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Wasserstein-1 Distance and Nonuniform Berry-Esseen Bound for a Supercritical Branching Process in a Random Environment

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Abstract Let $(Z_n)_{n\geq 0}$ be a supercritical branching process in an independent and identically distributed random environment. We establish an optimal convergence rate in the Wasserstein-1 distance for the process $(Z_n)_{n\geq 0}$, which completes a result of Grama et al. [Stochastic Process. Appl., 2017, **127**(4): 1255–1281]. Moreover, an exponential nonuniform Berry-Esseen bound is also given. At last, some applications of the main results to the confidence interval estimation for the criticality parameter and the population size Z_n are discussed.

Keywords Branching processes; Random environment; Wasserstein-1 distance; Nonuniform Berry-Esseen bounds

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1. Introduction

Branching process in a random environment (BPRE) initially introduced by Smith and Wilkinson [1] is a generalization of the Galton-Watson process. Denote by $\xi = (\xi_0, \xi_1, \ldots)$ a sequence of independent and identically distributed (i.i.d.) random variables, where ξ_n stands for the random environment in the n-th generation. Then the branching process $(Z_n)_{n\geq 0}$ in the random environment ξ is defined as follows:

$$Z_0 = 1,\ Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}, \quad n \geq 0,$$

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where $X_{n,i}$ represents the number of offspring produced by the i-th particle in the n-th generation. The distribution of $X_{n,i}$ depending on the environment ξ_n is denoted by

$$p(\xi_n) = \{p_k(\xi_n) = \mathbb{P}(X_{n,i} = k|\xi_n) : k \in \mathbb{N}\}.$$

Suppose that given ξ_n, $(X_{n,i})_{i\geq 1}$ is a sequence of i.i.d. random variables; moreover, $(X_{n,i})_{i\geq 1}$ is independent of $(Z_1, \dots, Z_n)$. Let $(\Gamma, \mathbb{P}_\xi)$ be the probability space under which the process is defined when the environment ξ is given, where $\mathbb{P}_\xi$ is usually called the quenched law. The state space of the random environment ξ is denoted by Θ and the total probability space can be regarded as the product space $(\Theta^{\mathbb{N}} \times \Gamma, \mathbb{P})$, where $\mathbb{P}(\mathrm{d}x, \mathrm{d}\xi) = \mathbb{P}_\xi(\mathrm{d}x)\tau(\mathrm{d}\xi)$. That is, for any measurable positive function g defined on $\Theta^{\mathbb{N}} \times \Gamma$, we have

$$\int g(x,\xi)\mathbb{P}(\mathrm{d}x, \mathrm{d}\xi) = \int\int g(x,\xi)\mathbb{P}_\xi(\mathrm{d}x)\tau(\mathrm{d}\xi),$$

where τ represents the distribution law of the random environment ξ. We usually call $\mathbb{P}$ the annealed law. And $\mathbb{P}_\xi$ can be regarded as the conditional probability of $\mathbb{P}$ given the environment ξ. The expectations with respect to $\mathbb{P}_\xi$ and $\mathbb{P}$ are denoted by $\mathbb{E}_\xi$ and $\mathbb{E}$, respectively. For any environment ξ, integer $n \geq 0$ and real number $p > 0$, define

$$m_n^{(p)} = m_n^{(p)}(\xi) = \sum_{i=0}^{\infty} i^p p_i(\xi_n), \quad m_n = m_n(\xi) = m_n^{(1)}(\xi).$$

Then

$$m_0^{(p)} = \mathbb{E}_\xi Z_1^p, \quad m_n = \mathbb{E}_\xi X_{n,i}, \quad i \geq 1.$$

Consider the following random variables

$$\Pi_0 = 1, \ \Pi_n = \Pi_n(\xi) = \prod_{i=0}^{n-1} m_i, \quad n \geq 1.$$

It is easy to see that

$$\mathbb{E}_\xi Z_{n+1} = \mathbb{E}_\xi\Big[\sum_{i=1}^{Z_n} X_{n,i}\Big] = \mathbb{E}_\xi\Big[\mathbb{E}_\xi\Big(\sum_{i=1}^{Z_n} X_{n,i}\Big|Z_n\Big)\Big] = \mathbb{E}_\xi\Big[\sum_{i=1}^{Z_n} m_n\Big] = m_n\mathbb{E}_\xi Z_n.$$

Then by recursion, we get $\Pi_n = \mathbb{E}_\xi Z_n$. Denote

$$X = \log m_0, \ \mu = \mathbb{E}X, \quad \sigma^2 = \mathbb{E}(X-\mu)^2.$$

The branching process $(Z_n)_{n\geq 0}$ is called supercritical, critical or subcritical according to $\mu > 0$, $\mu = 0$ or $\mu < 0$, respectively. Hence μ is known as the criticality parameter. Over all the paper, assume that $\mu \in (0, \infty)$, which means $(Z_n)_{n\geq 0}$ is a supercritical BPRE; and assume also

$$p_0(\xi_0) = 0 \text{ a.s.} \tag{1.1}$$

The condition (1.1) means that each particle has at least one offspring, which implies $X \geq 0$ a.s. and hence $Z_n \geq 1$ a.s., for any $n \geq 1$.

Limit theorems for BPRE have attracted a lot of interests since the seminal work of Smith and Wilkinson [1]. For critical and subcritical BPRE, the study mainly focuses on the survival probability and conditional limit theorems for the branching processes, see, for instance, Vatutin

[2] and Afanasyev et al. [3, 4]. For the supercritical BPRE, a number of researches focused on the central limit theorem, moderate and large deviations [5–12].

The aim of this paper is to establish optimal convergence rates in the Wasserstein-1 distance for the branching process in a random environment. We first recall the definition of the Wasserstein-1 distance. Let $\mathcal{L}(\mu,\nu)$ be a set of probability laws on $\mathbb{R}^2$ with marginals μ and ν. The Wasserstein-1 distance between μ and ν is defined as follows:

$$W_1(\mu,\nu) = \inf\Big\{ \int |x-y|\mathbb{P}(\mathrm{d}x,\mathrm{d}y) : \ \mathbb{P}\in\mathcal{L}(\mu,\nu)\Big\}.$$

In particular, if μ_X is the distribution of X and ν is the standard normal distribution, then we have

$$W_1(\mu_X,\nu) = d_w(X) := \int_{-\infty}^{+\infty} |\mathbb{P}(X\le x)-\Phi(x)|\mathrm{d}x.$$

