*Journal of Mathematical Research with Applications*
Mar., 2024, Vol. 44, No. 2, pp. 143–151
DOI:10.3770/j.issn:2095-2651.2024.02.001
Http://jmre.dlut.edu.cn

# Equitable Total Coloring of Fibonacci Graphs

**Yong LI[1], Chunling TONG[1,2,*], Senyuan SU[1], Yanan SU[1]**
1. *School of Information Science and Electricity Engineering, Shandong Jiaotong University, Shandong* 250357*, P. R. China;*
2. *Department of Mathematics, Simon Fraser University, BC V*5*A* 1*S*6*, Canada*

**Abstract** The equitable total coloring of a graph $G$ is a total coloring such that the numbers of elements in any two colors differ by at most one. The smallest number of colors needed for an equitable total coloring is called the equitable total chromatic number. This paper contributes to the equitable total coloring of Fibonacci graphs $F_{\Delta,n}$. We determine the equitable total chromatic numbers of $F_{\Delta,n}$ for $\Delta = 3, 4, 5$ and propose a conjecture on that for $\Delta >= 6$.

**Keywords** Fibonacci graph; equitable total coloring; equitable total chromatic number

**MR(2020) Subject Classification** 05C15

## 1. Introduction

Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. A total $k$-coloring of a graph $G$ is a map $\sigma$: $V(G) \cup E(G) \to \{1, 2, \dots, k\}$ such that no two adjacent or incident elements of $V(G) \cup E(G)$ receive the same color. The smallest number of colors needed for a total coloring of $G$ is known as the total chromatic number, denoted as $\chi''(G)$. Determining total chromatic number is NP-complete [1], and NP-hard even for $k$-regular bipartite graphs with $k \geq 3$ (see [2]).

A total $k$-coloring is said to be equitable if the numbers of elements in any two colors differ by at most one, i.e., $||\sigma^{-1}(i)| - |\sigma^{-1}(j)|| \leq 1$ for each pair of distinct colors $i$ and $j$ $(1 \leq i, j \leq k)$, where $|\sigma^{-1}(i)|$ denotes the number of elements in color $i$. The smallest number of colors needed for an equitable total coloring of $G$ is specially called the equitable total chromatic number of $G$, denoted as $\chi''_{=}(G)$.

Fu [3] first introduced the concept of equitable total coloring and conjectured that any graph $G$ has an equitable total $k$-coloring for each $k \geq \max(\chi''(G), \Delta(G) + 2)$. Wang [4] further conjectured that $\chi''_{=}(G) \leq \Delta(G) + 2$ for any graph $G$. These conjectures are verified by trees [3], complete $r$-partite graphs [5–7], subcubic graphs [8], coronas of cubic graphs [9] and all multigraphs $G$ with $\Delta(G) \leq 3$ (see [4]). Exact values of equitable total chromatic number have

Received January 8, 2023; Accepted November 15, 2023
Supported by the National Natural Science Foundation of China (Grant No. 62072292) and the Natural Science Foundation of Shandong Province (Grant No. ZR2020KF010).
* Corresponding author
E-mail address: liy@stu.sdjtu.edu.cn (Yong LI); tongcl@sdjtu.edu.cn (Chunling TONG); susy@stu.sdjtu.edu.cn (Senyuan SU); suyn@stu.sdjtu.edu.cn (Yanan SU)

been determined for some particular classes of graphs, such as generalized Petersen graphs [10], Knödel graphs [11], snarks [12], Cartesian product graphs [13], join graphs [14, 15], and some others [16–18]. However, the equitable total chromatic numbers for a great mass of graphs, including Fibonacci graphs, remain open even after many efforts.

In this paper, we study the equitable total coloring of Fibonacci graphs $F_{\Delta,n}$. We aim to determine the equitable total chromatic numbers of $F_{\Delta,n}$ for $\Delta = 3, 4, 5$.

The paper is organized as follows. In Section 2, we recapitulate the definition of Fibonacci graphs with examples. In Sections 3–5, we determine the equitable total chromatic numbers of Fibonacci graphs for $\Delta = 3, 4, 5$, respectively. Section 6 is our conclusion.

## 2. Fibonacci graphs

The Fibonacci graphs $F_{\Delta,n}$ are an important class of bipartite graphs. They are extensively studied for the purpose of fast communication in networks [19–21] and deserve a lot of attention in this context.

