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Equitable Cluster Partition of Planar Graphs with Girth at Least 12

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Abstract An equitable $(\mathcal{O}_k^1, \mathcal{O}_k^2, \ldots, \mathcal{O}_k^m)$-partition of a graph G, which is also called a k cluster m-partition, is the partition of $V(G)$ into m non-empty subsets $V_1, V_2, \ldots, V_m$ such that for every integer i in $\{1, 2, \ldots, m\}$, $G[V_i]$ is a graph with components of order at most k, and for each distinct pair i, j in $\{1, \ldots, m\}$, there is $-1 \le |V_i| - |V_j| \le 1$. In this paper, we proved that every planar graph G with minimum degree $\delta(G) \ge 2$ and girth $g(G) \ge 12$ admits an equitable $(\mathcal{O}_7^1, \mathcal{O}_7^2, \ldots, \mathcal{O}_7^m)$-partition, for any integer $m \ge 2$.

Keywords equitable cluster partition; planar graph; girth; discharging

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1. Introduction

All graphs considered in this paper are finite, simple and undirected. For a graph G, we use $V(G)$ to denote the vertex set. An equitable k-partition of a graph G is a partition of $V(G)$ into $(V_1, \ldots, V_k)$ such that $-1 \le |V_i| - |V_j| \le 1$ for all $1 \le i < j \le k$. Let $\mathcal{G}_i$ be a class of graphs for $1 \le i \le k$, given a graph G, an equitable $(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k)$-partition of graph G is an equitable k-partition of G such that for all $1 \le i \le k$, the induced subgraph $G[V_i]$ belongs to $\mathcal{G}_i$.

The $\mathcal{G}$-equitable partition number of a graph G, denoted by $\chi_{e\mathcal{G}}(G)$, is the smallest integer k such that G has an equitable $(\mathcal{G}_1, \ldots, \mathcal{G}_k)$-partition with $\mathcal{G}_1 = \mathcal{G}_2 = \cdots = \mathcal{G}_k = \mathcal{G}$. In contrast to the ordinary vertex partition, a graph may have an equitable $(\mathcal{G}_1, \ldots, \mathcal{G}_k)$-partition, but no equitable $(\mathcal{G}_1, \ldots, \mathcal{G}_k, \mathcal{G}_{k+1})$-partition with $\mathcal{G}_1 = \cdots = \mathcal{G}_k = \mathcal{G}_{k+1} = \mathcal{G}$. The $\mathcal{G}$-equitable partition threshold of G, denoted by $\chi_{e\mathcal{G}}^*(G)$, is the smallest integer k such that G has an equitable $(\mathcal{G}_1, \ldots, \mathcal{G}_m)$-partition for all $m \ge k$ with $\mathcal{G}_1 = \mathcal{G}_2 = \cdots = \mathcal{G}_m = \mathcal{G}$.

It is clear that $\chi_{e\mathcal{G}}(G) \le \chi_{e\mathcal{G}}^*(G)$. In fact, the gap between the two parameters can be arbitrarily large. Let $\mathcal{I}$, $\mathcal{O}_k$ denote the class of independent sets, the class of graphs whose components have order at most k, respectively. Let $g(G)$ denote the girth of G, which is the length of the shortest cycle of G.

There are some results in the field of equitable partition of graphs. Hajnal and Szemerédi [1] proved that for any graph G with maximum degree $\Delta(G)$, there is $\chi_{e\mathcal{I}}^*(G) \le \Delta(G) + 1$. Chen,