The following convergence rate in the Wasserstein-1 distance between $\frac{\log Z_n - n\mu}{\sigma\sqrt{n}}$ and the standard normal random variable is due to Grama et al. [9]. Assume that there exists a constant $\varepsilon>0$ such that

$$\mathbb{E}|X|^{3+\varepsilon} < \infty. \tag{1.2}$$

Grama et al. [9] have established the following convergence rate in the Wasserstein-1 distance:

$$d_w\big(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}}\big) \le \frac{C}{\sqrt{n}}, \tag{1.3}$$

where C is a positive constant. In this paper, we are going to extend this result to the case X having a moment of order $2+\delta$, $\delta\in(0,1]$. We get the following convergence rate in the Wasserstein-1 distance for $\frac{\log Z_n - n\mu}{\sigma\sqrt{n}}$. Assume that there exist two constants $p>1$ and $\delta\in(0,1]$ such that

$$\mathbb{E}X^{2+\delta} < \infty, \quad \mathbb{E}\big(\frac{Z_1}{m_0}\big)^p < \infty, \tag{1.4}$$

then it holds

$$d_w\big(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}}\big) \le \frac{C}{n^{\delta/2}}. \tag{1.5}$$

Moreover, the same result holds when $\frac{\log Z_n - n\mu}{\sigma\sqrt{n}}$ is replaced by $\frac{n\mu - \log Z_n}{\sigma\sqrt{n}}$. Notice that the last convergence rate on the right-hand side of (1.5) coincides with the best possible convergence rate in the Wasserstein-1 distance for random walks with finite $2+\delta$ moments, and therefore inequality (1.5) is also the best possible. In addition, when X has an exponential moment and satisfies

$$\mathbb{E}\frac{Z_1^p}{m_0} < \infty,$$

we also establish a nonuniform Berry-Esseen bound with exponential decaying rate (see Theorem 2.2).

The paper is organized as follows. The main results are stated in Section 2. The applications and the proofs of the main results are given in Sections 3 and 4, respectively.

2. Main results

Throughout the paper, denote by $X_i = \log m_i,\ i \geq 0$. Evidently, $(X_i)_{i\geq 0}$ is a sequence of i.i.d. random variables depending only on the environment ξ. Let $(S_n)_{n\geq 0}$ be the random walk associated with the branching process, which is defined as follows:

$$S_0 = 0,\ S_n = \log \Pi_n = \sum_{i=1}^{n-1} X_i, \quad n \geq 1.$$

Then we have the following decomposition of $\log Z_n$, that is

$$\log Z_n = S_n + \log W_n, \tag{2.1}$$

where $W_n = \frac{Z_n}{\Pi_n}$. The normalized population size $(W_n)_{n\geq 0}$ is a non-negative martingale under both $\mathbb{P}$ and $\mathbb{P}_\xi$, with respect to the natural filtration $(\mathscr{F}_n)_{n\geq 0}$, defined by

$$\mathscr{F}_0 = \sigma\{\xi\},\ \mathscr{F}_n = \sigma\{\xi, X_{k,i}, 0 \leq k \leq n-1, i \geq 1\}, \quad n \geq 1.$$

By Doob's martingale convergence theorem and Fatou's lemma, we can obtain that W_n converges a.s. to a finite limit W and $\mathbb{E}W \leq 1$. We assume the following conditions throughout this paper:

$$\sigma \in (0, \infty) \quad \text{and} \quad \mathbb{E}\frac{Z_1}{m_0}\log Z_1 < \infty. \tag{2.2}$$

The first condition above together with (1.1) imply in particular that

$$Z_1 \geq 1 \ \text{ a.s. and } \ \mathbb{P}(Z_1 = 1) = \mathbb{E}p_1(\xi_0) < 1.$$

The second condition in (2.2) implies that W_n converges to W in $\mathbb{L}^1$ and

$$\mathbb{P}(W > 0) = \mathbb{P}(Z_n \stackrel{n\to\infty}{\longrightarrow} \infty) = \lim_{n\to\infty} \mathbb{P}(Z_n > 0) > 0.$$

(See Tanny [13] and Grama et al. [9]). Therefore, it follows with the assumption (1.1) that $W > 0$ and $Z_n \stackrel{n\to\infty}{\longrightarrow} \infty$ a.s.

Consider the following assumptions:

(A1) There exists a positive constant $\delta \in (0, 1]$, such that

$$\mathbb{E}X^{2+\delta} = \mathbb{E}(\log m_0)^{2+\delta} < \infty.$$

(A2) There exists constant $p > 1$ such that

$$\mathbb{E}\frac{m_0^{(p)}}{m_0^p} < \infty.$$

Under the conditions above, we obtain the following bound for the Wasserstein-1 distance between the distribution of $\frac{\log Z_n - n\mu}{\sigma\sqrt{n}}$ and the standard normal distribution.

Theorem 2.1 *Suppose that the conditions (A1) and (A2) are satisfied. Then*

$$d_w\Big(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}}\Big) \leq \frac{C}{n^{\delta/2}}.$$

Moreover, the same inequality holds when $\frac{\log Z_n - n\mu}{\sigma\sqrt{n}}$ *is replaced by* $-\frac{\log Z_n - n\mu}{\sigma\sqrt{n}}$.

Next, we consider the following stronger conditions than (A1) and (A2):

(A3) There exists a constant $\lambda_0 > 0$ such that

$$\mathbb{E}e^{\lambda_0 X} = \mathbb{E}m_0^{\lambda_0} < \infty.$$

(A4) There exists a constant $p > 1$ such that

$$\mathbb{E}\frac{Z_1^p}{m_0} = \mathbb{E}\frac{m_0^{(p)}}{m_0} < \infty.$$

We have the following nonuniform Berry-Esseen bound with exponential decay rate under the conditions (A3) and (A4), which is of independent interest. Such type of result can be found in Fan et al. [14], where an exponential nonuniform Berry-Esseen bound for martingales has been established.

Theorem 2.2 *Suppose that the conditions (A3) and (A4) are satisfied. Then for any $x \in \mathbb{R}$,*

$$|\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x) - \Phi(x)| \le C\frac{1}{\sqrt{n}}(1+x^2)\exp\{-\frac{\hat{x}^2}{2}\}, \tag{2.3}$$

where

$$\hat{x} = \frac{|x|}{\sqrt{1+c|x|/\sqrt{n}}}.$$

In Grama et al. [9], Cramér moderate deviations (cf. Theorem 1.3) have been obtained under the conditions (A3) and (A4). Compared to their results, the interesting feature of (2.3) is that it holds for any $x \in \mathbb{R}$ rather than only for $0 < |x| = o(\sqrt{n})$.

3. Application to interval estimation

In this section, we will discuss some applications of the main results to the confidence interval estimation for the criticality parameter μ and the population size Z_n.

3.1. Confidence intervals for μ

When σ is known, Theorem 2.1 can be applied to construct the confidence interval for μ.

Proposition 3.1 *Suppose that the conditions (A1) and (A2) are satisfied. Let $\kappa_n \in (0,1)$, such that*

$$|\log \kappa_n| = o(\log n), \quad n \to \infty. \tag{3.1}$$

Then for n large enough, $[A_n, B_n]$, with

$$A_n = \frac{\log Z_n}{n} - \frac{\sigma\Phi^{-1}(1-\frac{\kappa_n}{2})}{\sqrt{n}}, \quad B_n = \frac{\log Z_n}{n} + \frac{\sigma\Phi^{-1}(1-\frac{\kappa_n}{2})}{\sqrt{n}},$$

is a $1-\kappa_n$ confidence interval for μ.