By definition [22, 23], $F_{\Delta,n}$ is a graph defined on $2n$ ($n \geq 1$) vertices with $V(F_{\Delta,n}) = \{v_i, u_i : 0 \leq i \leq n-1\}$ and $E(F_{\Delta,n}) = \{v_i u_{(i+F(j)-1) \bmod n} : 0 \leq i \leq n-1, 1 \leq j \leq F^{-1}(n)\}$, where $F(j)$ denotes the $j$-th Fibonacci number ($F(0) = F(1) = 1$, and $F(j) = F(j-1) + F(j-2)$ for $j \geq 2$) and $F^{-1}(n)$ denotes the integer $j$ for which $n \geq F(j)$. For example, $\Delta = 3$, there is $V(F_{3,n}) = \{v_i, u_i : 0 \leq i \leq n-1\}$ and $E(F_{3,n}) = \{v_i u_i, v_i u_{(i+1) \bmod n}, v_i u_{(i+2) \bmod n} : 0 \leq i \leq n-1\}$. Figure 1 shows Fibonacci graphs $F_{3,4}$ and $F_{3,5}$.

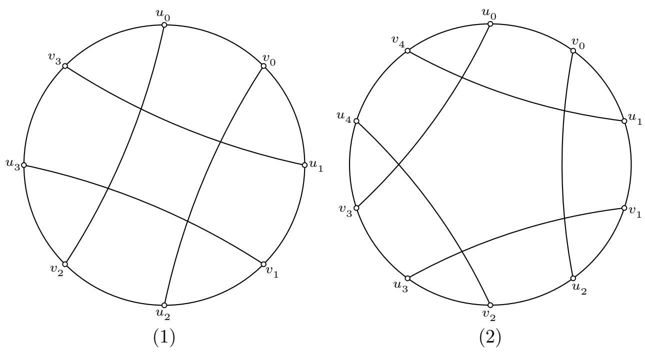


Figure 1 The Fibonacci graphs $F_{3,4}$ and $F_{3,5}$

## 3. Equitable total coloring of $F_{3,n}$

We determine the equitable total chromatic number of $F_{3,n}$ in this section.

**Lemma 3.1** *$\chi''_{=}(F_{3,n}) = 4$ for $n \pmod 2 = 0$ and 5 for $n \pmod 2 = 1$.*

**Proof** We first show $\chi''_{=}(F_{3,n}) = 4$ for $n \pmod 2 = 0$. Let $V_1 = \{u_i : 0 \leq i \leq n-1\}$, $V_2 = \{v_i : 0 \leq i \leq n-1\}$, $E_1 = \{v_i u_i : 0 \leq i \leq n-1\}$, $E_2 = \{v_i u_{(i+1) \bmod n} : 0 \leq i \leq n-1\}$, $E_3 = \{v_i u_{(i+2) \bmod n} : 0 \leq i \leq n-1\}$ and $n_4 = n \bmod 4$. We construct a 4-total coloring of

$F_{3,n}$ for $n \pmod 2 = 0$ as follows:

$$\sigma(V_1) = \begin{cases} (4231)^{\frac{n}{4}} & n_4 = 0, \\ (4231)^{\frac{n-6}{4}}443211 & n_4 = 2; \end{cases} \quad \sigma(V_2) = \begin{cases} (1423)^{\frac{n}{4}} & n_4 = 0, \\ (1423)^{\frac{n-6}{4}}114423 & n_4 = 2; \end{cases}$$
$$\sigma(E_1) = \begin{cases} (3344)^{\frac{n}{4}} & n_4 = 0, \\ (3344)^{\frac{n-6}{4}}331144 & n_4 = 2; \end{cases} \quad \sigma(E_2) = \begin{cases} (4132)^{\frac{n}{4}} & n_4 = 0, \\ (4132)^{\frac{n-6}{4}}223332 & n_4 = 2; \end{cases}$$
$$\sigma(E_3) = \begin{cases} (2211)^{\frac{n}{4}} & n_4 = 0, \\ (2211)^{\frac{n-6}{4}}442211 & n_4 = 2. \end{cases}$$

Further, we calculate the numbers of each colors and have

$$\begin{array}{ll} |\sigma^{-1}(1)| = |\sigma^{-1}(2)| = |\sigma^{-1}(3)| = |\sigma^{-1}(4)| = \frac{5n}{4}, & n_4 = 0, \\ |\sigma^{-1}(1)| = |\sigma^{-1}(4)| = \frac{5n+2}{4}, |\sigma^{-1}(2)| = |\sigma^{-1}(3)| = \frac{5n-2}{4}, & n_4 = 2, \end{array}$$

which shows that $||\sigma^{-1}(i)| - |\sigma^{-1}(j)|| \le 1$ $(1 \le i < j \le 4)$ for $n \pmod 2 = 0$.

The above construction implies $\chi''_{=}(F_{3,n}) \le 4$ for $n \pmod 2 = 0$. On the other hand, by the definition of equitable total chromatic number, there is $\chi''_{=}(F_{3,n}) \ge 4$. Hence, $\chi''_{=}(F_{3,n}) = 4$ for $n \pmod 2 = 0$. Figure 2 shows $\sigma(F_{3,n})$ for $n = 4, 6$.