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Lih and Wu [2] conjectured that for any connected graph G different from K_m, C_{2m+1} and $K_{2m+1,2m+1}$, there is $\chi^*_{e\mathcal{I}}(G) \leq \Delta(G)$. If this conjecture is true, it will prove the former result. For the planar graphs, Zhang, Yap [3] proved that for every planar graph with $\Delta(G) \geq 13$, there is $\chi^*_{e\mathcal{I}}(G) \leq \Delta(G)$. Wu, Wang [4] proved that for every planar graph with $\delta(G) \geq 2$, $g(G) \geq 26$, there is $\chi^*_{e\mathcal{I}}(G) \leq 3$ and for every planar graph with $\delta(G) \geq 2$, $g(G) \geq 14$, there is $\chi^*_{e\mathcal{I}}(G) \leq 4$. Later, Luo, Sébastien, Stephens and et al. [5] improved the above results by proving that for every planar graph with $\delta(G) \geq 2$, $g(G) \geq 14$, there is $\chi^*_{e\mathcal{I}}(G) \leq 3$ and for every planar graph with $\delta(G) \geq 2$, $g(G) \geq 10$, there is $\chi^*_{e\mathcal{I}}(G) \leq 4$.

We are interested in the equitable $(\mathcal{O}_k, \ldots, \mathcal{O}_k)$-partition. There are also some results.

Theorem 1.1 ([6]) *Every planar graph G with minimum degree $\delta(G) \geq 2$ and girth $g(G) \geq 10$ has an equitable $(\mathcal{O}_2^1, \ldots, \mathcal{O}_2^m)$-partition for any integer $m \geq 3$, that is $\chi^*_{e\mathcal{O}_2}(G) \leq 3$.*

Theorem 1.2 ([7]) *Every planar graph G with minimum degree $\delta(G) \geq 2$ and girth $g(G) \geq 8$ has an equitable $(\mathcal{O}_2^1, \ldots, \mathcal{O}_2^m)$-partition for any integer $m \geq 4$, that is $\chi^*_{e\mathcal{O}_2}(G) \leq 4$.*

Our main result is presented as follows:

Theorem 1.3 *Every planar graph G with minimum degree $\delta(G) \geq 2$ and girth $g(G) \geq 12$ admits an equitable $(\mathcal{O}_7^1, \mathcal{O}_7^2, \ldots, \mathcal{O}_7^m)$-partition for any integer $m \geq 2$, that is $\chi^*_{e\mathcal{O}_7}(G) = 2$.*

It is not hard to see that Theorem 1.3 gives a threshold of equitable tree partition of planar graphs by the condition $g(G) \geq 12$.

2. The structure of minimal counterexamples

By Theorem 1.1, we only need to show that every planar graph with minimum degree at least 2 and girth at least 12 has an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition. Let G be a counterexample in this case with smallest order. Before discussing the structure of G, we clarify some necessary definitions and notations firstly.

The degree of a vertex v in G, written by $d_G(v)$ or simply $d(v)$ when there is no confusion, is the number of edges incident with v in G. A k-vertex, k^+-vertex and k^--vertex is a vertex of degree k, at least k and at most k, respectively. A neighbor of the vertex v with degree k, at least k and at most k is called a k-neighbor, k^+-neighbor and k^--neighbor of v, respectively.

A chain of G is a maximal induced path whose internal vertices all have degree 2. A t-chain is a chain with t internal vertices. In a chain, the 3^+-vertex is called endvertex. Specially, a cycle with exactly one 3^+-vertex and all other vertices of degree 2 is also called a chain, in other words, the endvertices of chain are identical. Let x be an endvertex of a chain P, y be a vertex in P, if the distance between x and y is $l+1$, then we say that y is loosely l-adjacent to x. Thus "loosely 0-adjacent" is the same as usual "adjacent".

Let x be a vertex with $d(x) \geq 3$. Then x is the endvertex of $d(x)$ different chains. Set $T(x) = (a_3, a_2, a_1, a_0)$, where a_i is the number of i-chains incident with x, $i \in \{0, 1, 2, 3\}$. Let $t(x) = 3a_3 + 2a_2 + a_1$, $n(x) = t(x) + 1$, and $A(x)$ be the vertex set composed of all 2-vertices in its incident chains. We call a 3-vertex x *bad* 3-vertex if $d(x)=3$ with $t(x)=4$.