Proof Inequality (4.26) implies that

$$\frac{\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \ge x)}{1-\Phi(x)} = 1 + o(1), \quad \frac{\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le -x)}{\Phi(-x)} = 1 + o(1) \tag{3.2}$$

uniformly for $0 \le x = o(\sqrt{\log n})$. For $p \searrow 0$, the quantile function of the standard normal distribution has the following asymptotic expansion

$$\Phi^{-1}(p) = -\sqrt{\log \frac{1}{p^2} - \log\log \frac{1}{p^2} - \log(2\pi)} + o(1).$$

In particular, when κ_n satisfies the condition (3.1), the upper $(1-\kappa_n/2)$-th quantile of standard normal distribution satisfies

$$\Phi^{-1}(1-\frac{\kappa_n}{2}) = -\Phi^{-1}(\frac{\kappa_n}{2}) = O\big(\sqrt{|\log \kappa_n|}\big),$$

is of order $o(\sqrt{\log n})$. Then, applying the last equality to (3.2), we have as $n \to \infty$,

$$\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \ge \Phi^{-1}(1-\frac{\kappa_n}{2})) \sim \frac{\kappa_n}{2} \tag{3.3}$$

and

$$\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le -\Phi^{-1}(1-\frac{\kappa_n}{2})) \sim \frac{\kappa_n}{2}. \tag{3.4}$$

Therefore, as $n \to \infty$,

$$\mathbb{P}(-\Phi^{-1}(1-\frac{\kappa_n}{2}) \le \frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le \Phi^{-1}(1-\frac{\kappa_n}{2})) \sim 1-\kappa_n, \tag{3.5}$$

which implies $\mu \in [A_n,\, B_n]$ with probability $1-\kappa_n$ for n large enough. □

3.2. Confidence intervals for Z_n

When parameters μ and σ are known, we can also apply Theorems 2.1 and 2.2 to construct confidence intervals for Z_n. Such type of results can be used to predict the future population size Z_n.

Proposition 3.2 *Let $\kappa_n \in (0,1)$. Consider the following two groups of conditions:*

(B1) Suppose that the conditions (A1) and (A2) are satisfied, and that

$$|\log \kappa_n| = o(\log n), \quad n \to \infty. \tag{3.6}$$

(B2) Suppose that the conditions (A3) and (A4) are satisfied, and that

$$|\log \kappa_n| = o(n^{1/3}), \quad n \to \infty. \tag{3.7}$$

If (B1) or (B2) holds, then for n large enough, $[A_n, B_n]$ is the confidence interval of Z_n with confidence level $1-\kappa_n$, where

$$A_n = \exp\{n\mu - \sigma\sqrt{n}\Phi^{-1}(1-\frac{\kappa_n}{2})\}, \quad B_n = \exp\{n\mu + \sigma\sqrt{n}\Phi^{-1}(1-\frac{\kappa_n}{2})\}.$$

Proof Assume (B1). By inequality (4.26) and (3.5), we have

$$\begin{aligned}&\mathbb{P}(-\Phi^{-1}(1-\frac{\kappa_n}{2}) \le \frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le \Phi^{-1}(1-\frac{\kappa_n}{2}))\\ &\quad = \mathbb{P}\big(\exp\{n\mu - \sigma\sqrt{n}\Phi^{-1}(1-\frac{\kappa_n}{2})\} \le Z_n \le \exp\{n\mu + \sigma\sqrt{n}\Phi^{-1}(1-\frac{\kappa_n}{2})\}\big) \sim 1-\kappa_n,\end{aligned}$$

as $n \to \infty$. From the equality above, when n is large enough, $[A_n, B_n]$ is the confidence interval for Z_n with confidence level $1-\kappa_n$.

Now, assume (B2). By Theorem 2.2, we have

$$\frac{\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \geq x)}{1-\Phi(x)} = 1 + o(1), \quad \frac{\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \leq -x)}{\Phi(-x)} = 1 + o(1) \tag{3.8}$$

uniformly for $0 \leq x = o(n^{1/6})$, as $n \to \infty$. When κ_n satisfies the condition (3.7), the upper $(1-\kappa_n/2)$-th quantile of the standard normal distribution satisfies

$$\Phi^{-1}(1-\kappa_n/2) = -\Phi^{-1}(\kappa_n/2) = O(\sqrt{|\log \kappa_n|}\,),$$

which is of order $o(n^{1/6})$. Then by (3.8), we get

$$\mathbb{P}(-\Phi^{-1}(1-\frac{\kappa_n}{2}) \leq \frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \leq \Phi^{-1}(1-\frac{\kappa_n}{2})) \sim 1-\kappa_n,$$

as $n \to \infty$. From the equality above, the result still holds. □

4. Proofs of main results

In this section, we give the proofs of Theorems 2.1 and 2.2, respectively.

4.1. Preliminary lemmas for Theorem 2.1

For the sum of independent random variables, Bikelis [15] gave the following nonuniform Berry-Esseen bound.

Lemma 4.1 *Let $Y_1, \ldots, Y_n$ be independent random variables satisfying $\mathbb{E}Y_i = 0$ and $\mathbb{E}|Y_i|^{2+\delta} < \infty$, $i = 1, \ldots, n$, for some positive constant $\delta \in (0,1]$. Assume that $\sum_{i=1}^n \mathbb{E}Y_i^2 = 1$. Then for any $x \in \mathbb{R}$,*

$$|\mathbb{P}\Big(\sum_{i=1}^n Y_i \leq x\Big) - \Phi(x)| \leq \frac{C}{1+|x|^{2+\delta}} \sum_{i=1}^n \mathbb{E}|Y_i|^{2+\delta}.$$

For the convergence rate of W_n, we also have the following result. See Lemmas 2.3 and 2.4 in Grama et al. [9]. See also [11].

Lemma 4.2 *Assume (A1) and (A2). Then for any $q \in (1, 1+\delta)$,*

$$\mathbb{E}|\log W|^q < \infty \tag{4.1}$$

and

$$\sup_{n\in\mathbb{N}} \mathbb{E}|\log W_n|^q < \infty. \tag{4.2}$$

Lemma 4.3 *Assume (A1) and (A2). Then there exists constant $\gamma \in (0,1)$, such that for any $n \geq 0$,*

$$\mathbb{E}|\log W_n - \log W| \leq C, \gamma^n.$$

Using the lemmas above, we obtain the following lemma, which is a refinement of Lemma 2.5 in Grama et al. [9].