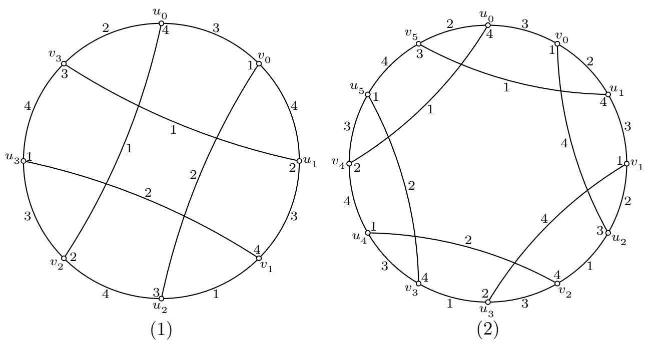

Figure 2 $\sigma(F_{3,n})$ for $n = 4, 6$

Since $F_{3,n}$ for $n \pmod 2 = 1$ is isomorphic to Möbius ladder $M_n$, of which the total chromatic number is 5 (see [24]), the total chromatic number of $F_{3,n}$ for $n \pmod 2 = 1$ is 5 too. Furthermore, we can construct an equitable total 5-coloring of $F_{3,n}$ for $n \pmod 2 = 1$. For example, $\sigma(V_1) = (1)^n$, $\sigma(V_2) = (2)^n$, $\sigma(E_1) = (3)^n$, $\sigma(E_2) = (4)^n$ and $\sigma(E_3) = (5)^n$. We then have $\chi''_{=}(F_{3,n}) = 5$ for $n \pmod 2 = 1$. This completes the proof of Lemma 3.1. □

## 4. Equitable total coloring of $F_{4,n}$

We determine the equitable total chromatic number of $F_{4,n}$ in this section.

**Lemma 4.1** $\chi''_{=}(F_{4,n}) = 5$ *for* $n \ge 5$ *except* $6, 7$, *and* $6$ *for the two exceptions.*

**Proof** We first show $\chi''_{=}(F_{4,n}) = 5$ for $n \ge 5$ except $6, 7$. We use the same expressions of $V_1$, $V_2$, $E_1$, $E_2$ and $E_3$ as those in the proof of Lemma 3.1, and let $E_4 = \{v_i u_{(i+4) \bmod n} : 0 \le i \le n-1\}$ and $n_5 = n \bmod 5$. We construct a 5-total coloring of $F_{4,n}$ as follows:

$$\sigma(V_1)=\begin{cases}(53412)^{\frac{n}{5}}, & n_5=0,\\ (53412)^{\frac{n-11}{5}}53534214112, & n_5=1,\ n\neq 6,\\ (53412)^{\frac{n-12}{5}}535344132112, & n_5=2,\ n\neq 7,\\ (53412)^{\frac{n-8}{5}}53415342, & n_5=3,\\ (53412)^{\frac{n-9}{5}}535344112, & n_5=4;\end{cases}$$

$$\sigma(V_2)=\begin{cases}(12534)^{\frac{n}{5}}, & n_5=0,\\ (12534)^{\frac{n-11}{5}}11213552434, & n_5=1,\ n\neq 6,\\ (12534)^{\frac{n-12}{5}}112132453534, & n_5=2,\ n\neq 7,\\ (12534)^{\frac{n-8}{5}}15342534, & n_5=3,\\ (12534)^{\frac{n-9}{5}}112553434, & n_5=4;\end{cases}$$

$$\sigma(E_1)=\begin{cases}(44341)^{\frac{n}{5}}, & n_5=0,\\ (44341)^{\frac{n-11}{5}}44441123541, & n_5=1,\ n\neq 6,\\ (44341)^{\frac{n-12}{5}}444411224241, & n_5=2,\ n\neq 7,\\ (44341)^{\frac{n-8}{5}}44534451, & n_5=3,\\ (44341)^{\frac{n-9}{5}}444422251, & n_5=4;\end{cases}$$

$$\sigma(E_2)=\begin{cases}(55253)^{\frac{n}{5}} & n_5=0,\\ (55253)^{\frac{n-11}{5}}53124414253, & n_5=1,\ n\neq 6,\\ (55253)^{\frac{n-12}{5}}531225115353, & n_5=2,\ n\neq 7,\\ (55253)^{\frac{n-8}{5}}53415243, & n_5=3,\\ (55253)^{\frac{n-9}{5}}531115343, & n_5=4;\end{cases}$$

$$\sigma(E_3)=\begin{cases}(23422)^{\frac{n}{5}}, & n_5=0,\\ (23422)^{\frac{n-11}{5}}22535235322, & n_5=1,\ n\neq 6,\\ (23422)^{\frac{n-12}{5}}225344342422, & n_5=2,\ n\neq 7,\\ (23422)^{\frac{n-8}{5}}22223322, & n_5=3,\\ (23422)^{\frac{n-9}{5}}225344522, & n_5=4;\end{cases}$$

$$\sigma(E_4)=\begin{cases}(31115)^{\frac{n}{5}}, & n_5=0,\\ (31115)^{\frac{n-11}{5}}35352341115, & n_5=1,\ n\neq 6,\\ (31115)^{\frac{n-12}{5}}353553531115, & n_5=2,\ n\neq 7,\\ (31115)^{\frac{n-8}{5}}31151115, & n_5=3,\\ (31115)^{\frac{n-9}{5}}353231115, & n_5=4.\end{cases}$$

