac{1}{2}$ and 1 in Eqs. (2.1) and (2.2), we get δ_1, ς_1, δ_* and ς_* which has been summarized in Table 5.

β	$\delta_1 = \varsigma_1$	$\delta_* = \varsigma_*$
0	2.581988897471611	1.233553491431155
$\frac{1}{2}$	1.490711984999860	1.041090224670559
1	1.254506483309379	0.919048899540282

Table 5 The value of δ_1, ς_1, δ_* and ς_*

Here, we get $\{v_n\}$ sequence which is generated by Chebyshev-Halley method with initial approximation v_0 converging to solution α of the operator f defined by (3.4). Here, $\phi''_*(\varsigma) = \phi''(\varsigma)$, then by Theorems 2.2 and 2.3 we present the existence and uniqueness domain of solution of Eq. (3.4) for $\beta = 0, 1$ and $\frac{1}{2}$ in Table 6.

β	Existence	Uniqueness
0	$U(0, 1.233553491431155)$	$U(0, 2.581988897471611)$
$\frac{1}{2}$	$U(0, 1.041090224670559)$	$U(0, 2.581988897471611)$
1	$U(0, 0.919048899540282)$	$U(0, 2.581988897471611)$

Table 6 Domain of existence and uniqueness of solution for (3.4)

Example 3.4 Consider the real analytic function $f : [0, 1) \longrightarrow \mathbb{R}$ defined by

$$f(v) = \frac{v}{1-v} - v \tag{3.5}$$

endowed with norm $\|\cdot\| = \|\cdot\|_\infty$. The Fréchet derivatives of Eq. (3.5) are

$$f'(v) = \frac{1}{(1-v)^2} - 2 \quad \text{and} \quad f''(v) = \frac{2}{(1-v)^3},$$

respectively. Clearly, $\alpha = 0$ is a solution of function f given by (3.5). Then $\|f'(\alpha)^{-1}\| = \|\Lambda_\alpha\| = 1$ and $f''(\alpha) = 2$. Now,

$$\|\Lambda_\alpha[f''(z) - f''(v)]\| \le 2\Big[\frac{1}{(1 - \|z-v\| - \|z-\alpha\|)^3} - \frac{1}{(1-\|v-\alpha\|)^3}\Big],$$

$$\|\Lambda_\alpha[f''(v) - f''(\alpha)]\| \le 2\Big[\frac{1}{(1-\|v-\alpha\|)^3} - 1\Big]$$

and

$$\|\Lambda_\alpha f''(\alpha)\| \le 2.$$

Thus, by Smale-type assumption 3.2, we get $\gamma_*=\gamma_0=\gamma_1=1$. Then, we have

$$\delta_0 = \varsigma_0 = 0.292893218813453.$$

For β=0, $\frac{1}{2}$ and 1 in Eqs. (2.1) and (2.2), we get δ_1, ς_1, δ_*, ς_* which are summarized in Table 7.

Here, we get the sequence $\{v_n\}$ generated by Chebyshev-Halley method with initial approximation v_0 converging to solution α of the operator f defined by (3.5). Here, $\phi''_*(\varsigma) = \phi''(\varsigma)$, then by Theorems 2.2 and 2.3, we get existence of the solution in $U(\alpha, 0.173268868573510)$, $U(\alpha, 0.132564545688955)$ and $U(\alpha, 0.108023558067875)$ converging to a unique solution α of the operator f in $U(\alpha, 0.292893218813453)$ for $\beta = 0, \frac{1}{2}$ and 1, respectively.

β	$\delta_1 = \varsigma_1$	$\delta_* = \varsigma_*$
0	0.292893218813453	0.173268868573510
$\frac{1}{2}$	0.154952231533193	0.132564545688955
1	0.124449648811983	0.108023558067875

Table 7 The value δ_1, ς_1, δ_* and ς_*

Example 3.5 Let $V_1= V_2= \mathbb{R}^2$, $\Omega = \bar{U}(0,1)$. Here, the function $f : V_1 \longrightarrow V_2$ is defined on Ω for $v = (v_1, v_2)^{\mathrm{T}}$,

$$f(v) = \big(\sin v_1,\ \frac{1}{4}(e^{v_2} + 3v_2 - 1)\big)^{\mathrm{T}} \tag{3.6}$$

endowed with the max norm $\|\cdot\| = \|\cdot\|_\infty$. Here, $\alpha = (\alpha_1, \alpha_2) = (0,0)^{\mathrm{T}}$ is a solution of (3.6). Now, Fréchet derivatives of Eq. (3.6) are

$$f'(v) = \begin{bmatrix} \cos v_1 & 0 \\ 0 & \frac{1}{4}e^{v_2} + 3 \end{bmatrix} \quad \text{and} \quad f''(v) = \left[\begin{array}{cc|cc} -\sin v_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{e^{v_2}}{4} \end{array}\right]$$

respectively. For $\alpha = (0,0)$, we get $\|\varLambda_\alpha\| = 1$. By using special case Kantorovich-type assumption 3.1, we get $\mu = \eta = 1$ and $\nu = 0.250000000000000$. By Remark 2.4 and Eq. (2.1), here we see that $\phi''_*(\varsigma) = \phi''(\varsigma)$ for Kantorovich-type assumption 3.1. Then, we get $\delta_0 = \varsigma_0 = 1.186140661634507$. For β=0, $\frac{1}{2}$, and 1 in Eqs. (2.1) and (2.2), we get δ_1, ς_1, δ_*, ς_* which are summarized in Table 8.

β	$\delta_1 = \varsigma_1$	$\delta_* = \varsigma_*$
0	1.186140661634507	0.640682921842484
$\frac{1}{2}$	0.646068014132433	0.510470854627500
1	0.530756245693649	0.431678755121912

Table 8 The value δ_1, ς_1, δ_* and ς_*

Here, we get $\{v_n\}$ sequence which is generated by Chebyshev-Halley method with initial approximation v_0 converging to the solution α of operator f defined by (3.6). Here, $\phi''_*(\varsigma) = \phi''(\varsigma)$, then by Theorems 2.2 and 2.3 we present the existence and uniqueness domain of solution for $\beta = 0, 1$ and $\frac{1}{2}$ in Table 9.

β	Existence	Uniqueness
0	$U(\alpha, 0.640682921842484)$	$U(\alpha, 1.186140661634507)$
$\frac{1}{2}$	$U(\alpha, 0.510470854627500)$	$U(\alpha, 1.186140661634507)$
1	$U(\alpha, 0.431678755121912)$	$U(\alpha, 1.186140661634507)$

Table 9 Domain of existence and uniqueness of solution for (3.6)

4. Conclusion

This article entirely lays its emphasis upon the study of local convergence of Chebyshev-Halley method for finding locally unique solution of non-linear operator $f(v) = 0$ by using majorizing function with majorant condition and their properties in $\mathbb{B}$-space. Here, under majorant condition (1)–(4), we presented a new local convergence theorem for Chebyshev-Halley method where we show that method is cubically convergent and obtained an error estimate. Beside, we specialize the majorant function ϕ and ϕ_* to obtain Kantorovich and Smale-type result and to show the efficiency of our method by some numerical examples.

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