Lemma 4.4 *Assume (A1) and (A2). Then for any $x \in \mathbb{R}$ and $\delta' \in (0,\delta)$,*

$$\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x, \frac{S_n - n\mu}{\sigma\sqrt{n}} \ge x) \le C\frac{1}{n^{\delta/2}}\frac{1}{1+|x|^{1+\delta'}} \tag{4.3}$$

and

$$\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \ge x, \frac{S_n - n\mu}{\sigma\sqrt{n}} \le x) \le C\frac{1}{n^{\delta/2}}\frac{1}{1+|x|^{1+\delta'}}. \tag{4.4}$$

Proof We only prove the inequality (4.3), as (4.4) can be proved in the same way. When $x > \frac{\sqrt{n}\mu}{\sigma}$, we have

$$\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x, \frac{S_n - n\mu}{\sigma\sqrt{n}} \ge x) \le \mathbb{P}(\frac{S_n - n\mu}{\sigma\sqrt{n}} \ge x),$$

which implies (4.3) by Lemma 4.1. When $x < -\frac{\sqrt{n}\mu}{\sigma}$, it follows from the fact that $Z_n \ge 1$ a.s.,

$$\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \ge -\frac{\sqrt{n}\mu}{\sigma}.$$

Then for any $x < -\frac{\sqrt{n}\mu}{\sigma}$,

$$\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x, \frac{S_n - n\mu}{\sigma\sqrt{n}} \ge x) = 0.$$

Thus for any $|x| > \frac{\sqrt{n}\mu}{\sigma}$, (4.3) holds.

Next, we prove that for any $|x| \le \frac{\sqrt{n}\mu}{\sigma}$, (4.3) holds. Let us consider the following notations

$$Y_{m,n} = \sum_{i=m+1}^{n} \frac{X_i - \mu}{\sigma\sqrt{n}},\ Y_n = Y_{0,n},\ V_m = \frac{\log W_m}{\sigma\sqrt{n}},\ D_{m,n} = V_n - V_m.$$

Set $\alpha_n = \frac{1}{\sqrt{n}}$ and $m = [\sqrt{n}]$, where $[t]$ stands for the largest integer less than t. From (2.1), we get

$$\begin{aligned}&\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x, \frac{S_n - n\mu}{\sigma\sqrt{n}} \ge x)\\ &\quad\le \mathbb{P}(Y_n + V_m \le x + \alpha_n, Y_n \ge x) + \mathbb{P}(|D_{m,n}| > \alpha_n).\end{aligned} \tag{4.5}$$

For the second term on the right-hand side of (4.5), using Markov's inequality and Lemma 4.3, for any $|x| \le \frac{\sqrt{n}\mu}{\sigma}$, there exists a constant $\gamma \in (0,1)$ such that

$$\begin{aligned}\mathbb{P}(|D_{m,n}| > \alpha_n) \le& \frac{\mathbb{E}|D_{m,n}|}{\alpha_n} = \frac{\sqrt{n}\,\mathbb{E}|\log W_n - \log W_m|}{\sigma\sqrt{n}} \le C\gamma^m\\ \le& C\frac{1}{n^{\delta/2}}\frac{1}{1+|x|^{2+\delta}}.\end{aligned} \tag{4.6}$$

The last line is obtained from the fact that $n^{\delta/2}(1+n^{(1+\delta)/2})\gamma^m = o(1)$, $n \to \infty$. For the first term on the right-hand side of (4.5), we use a decomposition to dominate it. Let

$$G_{m,n}(x) = \mathbb{P}(Y_{m,n} \le x),\ G_n(x) = \mathbb{P}(Y_n \le x),\ v_m(\mathrm{d}s,\mathrm{d}t) = \mathbb{P}(Y_m \in \mathrm{d}s, V_m \in \mathrm{d}t).$$

Since $Y_{m,n}$ and (Y_m, V_m) are independent, we have

$$\begin{aligned}&\mathbb{P}(Y_n + V_m \le x + \alpha_n, Y_n \ge x)\\ &\quad= \mathbb{P}(Y_{m,n} + Y_m + V_m \le x + \alpha_n, Y_{m,n} + Y_m \ge x)\end{aligned}$$

$$
\begin{aligned}
&= \iint \mathbb{P}(Y_{m,n}+s+t \le x+\alpha_n, Y_{m,n}+s \ge x) v_m(\mathrm{d}s,\mathrm{d}t) \\
&= \iint \mathbf{1}_{\{t\le \alpha_n\}} (G_{m,n}(x-s-t+\alpha_n) - G_{m,n}(x-s)) v_m(\mathrm{d}s,\mathrm{d}t). \qquad (4.7)
\end{aligned}
$$

Notice that $(X_i)_{i\ge 1}$ are i.i.d., so $\sum_{i=m+1}^{n} X_i$ and $\sum_{i=1}^{n-m} X_i$ follow the same distribution. We have

$$
G_{m,n}(x) = \mathbb{P}\Big(\sum_{i=m+1}^{n} \frac{X_i-\mu}{\sigma\sqrt{n}} \le x\Big) = G_{n-m}(\frac{x\sqrt{n}}{\sqrt{n-m}}) = G_{n-m}(x(1+R_n)), \qquad (4.8)
$$

where

$$
R_n = \sqrt{\frac{n}{n-m}} - 1
$$

satisfying $0 \le R_n \le \frac{C}{\sqrt{n}}$. By the mean value theorem, we deduce that for any $x \in \mathbb{R}$,

$$
\begin{aligned}
|\Phi(x(1+R_n)) - \Phi(x)| &\le R_n |x \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\}| \le \ C\frac{1}{\sqrt{n}} |x \exp\{-\frac{x^2}{2}\}| \\
&\le C\frac{1}{n^{\delta/2}} \frac{1}{1+|x|^{2+\delta}}. \qquad (4.9)
\end{aligned}
$$

It can be obtained from Lemma 4.1 that

$$
\begin{aligned}
&|G_{n-m}(x(1+R_n)) - \Phi(x(1+R_n))| = \Big|\mathbb{P}\Big(\sum_{i=1}^{n-m} \frac{X_i-\mu}{\sigma\sqrt{n-m}} \le \frac{x\sqrt{n}}{\sqrt{n-m}}\Big) - \Phi\Big(\frac{x\sqrt{n}}{\sqrt{n-m}}\Big)\Big| \\
&\le C\frac{1}{(n-m)^{\delta/2}} \frac{1}{1+|x(1+R_n)|^{2+\delta}} \le C\frac{1}{n^{\delta/2}} \frac{1}{1+|x|^{2+\delta}}. \qquad (4.10)
\end{aligned}
$$

Combining (4.8)–(4.10) together, we get

$$
|G_{m,n}(x) - \Phi(x)| \le C\frac{1}{n^{\delta/2}} \frac{1}{1+|x|^{2+\delta}}.
$$