Further, we calculate the numbers of each colors and have

$$\begin{array}{ll}|\sigma^{-1}(1)|=|\sigma^{-1}(2)|=|\sigma^{-1}(3)|=|\sigma^{-1}(4)|=|\sigma^{-1}(5)|=\frac{6n}{5}, & n_5=0,\\ |\sigma^{-1}(1)|=\frac{6n+4}{5},|\sigma^{-1}(2)|=|\sigma^{-1}(3)|=|\sigma^{-1}(4)|=|\sigma^{-1}(5)|=\frac{6n-1}{5}, & n_5=1,\\ |\sigma^{-1}(1)|=|\sigma^{-1}(3)|=\frac{6n+3}{5},|\sigma^{-1}(2)|=|\sigma^{-1}(4)|=|\sigma^{-1}(5)|=\frac{6n-2}{5}, & n_5=2,\\ |\sigma^{-1}(1)|=|\sigma^{-1}(2)|=\frac{6n-3}{5},|\sigma^{-1}(3)|=|\sigma^{-1}(4)|=|\sigma^{-1}(5)|=\frac{6n+2}{5}, & n_5=3,\\ |\sigma^{-1}(1)|=|\sigma^{-1}(3)|=|\sigma^{-1}(4)|=|\sigma^{-1}(5)|=\frac{6n+1}{5},|\sigma^{-1}(2)|=\frac{6n-4}{5}, & n_5=4,\end{array}$$

which shows that $||\sigma^{-1}(i)|-|\sigma^{-1}(j)||\le 1$ $(1\le i<j\le 5)$ for $n\ge 5$ and $n\neq 6,7$.

The above construction implies $\chi''_{=}(F_{4,n}) \leq 5$ for $n \geq 5$ except $6, 7$. On the other hand, by the definition of equitable total chromatic number, $\chi''_{=}(F_{4,n}) \geq 5$. Hence, $\chi''_{=}(F_{4,n}) = 5$ for $n \geq 5$ except $6, 7$. Figure 3 shows $\sigma(F_{4,n})$ for $n = 8, 9, 10, 11$ and 12.