Let H be a subgraph of G, for $x\in V(H)$, if x has no neighbor in $G-H$, then we call it free vertex, otherwise we call it non-free vertex, the neighbors of x in $G-H$ are called outer neighbors of x.

Lemma 2.1 *The graph G is connected.*

Proof On the contrary, let $H_1,H_2,\ldots,H_k$ be the connected components of G, where $k\geq 2$. By the minimality of G, both $H=H_1\cup H_2\cup\cdots\cup H_{k-1}$ and H_k have an equitable $(\mathcal{O}_7,\mathcal{O}_7)$-partition. An equitable $(\mathcal{O}_7,\mathcal{O}_7)$-partition of H with $|V_1(H)|\leq|V_2(H)|$ and an equitable $(\mathcal{O}_7,\mathcal{O}_7)$-partition of H_k with $|V_1(H_k)|\geq|V_2(H_k)|$ generate an equitable $(\mathcal{O}_7,\mathcal{O}_7)$-partition $(V_1(H)\cup V_1(H_k),V_2(H)\cup V_2(H_k))$ of G, which contradicts the choice of G. □

Lemma 2.2 *If G has a t-chain, then $t\leq 3$, and G has no chain whose endvertices are identical.*

Proof Suppose to the contrary that G has a t-chain $P=v_0v_1\cdots v_tv_{t+1}$ with $t\geq 4$, where $d(v_0),d(v_{t+1})\geq 3$. Let $G_1=G-\{v_1,\ldots,v_t\}$.

If $v_0\neq v_{t+1}$ or $d(v_0)\geq 4$, then $\delta(G_1)\geq 2$. By the minimality of G, the graph G_1 has an equitable $(\mathcal{O}_7,\mathcal{O}_7)$-partition. Let V_1, V_2 be the two sets with $|V_1|\leq|V_2|$. We can extend the partition of G_1 to an equitable $(\mathcal{O}_7,\mathcal{O}_7)$-partition of G as follows. First put the vertex v_i into the part V_1 if i is odd, into V_2 if i is even for each $i\in\{1,2,\ldots,t\}$. Swap the positions of v_1 and v_2 if v_0 and v_1 are put in the same part, and further swap the positions of v_{t-1} and v_t if v_t and v_{t+1} are put in the same part.

Now suppose that $v_0=v_{t+1}$ and $d(v_0)=3$. We know $g(G)\geq 12$, so $t\geq 11$. Let x be the neighbor of v_0 in G_1. If $d(x)\geq 3$, consider $G_2=G-\{v_0,v_1,\ldots,v_t\}$, then $\delta(G_2)\geq 2$. By the choice of G, the graph G_2 has an equitable $(\mathcal{O}_7,\mathcal{O}_7)$-partition with sets V_1, V_2 such that $|V_1|\leq|V_2|$. We can extend the partition of G_2 to an equitable $(\mathcal{O}_7,\mathcal{O}_7)$-partition of G as follows. First put the vertex v_i into the part V_1 if i is even, into V_2 if i is odd for each $i\in\{0,1,\ldots,t\}$. Swap the positions of v_0 and v_1 if the vertices v_0 and x are put in the same part (the partition of $\{v_0,v_1,\ldots,v_t\}$ generated in this way admits that the order of each component of each part is at most 2). If $d(x)=2$, then let $Q=x_0x_1x_2\cdots x_qx_{q+1}$ be the chain with $x_0=v_0$, $x_1=x$. Consider the graph $G_3=G-\{x_0,x_1,\ldots,x_q,v_1,\ldots,v_t\}$, then $\delta(G_3)\geq 2$. By the minimality of G, the graph G_3 has an equitable $(\mathcal{O}_7,\mathcal{O}_7)$-partition with sets V_1, V_2 such that $|V_1|\leq|V_2|$. We first extend the partition of G_3 to G_1 to obtain an equitable $(\mathcal{O}_7,\mathcal{O}_7)$-partition of $G-\{v_1,\ldots,v_t\}$ as follows. First put the vertex x_i into the part V_1 if i is even, into V_2 if i is odd for each $i\in\{0,1,\ldots,q\}$. If x_q and x_{q+1} are put in the same part, swap the positions of x_{q-1} and x_q. Next we further extend the partition to G similarly to the case that $d(v_0)\geq 4$. In any case, we can always get an equitable $(\mathcal{O}_7,\mathcal{O}_7)$-partition of G. This contradicts the choice of G. Hence, there is no t-chain with $t\geq 4$, and G has no chain whose endvertices are identical. □