Therefore, by (4.7), we have

$$
\mathbb{P}(Y_n + V_m \le x+\alpha_n, Y_n \ge x) \le J_1 + J_2 + J_3, \qquad (4.11)
$$

where

$$
\begin{aligned}
J_1 &= \iint \mathbf{1}_{\{t\le \alpha_n\}} |\Phi(x-s-t+\alpha_n) - \Phi(x-s)| v_m(\mathrm{d}s,\mathrm{d}t), \\
J_2 &= C\frac{1}{n^{\delta/2}} \iint \mathbf{1}_{\{t\le \alpha_n\}} \frac{1}{1+|x-s|^{2+\delta}} v_m(\mathrm{d}s,\mathrm{d}t)
\end{aligned}
$$

and

$$
J_3 = C\frac{1}{n^{\delta/2}} \iint \mathbf{1}_{\{t\le \alpha_n\}} \frac{1}{1+|x-s-t+\alpha_n|^{2+\delta}} v_m(\mathrm{d}s,\mathrm{d}t).
$$

For J_1, by the mean value theorem, it holds

$$
\begin{aligned}
&\mathbf{1}_{\{t\le \alpha_n\}} |\Phi(x-s-t+\alpha_n) - \Phi(x-s)| \\
&\le C|-t+\alpha_n| \exp\{-\frac{x^2}{8}\} + |-t+\alpha_n| \mathbf{1}_{\{|s|\ge 1+\frac{1}{4}|x|\}} + |-t+\alpha_n| \mathbf{1}_{\{|t|\ge 1+\frac{1}{4}|x|\}},
\end{aligned}
$$

and thus we have

$$
J_1 \le J_{11} + J_{12} + J_{13}, \qquad (4.12)
$$

where

$$J_{11} = C \iint |-t+\alpha_n| \exp\{-\frac{x^2}{8}\} v_m(\mathrm{d}s,\mathrm{d}t),$$
$$J_{12} = \iint |-t+\alpha_n| \mathbf{1}_{\{|s|\geq 1+\frac{1}{4}|x|\}} v_m(\mathrm{d}s,\mathrm{d}t)$$

and

$$J_{13} = \iint |-t+\alpha_n| \mathbf{1}_{\{|t|\geq 1+\frac{1}{4}|x|\}} v_m(\mathrm{d}s,\mathrm{d}t).$$

Firstly, considering J_{11}, it follows from Lemma 4.2 that for any $x \in \mathbb{R}$,

$$J_{11} \leq C \exp\{-\frac{x^2}{8}\}(\mathbb{E}|V_m| + \alpha_n) \leq C \frac{1}{n^{\delta/2}} \frac{1}{1+|x|^{2+\delta}}. \tag{4.13}$$

For J_{12}, let τ, $\iota > 1$ satisfy $\frac{1}{\tau}+\frac{1}{\iota}=1$. By Hölder's inequality, we have the following estimate: for any $x \in \mathbb{R}$,

$$\begin{aligned} J_{12} &\leq \iint |t| \mathbf{1}_{\{|s|\geq 1+\frac{1}{4}|x|\}} v_m(\mathrm{d}s,\mathrm{d}t) + \alpha_n \int \mathbf{1}_{\{|s|\geq 1+\frac{1}{4}|x|\}} v_m(\mathrm{d}s) \\ &\leq \Big(\int |t|^\tau v_m(\mathrm{d}t)\Big)^{\frac{1}{\tau}} \Big(\int \mathbf{1}_{\{|s|\geq 1+\frac{1}{4}|x|\}} v_m(\mathrm{d}s)\Big)^{\frac{1}{\iota}} + \frac{1}{\sqrt{n}}\, \mathbb{P}(|Y_m| \geq 1+\frac{1}{4}|x|) \\ &\leq C\frac{1}{\sqrt{n}} [\mathbb{P}(|Y_m| \geq 1+\frac{1}{4}|x|)]^{\frac{1}{\iota}} + \frac{1}{\sqrt{n}} \mathbb{P}(|Y_m| \geq 1+\frac{1}{4}|x|). \end{aligned} \tag{4.14}$$

From Lemma 4.1, we get

$$\begin{aligned} \mathbb{P}(|Y_m| \geq 1+\frac{1}{4}|x|) =& \mathbb{P}\big(Y_m \geq 1+\frac{1}{4}|x|\big) + \mathbb{P}\big(Y_m \leq -1-\frac{1}{4}|x|\big) \\ \leq& 1-\Phi\big(\big(1+\frac{1}{4}|x|\big)\frac{\sqrt{n}}{\sqrt{m}}\big) + \Phi\big(-\big(1+\frac{1}{4}|x|\big)\frac{\sqrt{n}}{\sqrt{m}}\big) + \\ & C\frac{1}{1+((1+|x|/4)\sqrt{n}/\sqrt{m})^{2+\delta}} \sum_{i=1}^m \mathbb{E}|\frac{X_i-\mu}{\sigma\sqrt{m}}|^{2+\delta} \\ \leq& C\frac{1}{n^{(1+\delta)/2}} \frac{1}{1+|x|^{2+\delta}}. \end{aligned} \tag{4.15}$$

If

$$\delta' = \frac{3+2\delta}{\iota} - \delta,$$

we can obtain

$$\delta' = 3+\delta - \frac{3+2\delta}{\tau}. \tag{4.16}$$

Substituting (4.15) into (4.14), we get, for any $|x| \leq \mu\sqrt{n}/\sigma$,

$$J_{12} \leq C\frac{1}{n^{\delta/2}} \frac{1}{1+|x|^{1+\delta'}}. \tag{4.17}$$

Next, we turn to J_{13}. Set $p = \frac{\delta'+\delta}{2}$. Using Markov's inequality and Lemma 4.2, we have for any $|x| \leq \frac{\sqrt{n}\mu}{\sigma}$,

$$\begin{aligned} J_{13} &\leq \int |t| \mathbf{1}_{\{|t|\geq 1+\frac{1}{4}|x|\}} v_m(\mathrm{d}t) + \int \alpha_n \mathbf{1}_{\{|t|\geq 1+\frac{1}{4}|x|\}} v_m(\mathrm{d}t) \\ &\leq \int \frac{|t|^p}{(1+|x|/4)^p}\, |t|\; v_m(\mathrm{d}t) + \frac{1}{\sqrt{n}} \mathbb{P}(|V_m| \geq 1+\frac{1}{4}|x|) \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{(1+|x|/4)^p}\mathbb{E}|V_m|^{1+p} + \frac{1}{\sqrt{n}}\frac{1}{(1+|x|/4)^{1+p}}\mathbb{E}|V_m|^{1+p} \\
&\leq C\frac{1}{n^{\delta/2}}\frac{1}{1+|x|^{1+\delta'}}. \qquad (4.18)
\end{aligned}
$$