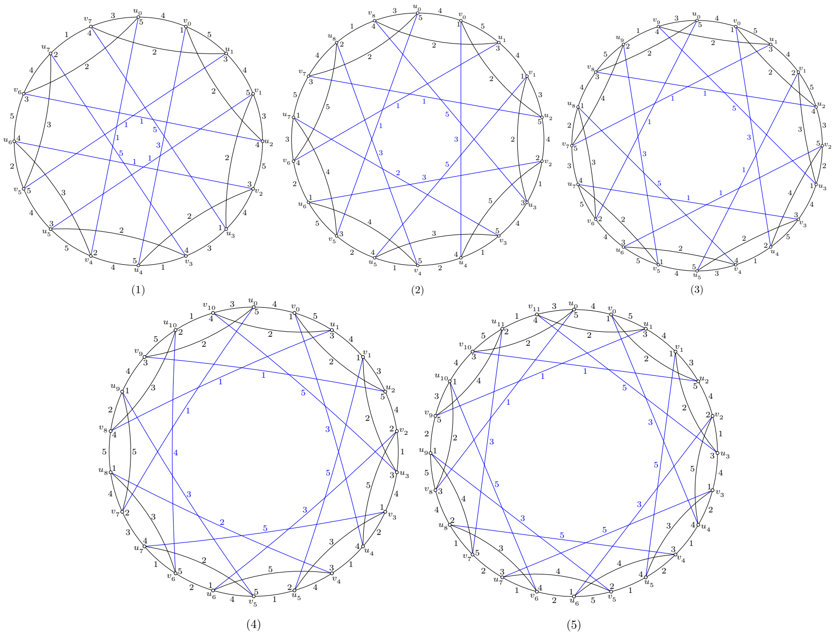

Figure 3 $\sigma(F_{4,n})$ for $n = 8, 9, 10, 11$ and 12

We now prove $\chi''_{=}(F_{4,6}) = 6$. Suppose that $F_{4,6}$ has a 5-total coloring. By [24, Lemmas 3 and 4], if $F_{4,6}$ has a 5-total coloring, then it has a vertex 5-coloring $\Phi$ with colors $\{1, \ldots, 5\}$ and $|\Phi_1^{-1}(j)| = |\Phi_2^{-1}(j)|$ $(1 \leq j \leq 5)$ where $|\Phi_1^{-1}(j)|$ and $|\Phi_2^{-1}(j)|$ are the number of vertices of $V_1$ and $V_2$, respectively in color $j$. Since there are 6 vertices in both $V_1$ and $V_2$, there must be one $j$ such that $|\Phi_1^{-1}(j)| = |\Phi_2^{-1}(j)| = 2$. However, there is no independent set consisting of two vertices of $V_1$ and two vertices of $V_2$ in $F_{4,6}$. This is a contradiction. Thus, $\chi''_{=}(F_{4,6}) \geq 6$. Furthermore, we can construct an equitable total 6-coloring of $F_{4,6}$. For example, $\sigma(V_1) = (1)^6$, $\sigma(V_2) = (2)^6$, $\sigma(E_1) = (3)^6$, $\sigma(E_2) = (4)^6$, $\sigma(E_3) = (5)^6$ and $\sigma(E_4) = (6)^6$. Hence, we have $\chi''_{=}(F_{4,6}) = 6$. Similarly, we can prove that $\chi''_{=}(F_{4,7}) = 6$. This completes the proof of Lemma 4.1. □

## 5. Equitable total coloring of $F_{5,n}$

We determine the equitable total chromatic number of $F_{5,n}$ in this section.

**Lemma 5.1** $\chi''_{=}(F_{5,n}) = 6$ *for* $n \geq 9$ *and 7 for* $n = 8$.

**Proof** We first show $\chi''_{=}(F_{5,n}) = 6$ for $n \geq 9$. Again, we use the same expressions of $V_1$, $V_2$, $E_1$, $E_2$, $E_3$ and $E_4$ as those in the proofs of Lemmas 3.1 and 4.1, and let $E_5 = \{v_i u_{(i+7) \bmod n} : 0 \leq i \leq n-1\}$ and $n_6 = n \bmod 6$. We construct a 6-total coloring of $F_{5,n}$ as follows:

$$\sigma(V_1)=\begin{cases}(612453)^{\frac{n}{6}}, & n_6=0,\\ 6342421651153(612453)^{\frac{n-13}{6}}, & n_6=1,\\ 41215643(612453)^{\frac{n-8}{6}}, & n_6=2,\ n\neq 8,\\ 634212153(612453)^{\frac{n-9}{6}}, & n_6=3,\\ 6321361454(612453)^{\frac{n-10}{6}}, & n_6=4,\\ 63413625124(612453)^{\frac{n-11}{6}}, & n_6=5;\end{cases}$$

$$\sigma(V_2)=\begin{cases}(463215)^{\frac{n}{6}}, & n_6=0,\\ 1153642463215(463215)^{\frac{n-13}{6}}, & n_6=1,\\ 64341215(463215)^{\frac{n-8}{6}}, & n_6=2,\ n\neq 8,\\ 213463215(463215)^{\frac{n-9}{6}}, & n_6=3,\\ 1462433615(463215)^{\frac{n-10}{6}}, & n_6=4,\\ 12624433615(463215)^{\frac{n-11}{6}}, & n_6=5;\end{cases}$$

$$\sigma(E_1)=\begin{cases}(544626)^{\frac{n}{6}}, & n_6=0,\\ 5511116114346(544626)^{\frac{n-13}{6}}, & n_6=1,\\ 33166326(544626)^{\frac{n-8}{6}}, & n_6=2,\ n\neq 8,\\ 522126646(544626)^{\frac{n-9}{6}}, & n_6=3,\\ 5536244146(544626)^{\frac{n-10}{6}}, & n_6=4,\\ 55232241246(544626)^{\frac{n-11}{6}}, & n_6=5;\end{cases}$$

$$\sigma(E_2)=\begin{cases}(211344)^{\frac{n}{6}}, & n_6=0,\\ 226243333662(421134)^{\frac{n-13}{6}}4, & n_6=1,\\ 2643464(421134)^{\frac{n-8}{6}}2, & n_6=2,\ n\neq 8,\\ 43451432(421134)^{\frac{n-9}{6}}4, & n_6=3,\\ 214416622(421134)^{\frac{n-10}{6}}4, & n_6=4,\\ 2146312412(421134)^{\frac{n-11}{6}}4, & n_6=5;\end{cases}$$

$$\sigma(E_3)=\begin{cases}(336133)^{\frac{n}{6}}, & n_6=0,\\ 34365245211(333361)^{\frac{n-13}{16}}31, & n_6=1,\\ 452131(333361)^{\frac{n-8}{6}}64, & n_6=2,\ n\neq 8,\\ 1663311(333361)^{\frac{n-9}{6}}31, & n_6=3,\\ 43135213(333361)^{\frac{n-10}{6}}31, & n_6=4,\\ 361163663(333361)^{\frac{n-11}{6}}31, & n_6=5;\end{cases}$$