Combining (4.12)–(4.18), we get

$$
J_1 \leq C\frac{1}{n^{\delta/2}}\frac{1}{1+|x|^{1+\delta'}}. \qquad (4.19)
$$

Now we discuss the upper bounds for J_2 and J_3, respectively. By an argument similar to that of (4.15), we can obtain for any $x \in \mathbb{R}$,

$$
\begin{aligned}
J_2 =& C\frac{1}{n^{\delta/2}}\iint \mathbf{1}_{\{t\leq\alpha_n\}}\frac{1}{1+|x-s|^{2+\delta}}v_m(\mathrm{d}s,\mathrm{d}t) \\
\leq& C\frac{1}{n^{\delta/2}}\Big(\int_{|s|<1+|x|/2}\frac{1}{1+|x-s|^{2+\delta}}v_m(\mathrm{d}s) + \int_{|s|\geq 1+|x|/2}\frac{1}{1+|x-s|^{2+\delta}}v_m(\mathrm{d}s)\Big) \\
\leq& C\frac{1}{n^{\delta/2}}\Big[\frac{1}{1+|x/2|^{2+\delta}} + \mathbb{P}\big(|Y_m| \geq 1+\frac{1}{2}|x|\big)\Big] \\
\leq& C\frac{1}{n^{\delta/2}}\frac{1}{1+|x|^{2+\delta}}. \qquad (4.20)
\end{aligned}
$$

For J_3, using an argument similar to that of (4.15) and (4.18), we have for any $|x| \leq \frac{\sqrt{n}\mu}{\sigma}$,

$$
\begin{aligned}
J_3 =& C\frac{1}{n^{\delta/2}}\iint \mathbf{1}_{\{t\leq\alpha_n\}}\frac{1}{1+|x-s-t|^{2+\delta}}v_m(\mathrm{d}s,\mathrm{d}t) \\
\leq& C\frac{1}{n^{\delta/2}}\Big(\iint_{|s+t|<2+|x|/2}\frac{1}{1+|x/2|^{2+\delta}}v_m(\mathrm{d}s,\mathrm{d}t) + \\
&\int_{|s|\geq 1+|x|/4}v_m(\mathrm{d}s) + \int_{|t|\geq 1+|x|/4}v_m(\mathrm{d}t)\Big) \\
\leq& C\frac{1}{n^{\delta/2}}\Big[\frac{1}{1+|x/2|^{2+\delta}} + \mathbb{P}(|Y_m| \geq 1+\frac{1}{4}|x|) + \mathbb{P}\big(|V_m| \geq 1+\frac{1}{4}|x|\big)\Big] \\
\leq& C\frac{1}{n^{\delta/2}}\frac{1}{1+|x|^{2+\delta}}. \qquad (4.21)
\end{aligned}
$$

Then, substituting (4.19)–(4.21) into (4.11), we get

$$
\mathbb{P}(Y_n + V_m \leq x + \alpha_n, Y_n \geq x) \leq C\frac{1}{n^{\delta/2}}\frac{1}{1+|x|^{1+\delta'}}. \qquad (4.22)
$$

Finally, from (4.5), (4.6) and (4.22), we have (4.3) holds for any $|x| \leq \frac{\sqrt{n}\mu}{\sigma}$. $\square$

4.2. Proof of Theorem 2.1

From (2.1), it holds

$$
\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} = \frac{S_n - n\mu}{\sigma\sqrt{n}} + \frac{\log W_n}{\sigma\sqrt{n}}.
$$

By Lemma 4.1 and (A1), we have

$$
|\mathbb{P}(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x) - \Phi(x)| \leq C\sum_{i=1}^{n}\mathbb{E}|\frac{X_i-\mu}{\sigma\sqrt{n}}|^{2+\delta}\frac{1}{(1+|x|)^{2+\delta}} \leq C\frac{1}{n^{\delta/2}}\frac{1}{1+|x|^{2+\delta}}. \qquad (4.23)
$$

Notice that

$$\begin{aligned}
&\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x) \\
&= \mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x, \frac{S_n - n\mu}{\sigma\sqrt{n}} \le x) + \mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x, \frac{S_n - n\mu}{\sigma\sqrt{n}} > x) \\
&= \mathbb{P}(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x) + \mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x, \frac{S_n - n\mu}{\sigma\sqrt{n}} > x) - \\
&\quad \mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} > x, \frac{S_n - n\mu}{\sigma\sqrt{n}} \le x). \qquad (4.24)
\end{aligned}$$

Applying Lemma 4.4 to the last equality, we get for any $x \in \mathbb{R}$,

$$|\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x) - \mathbb{P}(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x)| \le C\frac{1}{n^{\delta/2}}\frac{1}{1+|x|^{1+\delta'}}. \qquad (4.25)$$

Combining (4.23) and (4.25), we have for any $x \in \mathbb{R}$,

$$\begin{aligned}
&|\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x) - \Phi(x)| \le |\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x) - \mathbb{P}(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x)| + \\
&\quad |\mathbb{P}(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x) - \Phi(x)| \le C\frac{1}{n^{\delta/2}}\frac{1}{1+|x|^{1+\delta'}}. \qquad (4.26)
\end{aligned}$$

Thus

$$\begin{aligned}
d_w(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}}) &= \int_{-\infty}^{\infty} |\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x) - \Phi(x)|\mathrm{d}x \\
&\le \frac{C}{n^{\delta/2}}\int_{-\infty}^{\infty} \frac{1}{1+|x|^{1+\delta'}}\mathrm{d}x \le \frac{C}{n^{\delta/2}},
\end{aligned}$$

which gives the first desired inequality of Theorem 2.1. By a similar argument, it is easy to show that the same result holds when $\frac{\log Z_n - n\mu}{\sigma\sqrt{n}}$ is replaced by $-\frac{\log Z_n - n\mu}{\sigma\sqrt{n}}$. $\square$

4.3. Preliminary Lemmas for Theorem 2.2

To prove Theorem 2.2, we shall make use of the following lemma [9, Theorem 3.1]. The lemma shows that conditions (A3) and (A4) imply the existence of a harmonic moment of positive order a_0.

Lemma 4.5 *Assume (A3) and (A4). Then there exists a positive a_0 such that*

$$\mathbb{E}W^{-a_0} < \infty, \quad \sup_{n\in\mathbb{N}} \mathbb{E}W_n^{-a_0} < \infty. \qquad (4.27)$$

The next lemma shows that in the case of i.i.d., Cramér's condition (A3) and Bernstein's condition (A3′) are equivalent [16].

Lemma 4.6 *Condition (A3) is equivalent to the following condition (A3′). There exists a constant $H > 0$ such that for any $k \ge 2$,*

$$\mathbb{E}(X-\mu)^k \le \frac{1}{2}k!H^{k-2}\mathbb{E}(X-\mu)^2. \qquad (4.28)$$

The following lemma gives two Bernstein type inequalities for $\log Z_n$.

Lemma 4.7 *Suppose that the conditions (A3) and (A4) are satisfied. Then for any $x \geq 0$,*

$$\mathbb{P}\Big(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \geq x\Big) \leq 2\exp\{-\frac{x^2}{2(1+cx/\sqrt{n})}\} \tag{4.29}$$

and

$$\mathbb{P}\Big(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \leq -x\Big) \leq C\exp\{-\frac{x^2}{2(1+cx/\sqrt{n})}\}, \tag{4.30}$$

where c, C are two positive constants.

Proof Since the Cramér's condition (A3) is equivalent to the Bernstein condition (A3′), we only need to prove Lemma 4.7 under the conditions (A3′) and (A4).

We first give a proof for (4.29). Denote

$$\eta_{n,i} = \frac{X_i - \mu}{\sigma\sqrt{n}}, \quad i = 1, \ldots, n.$$

It is easy to see that

$$\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} = \sum_{i=1}^n \eta_{n,i} + \frac{\log W_n}{\sigma\sqrt{n}},$$

where $\sum_{i=1}^n \eta_{n,i}$ is a sum of i.i.d. random variables. Then we have for any $x \geq 0$,

$$\mathbb{P}\Big(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \geq x\Big) = \mathbb{P}\Big(\sum_{i=1}^n \eta_{n,i} + \frac{\log W_n}{\sigma\sqrt{n}} \geq x\Big) \leq I_1 + I_2, \tag{4.31}$$

where

$$I_1 = \mathbb{P}\Big(\sum_{i=1}^n \eta_{n,i} \geq x - \frac{x^2}{\sigma\sqrt{n}}\Big), \quad I_2 = \mathbb{P}\Big(\frac{\log W_n}{\sigma\sqrt{n}} \geq \frac{x^2}{\sigma\sqrt{n}}\Big).$$

Applying Bernstein's inequality to I_1, we obtain for any $x \in [0, \frac{\sigma\sqrt{n}}{2})$,

$$I_1 \leq \exp\Big\{-\frac{x^2(1-\frac{x}{\sigma\sqrt{n}})^2}{2(1+\frac{H}{\sigma\sqrt{n}}x(1-\frac{x}{\sigma\sqrt{n}}))}\Big\} \leq \exp\Big\{-\frac{x^2}{2(1+cx/\sqrt{n})}\Big\}. \tag{4.32}$$

By Markov's inequality and the fact $\mathbb{E}W_n = 1$, we have for any $x \in [0, \frac{\sigma\sqrt{n}}{2})$,

$$I_2 = \mathbb{P}(W_n \geq \exp\{x^2\}) \leq \exp\{-x^2\}\mathbb{E}W_n = \exp\{-x^2\}. \tag{4.33}$$

Combining (4.31)–(4.33) together, we find that (4.29) holds for any $x \in [0, \frac{\sigma\sqrt{n}}{2})$. When $x \geq \frac{\sigma\sqrt{n}}{2}$, it holds that

$$\mathbb{P}\Big(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \geq x\Big) \leq I_3 + I_4, \tag{4.34}$$

where

$$I_3 = \mathbb{P}\Big(\sum_{i=1}^n \eta_{n,i} \geq \frac{x}{2}\Big), \quad I_4 = \mathbb{P}\Big(\frac{\log W_n}{\sigma\sqrt{n}} \geq \frac{x}{2}\Big).$$

By the same arguments as the proofs of I_1 and I_2, we have for any $x \geq \frac{\sigma\sqrt{n}}{2}$,

$$I_3 \leq \exp\{-\frac{(x/2)^2}{2(1+\frac{H}{\sigma\sqrt{n}}\frac{x}{2})}\} \leq \exp\{-\frac{x^2}{2(1+cx/\sqrt{n})}\} \tag{4.35}$$

and

$$I_4 \le \exp\{-\frac{x\sigma\sqrt{n}}{2}\}\mathbb{E}W_n \le \exp\{-\frac{x^2}{2(1+cx/\sqrt{n})}\} \tag{4.36}$$

with c large enough. Combining (4.34)–(4.36) together, we get (4.29) for $x \ge \frac{\sigma\sqrt{n}}{2}$.

Next, we will prove (4.30). Let a_0 be a positive constant given by Lemma 4.5. Then it holds for any $x \ge 0$,

$$\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le -x) = \mathbb{P}\Big(-\sum_{i=1}^{n}\eta_{n,i} - \frac{\log W_n}{\sigma\sqrt{n}} \ge x\Big) \le I_5 + I_6, \tag{4.37}$$

where

$$I_5 = \mathbb{P}\Big(-\sum_{i=1}^{n}\eta_{n,i} \ge x - \frac{x^2}{a_0\sigma\sqrt{n}}\Big), \quad I_6 = \mathbb{P}(-\frac{\log W_n}{\sigma\sqrt{n}} \ge \frac{x^2}{a_0\sigma\sqrt{n}}).$$

The upper bounds for I_5 and I_6 are given respectively as follows. Using Bernstein's inequality, we get for any $x \in [0, \frac{a_0\sigma\sqrt{n}}{2})$,

$$I_5 \le \exp\Big\{-\frac{x^2(1-\frac{x}{a_0\sigma\sqrt{n}})^2}{2\big(1+\frac{H}{\sigma\sqrt{n}}x\big(1-\frac{x}{a_0\sigma\sqrt{n}}\big)\big)}\Big\} \le \exp\{-\frac{x^2}{2(1+cx/\sqrt{n})}\}. \tag{4.38}$$

And by Markov's inequality, we get for any $x \in [0, \frac{a_0\sigma\sqrt{n}}{2})$,

$$\begin{aligned} I_6 =& \mathbb{P}(\log W_n \le -\frac{x^2}{a_0}) = \mathbb{P}(W_n^{-a_0} \ge \exp\{x^2\}) \\ \le& \exp\{-x^2\}\mathbb{E}W_n^{-a_0} < \infty. \end{aligned} \tag{4.39}$$

Combining (4.37)–(4.39) together, we get (4.30) for any $x \in [0, \frac{a_0\sigma\sqrt{n}}{2})$. When $x \ge \frac{a_0\sigma\sqrt{n}}{2}$, it holds that

$$\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le -x) \le I_7 + I_8,$$

where

$$I_7 = \mathbb{P}\Big(-\sum_{i=1}^{n}\eta_{n,i} \ge \frac{x}{2}\Big), \quad I_8 = \mathbb{P}(-\frac{\log W_n}{\sigma\sqrt{n}} \ge \frac{x}{2}).$$

Again by Bernstein's inequality, we obtain for any $x \ge \frac{a_0\sigma\sqrt{n}}{2}$,

$$I_7 \le \exp\Big\{-\frac{(x/2)^2}{2\big(1+\frac{H}{\sigma\sqrt{n}}\frac{x}{2}\big)}\Big\} \le \exp\{-\frac{x^2}{2(1+cx/\sqrt{n})}\}.$$

And by the same arguments as the proof of I_6, we can get for any $x \ge \frac{a_0\sigma\sqrt{n}}{2}$,

$$I_8 = \mathbb{P}(\log W_n \le -\frac{\sigma\sqrt{n}x}{2}) \le \exp\{-\frac{a_0\sigma\sqrt{n}x}{2}\}\mathbb{E}W_n^{-a_0} \le C\exp\{-\frac{x^2}{2(1+cx/\sqrt{n})}\}.$$

This completes the proof of lemma. □

The following lemma is a direct consequence of Theorem 1.3 in [9] and its detailed proof can be found in [9].