$$\sigma(E_4)=\begin{cases}(122562)^{\frac{n}{6}}, & n_6=0,\\ 634526524(256212)^{\frac{n-13}{6}}2463, & n_6=1,\\ 1252(256212)^{\frac{n-8}{6}}5533, & n_6=2,\ n\neq 8,\\ 34524(256212)^{\frac{n-9}{6}}2563, & n_6=3,\\ 622531(256212)^{\frac{n-10}{6}}2462, & n_6=4,\\ 4454551(256212)^{\frac{n-11}{6}}2462, & n_6=5;\end{cases}$$

$$\sigma(E_5)=\begin{cases}(655451)^{\frac{n}{6}}, & n_6=0,\\ 462435(165545)^{\frac{n-13}{6}}1655552, & n_6=1,\\ 5(165545)^{\frac{n-8}{6}}1652451, & n_6=2,\ n\neq 8,\\ 65(165545)^{\frac{n-9}{6}}1655452, & n_6=3,\\ 365(165545)^{\frac{n-10}{6}}1655553, & n_6=4,\\ 6335(165545)^{\frac{n-11}{6}}1655553, & n_6=5.\end{cases}$$

Further, we calculate the numbers of each colors and have

$$\begin{array}{ll}|\sigma^{-1}(1)|=|\sigma^{-1}(2)|=|\sigma^{-1}(3)|=|\sigma^{-1}(4)|=|\sigma^{-1}(5)|=|\sigma^{-1}(6)|=\frac{7n}{6}, & n_6=0,\\ |\sigma^{-1}(1)|=\frac{7n+5}{6},|\sigma^{-1}(2)|=|\sigma^{-1}(3)|=|\sigma^{-1}(4)|=|\sigma^{-1}(5)|=|\sigma^{-1}(6)|=\frac{7n-1}{6}, & n_6=1,\\ |\sigma^{-1}(1)|=|\sigma^{-1}(4)|=\frac{7n+4}{6},|\sigma^{-1}(2)|=|\sigma^{-1}(3)|=|\sigma^{-1}(5)|=|\sigma^{-1}(6)|=\frac{7n-2}{6}, & n_6=2,\\ |\sigma^{-1}(1)|=|\sigma^{-1}(2)|=|\sigma^{-1}(3)|=\frac{7n+3}{6},|\sigma^{-1}(4)|=|\sigma^{-1}(5)|=|\sigma^{-1}(6)|=\frac{7n-3}{6}, & n_6=3,\\ |\sigma^{-1}(1)|=|\sigma^{-1}(3)|=|\sigma^{-1}(4)|=|\sigma^{-1}(6)|=\frac{7n+2}{6},|\sigma^{-1}(2)|=|\sigma^{-1}(5)|=\frac{7n-4}{6}, & n_6=4,\\ |\sigma^{-1}(1)|=|\sigma^{-1}(2)|=|\sigma^{-1}(3)|=|\sigma^{-1}(4)|=|\sigma^{-1}(6)|=\frac{7n+1}{6},|\sigma^{-1}(5)|=\frac{7n-5}{6}, & n_6=5,\end{array}$$

which shows that $||\sigma^{-1}(i)|-|\sigma^{-1}(j)||\leq 1$ $(1\leq i<j\leq 6)$ for $n\geq 9$.

The above construction implies $\chi''_{=}(F_{5,n})\leq 6$ for $n\geq 9$. On the other hand, by the definition of equitable total chromatic number, $\chi''_{=}(F_{5,n})\geq 6$. Hence, $\chi''_{=}(F_{5,n})=6$ for $n\geq 9$. Figure 4 shows $\sigma(F_{5,n})$ for $n=9,10,11,12,13$ and 14.