Lemma 4.8 *Assume (A3) and (A4). Then for any* $x \in [0, n^{1/4}]$,

$$\Big|\log\frac{\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} > x)}{1-\Phi(x)}\Big| \le C\frac{1+x^3}{\sqrt{n}}$$

and

$$|\log \frac{\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le -x)}{\Phi(-x)}| \le C\frac{1+x^3}{\sqrt{n}}.$$

4.4. Proof of Theorem 2.2

When $x \in [0, n^{1/4}]$, by the first inequality in Lemma 4.8, we get

$$\begin{aligned}
&\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x) - \Phi(x) = -[\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} > x) - (1-\Phi(x))] \\
&\ge -[(1-\Phi(x))\exp\{C\frac{1+x^3}{\sqrt{n}}\} - (1-\Phi(x))] \\
&= -(1-\Phi(x))(\exp\{C\frac{1+x^3}{\sqrt{n}}\} - 1). \qquad (4.40)
\end{aligned}$$

From the inequality $e^x - 1 \le xe^x$ $(x \ge 0)$, we have

$$\exp\{C\frac{1+x^3}{\sqrt{n}}\} - 1 \le C\frac{1+x^3}{\sqrt{n}}\exp\{C\frac{1+x^3}{\sqrt{n}}\}. \qquad (4.41)$$

Using the inequalities

$$\frac{e^{-x^2/2}}{\sqrt{2\pi}(1+x)} \le 1-\Phi(x) \le \frac{e^{-x^2/2}}{\sqrt{\pi}(1+x)}, \quad x \ge 0, \qquad (4.42)$$

and the inequalities (4.40) and (4.41), we get

$$\begin{aligned}
&\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x) - \Phi(x) \ge -\frac{C(x^2+1-x)}{\sqrt{2\pi n}}\exp\{-\frac{x^2}{2} + C\frac{1+x^3}{\sqrt{n}}\} \\
&\ge -C\frac{1}{\sqrt{n}}(1+x^2)\exp\{-\frac{x^2}{2} + C\frac{x^3}{\sqrt{n}}\}. \qquad (4.43)
\end{aligned}$$

For any $x \in [0, n^{1/4}]$, it holds

$$1 - C\frac{x}{\sqrt{n}} = \frac{1}{1+\sum_{k=1}^{\infty}(Cx/\sqrt{n})^k} = \frac{1}{1+Cx/\sqrt{n}(\frac{1}{1-Cx/\sqrt{n}})} \ge \frac{1}{1+Cx/\sqrt{n}}.$$

Thus, we have for any $x \in [0, n^{1/4}]$,

$$\exp\{-\frac{x^2}{2} + C\frac{x^3}{\sqrt{n}}\} = \exp\{-\frac{x^2}{2}(1-C\frac{x}{\sqrt{n}})\} \le \exp\{-\frac{x^2}{2(1+Cx/\sqrt{n})}\}. \qquad (4.44)$$

Applying (4.44) to (4.43), we obtain for any $x \in [0, n^{1/4}]$,

$$\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x) - \Phi(x) \ge -C\frac{1}{\sqrt{n}}(1+x^2)\exp\{-\frac{x^2}{2(1+cx/\sqrt{n})}\}. \qquad (4.45)$$

Next, we prove the upper bound of $\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x) - \Phi(x)$. Again using the first inequality in Lemma 4.8, we get

$$\begin{aligned}
&\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x) - \Phi(x) = -[\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} > x) - (1-\Phi(x))] \\
&\le -[(1-\Phi(x))\exp\{-C\frac{1+x^3}{\sqrt{n}}\} - (1-\Phi(x))] \\
&= -(1-\Phi(x))(\exp\{-C\frac{1+x^3}{\sqrt{n}}\} - 1). \qquad (4.46)
\end{aligned}$$

From the inequality $e^x - 1 \geq x$, $x \leq 0$, we have for any $x \leq 0$,

$$\exp\{-C\frac{1+x^3}{\sqrt{n}}\} - 1 \geq -C\frac{1+x^3}{\sqrt{n}}. \tag{4.47}$$

Applying (4.42) and (4.47) to (4.46), we obtain

$$\begin{aligned}\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \leq x) - \Phi(x) &\leq \frac{C(1+x^2-x)}{\sqrt{\pi n}}\exp\{-\frac{x^2}{2}\} \\ &\leq C\frac{1}{\sqrt{n}}(1+x^2)\exp\{-\frac{x^2}{2(1+cx/\sqrt{n})}\}.\end{aligned} \tag{4.48}$$

Combining (4.45) and (4.48) together, we get for any $x \in [0,\, n^{1/4}]$,

$$|\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \leq x) - \Phi(x)| \leq C\frac{1}{\sqrt{n}}(1+x^2)\exp\{-\frac{x^2}{2(1+cx/\sqrt{n})}\}.$$

When $x > n^{1/4}$, we have

$$\begin{aligned}|\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \leq x) - \Phi(x)| &= |\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} > x) - (1-\Phi(x))| \\ &\leq \mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} > x) + 1 - \Phi(x).\end{aligned} \tag{4.49}$$

By the first inequality in Lemmas 4.7 and 4.6, it follows that for any $x > n^{1/4}$,

$$\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} > x) \leq C\exp\{-\frac{x^2}{2(1+cx/\sqrt{n})}\}. \tag{4.50}$$

Notice that

$$1 - \Phi(x) \leq \frac{e^{-x^2/2}}{\sqrt{\pi}(1+x)} \leq \frac{1}{1+x}\exp\{-\frac{x^2}{2(1+cx/\sqrt{n})}\}. \tag{4.51}$$

And it holds for any $x > n^{1/4}$,

$$\frac{1}{\sqrt{n}}(1+x^2) \geq 1 \quad \text{and} \quad \frac{1}{1+x} \leq \frac{1}{\sqrt{n}}(1+x^2). \tag{4.52}$$

Combining (4.42) and (4.49)–(4.52) together, we get

$$|\mathbb{P}(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \leq x) - \Phi(x)| \leq C\frac{1}{\sqrt{n}}(1+x^2)\exp\{-\frac{x^2}{2(1+cx/\sqrt{n})}\},$$

which gives the desired inequality for $x > n^{1/4}$.

For the case when $x < 0$, it can be proved in a similar way, but (4.42) is replaced by

$$\frac{e^{-x^2/2}}{\sqrt{2\pi}(1+|x|)} \leq \Phi(x) \leq \frac{e^{-x^2/2}}{\sqrt{\pi}(1+|x|)}, \quad x \leq 0,$$

and the second inequalities in Lemmas 4.7 and 4.8 are used for the cases $x \in (-\infty,\, -n^{1/4})$ and $x \in [-n^{1/4},\, 0]$, respectively. □

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