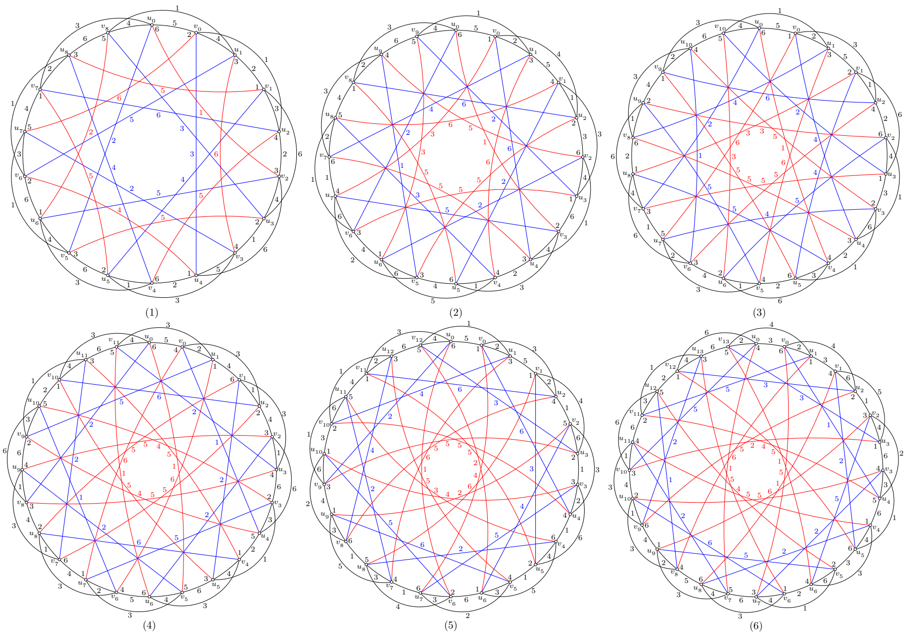

Figure 4 $\sigma(F_{5,n})$ for $n=9,10,11,12,13$ and 14

By an argument similar to $F_{4,6}$, we can prove that $\chi''_{=}(F_{5,8})\geq 7$. Furthermore, we can construct an equitable total 7-coloring of $F_{5,8}$. For example, $\sigma(V_1)=(1)^8$, $\sigma(V_2)=(2)^8$, $\sigma(E_1)=(3)^8$, $\sigma(E_2)=(4)^8$, $\sigma(E_3)=(5)^8$, $\sigma(E_4)=(6)^8$ and $\sigma(E_5)=(7)^8$. Hence, $\chi''_{=}(F_{5,8})=7$. This completes the proof of Lemma 5.1. □

## 6. Conclusion and conjecture

In summary, we have presented equitable total coloring of $F_{3,n}$, $F_{4,n}$ and $F_{5,n}$. By combining Lemmas 3.1, 4.1 and 5.1, we finally obtain the following theorem.

**Theorem 6.1** *The equitable total chromatic numbers of Fibonacci graphs* $F_{\Delta,n}$ *are* $\Delta+1$ *for* $\Delta=3$ *and* $n \pmod 2 = 0$, $\Delta=4$ *and* $n\neq 6,7$, *and* $\Delta=5$ *and* $n\neq 8$, *and* $\Delta+2$ *for* $\Delta=3$ *and* $n \pmod 2 = 1$, $\Delta=4$ *and* $n=6,7$ *and* $\Delta=5$ *and* $n=8$.

Finally, we propose a conjecture on the equitable total chromatic number of $F_{\Delta,n}$ for $\Delta\geq 6$.

**Conjecture 6.2** The equitable total chromatic number of $F_{\Delta,n}$ for $\Delta\geq 6$ is $\Delta+1$ or at most with very few exceptions.

Note. We further examine the graphs $F_{6,n}$ for $n \pmod 7 = 0$, and find its equitable total chromatic number is 7, fulfilling the conjecture $\Delta+1$. Indeed, using the same expressions of $V_1$, $V_2$, $E_1$, $E_2$, $E_3$, $E_4$ and $E_5$ as those in the proofs of Lemmas 3.1, 4.1 and 5.1, and letting $E_6=\{v_iu_{(i+12) \bmod n}: 0\leq i\leq n-1\}$, we can construct a total 7-coloring of $F_{6,n}$ for $n \pmod 7 = 0$ as follows: $\sigma(V_1)=(7562341)^{\frac{n}{7}}$, $\sigma(V_2)=(2341756)^{\frac{n}{7}}$, $\sigma(E_1)=(1113113)^{\frac{n}{7}}$, $\sigma(E_2)=(3252362)^{\frac{n}{7}}$, $\sigma(E_3)=(7475434)^{\frac{n}{7}}$, $\sigma(E_4)=(4724241)^{\frac{n}{7}}$, $\sigma(E_5)=(5636627)^{\frac{n}{7}}$, $\sigma(E_6)=(6567575)^{\frac{n}{7}}$. There is $|\sigma^{-1}(1)|=|\sigma^{-1}(2)|=|\sigma^{-1}(3)|=|\sigma^{-1}(4)|=|\sigma^{-1}(5)|=|\sigma^{-1}(6)|=|\sigma^{-1}(7)|=\frac{8n}{7}$. Figure 5 shows $\sigma(F_{6,14})$.

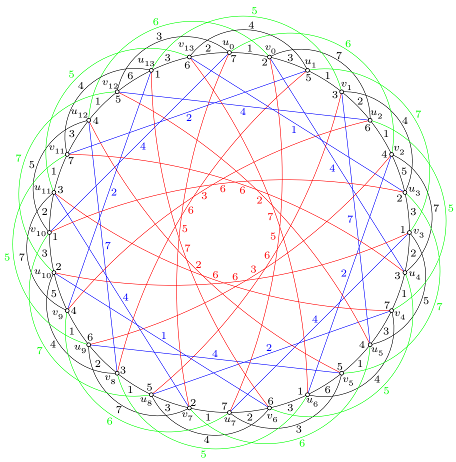


Figure 5 $\sigma(F_{6,14})$

**Acknowledgements** We thank the referees for their time and comments.

## References

[1] A. SÁNCHEZ-ARROYO. *Determining the total colouring number is NP-hard.* Discrete Math., 1989, **78**(3): 315–319.

[2] C. J. H. MCDIARMID, A. SÁNCHEZ-ARROYO. *Total colouring regular bipartite graphs is NP-hard.* Discrete Math., 1994, **124**(1-3): 155–162.

[3] Hunglin FU. *Some results on equalized total coloring.* Congr. Numer., 1994, **102**: 111–119.

[4] Weifan WANG. *Equitable total coloring of graphs with maximum degree 3*. Graphs Combin., 2002, **18**(3): 677–685.

[5] A. G. DA SILVA, S. DANTAS, D. SASAKI. *Equitable total coloring of complete $r$-partite $p$-balanced graphs*. Discrete Appl. Math., 2019, **261**: 123–135.

[6] A. G. DA SILVA, S. DANTAS, D. SASAKI. *Equitable total chromatic number of $K_{r\times p}$ for p even*. Electron Notes Theoy Comput. Sci., 2019, **346**: 685–697.

[7] A. G. DA SILVA, S. DANTAS, D. SASAKI. *Determining equitable total chromatic number for infinite classes of complete $r$-partite graphs*. Discrete Appl. Math., 2021, **296**: 56–67.

[8] Hao GUI, Weifan WANG, Yiqiao WANG, et al. *Equitable total-coloring of subcubic graphs*. Discrete Appl. Math., 2015, **184**: 167–170.

[9] H. FURMAŃCZYK, R. ZUAZUA. *Equitable total coloring of corona of cubic graphs*. Discuss. Math. Graph Theory, 2021, **41**(4): 1147–1163.

[10] Chunling TONG, Xiaohui LIN, Yuansheng YANG. *Equitable total coloring of generalized Petersen graphs $P(n,k)$*. Ars Combin., 2019, **143**: 321–336.

[11] Chunling TONG, Aijun DONG, Shouqiang WANG, et al. *Equitable total coloring of Knödel graph $W_{\Delta,n}$*. J. Comput. Inform. Sys., 2014, **10**(22): 9821–9829.

[12] I. F. A. GONÇALVES, S. DANTAS, D. SASAKI. *On equitable total coloring of snarks*. Proc. Comput. Sci., 2021, **195**: 334–342.

[13] Chunling TONG, Xiaohui LIN, Yansheng YANG, et al. *Equitable total coloring of $C_m\square C_n$*. Discrete Appl. Math., 2009, **157**(4): 596–601.

[14] Kun GONG, Zhongfu ZHANG, Jianfang WANG. *Equitable total coloring of $F_n \vee W_n$*. Acta Math. Appl. Sin. Engl. Ser., 2009, **25**(1): 83–86.

[15] Zhongfu ZHANG, Weifan WANG, Bau SHENG, et al. *On the equitable total colorings of some join graphs*. J. Inform. Comput. Sci., 2005, **2**(4): 829–834.

[16] G. JAYARAMAN, D. MUTHURAMAKRISHNAN, K. MANIKANDAN. *Equitable total chromatic number of splitting graph*. Proyecciones, 2019, **38**(4): 699–705.

[17] K. PRAVEENA, M. VENKATACHALAM. *On equitable chromatic number of Tadpole graph $T_{m,n}$*. Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat., 2019, **68**(2): 1638–1646.

[18] J. V. VIVIK. *An algorithmic approach to equitable total chromatic number of graphs*. Proyecciones, 2017, **36**(2): 307–324.

[19] G. CYBENKO, D. KRUMME, K. VENKATARAMAN. *Gossiping in minimum time*. SIAM J. Comput., 1992, **21**(1): 111–139.

[20] S. EVEN, B. MONIEN. *On the number of rounds necessary to disseminate information*. First ACM Symposium on Parallel Algorithms and Architectures (SPAA), 1989.

[21] R. LABAHN, I. WARNKE. *Quick gossiping by telegraphs*. Discrete Math., 1994, **126**(1-3): 421–424.

[22] J. COHEN, P. FRAIGNIAUD, C. GAVOILLE. *Recognizing Knödel and Fibonacci graphs*. Technical Report LRI-1238, 2000.

[23] J. COHEN, P. FRAIGNIAUD, C. GAVOILLE. *Recognizing knödel graphs*. Discrete Math., 2002, **250**(1-3): 41–62.

[24] A. G. CHETWYND, A. J. W. HILTON. *Some refinements of the total chromatic number conjecture*. Congr. Numer., 1988, **66**: 195–216.