Lemma 2.3 *If x is a 3-vertex, then $t(x)\leq 4$.*

Proof On the contrary, suppose that x is a 3-vertex with $t(x) \geq 5$. Lemma 2.2 implies that x is not incident with any t-chains, where $t \geq 4$. Since $t(x) \geq 5$, the vertex x is incident with at least one 3-chain or at least two 2-chains, then $6 \leq n(x) \leq 10$. Let $A(x)$ be the vertex set composed of all 2-vertices in its incident chains and $A = A(x) \cup \{x\}$. Let N be the set of the three non-free vertices in A. Then every vertex in N has exactly one outer neighbor in $G - A$. Since $g(G) \geq 12$, the chains do not share endvertices other than x. So $\delta(G - A) \geq 2$. By the minimality of G, the graph $G - A$ has an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition. We can extend the partition of $G - A$ to G as follows. First, we put the non-free vertices into the part that its neighbor in $G - A$ is not in. If there are i non-free vertices in V_1, then we put arbitrary $\lceil \frac{n(x)}{2} \rceil - i$ vertices in $A - N$ into V_1, where $i \in \{0, 1, 2, 3\}$. Then put the other vertices in A into V_2. There are at most five vertices that are put into the same part. In this way, we get an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition of G, this leads to a contradiction. □

Lemma 2.4 *If x is a 4-vertex, then $t(x) \leq 6$.*

Proof On the contrary, suppose that x is a 4-vertex with $t(x) \geq 7$. Lemma 2.2 implies that x is not incident with any t-chains, where $t \geq 4$. Since $t(x) \geq 7$, the vertex x is incident with at least three 2^+-chains, or two 3^+-chains, or two 2^+-chains and two 1-chains, then $8 \leq n(x) \leq 13$. Let $A(x)$ be the vertex set composed of all 2-vertices in its incident chains and $A = A(x) \cup \{x\}$. Let N be the set of the four non-free vertices in A. Then every vertex in N has exactly one outer neighbor in $G - A$. Since $g(G) \geq 12$, the chains do not share endvertices other than x. So $\delta(G - A) \geq 2$. By the minimality of G, the graph $G - A$ has an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition. We can extend the partition of $G - A$ to G as follows. First, we put the non-free vertices into the part that its neighbor in $G - A$ is not in. If there are i non-free vertices in V_1, then we put arbitrary $\lceil \frac{n(x)}{2} \rceil - i$ vertices in $A - N$ into V_1, where $i \in \{0, 1, 2, 3, 4\}$. Then put the other vertices in A into V_2. There are at most seven vertices that are put into the same part. In this way, we get an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition of G, this leads to a contradiction. □

Lemma 2.5 *If x is a 5-vertex, then $t(x) \leq 8$ or $T(x) = (3, 0, 0, 2)$.*

Proof On the contrary, suppose that x is a 5-vertex with $t(x) \geq 9$ and $T(x) \neq (3, 0, 0, 2)$. Lemma 2.2 implies that x is not incident with any t-chains, where $t \geq 4$. So $10 \leq n(x) \leq 16$. Let $A(x)$ be the vertex set composed of all 2-vertices in its incident chains and $A = A(x) \cup \{x\}$. Let N be the set of the five non-free vertices in A. Then every vertex in N has exactly one outer neighbor and x has at most one outer neighbor. Since $g(G) \geq 12$, the chains do not share endvertices other than x. So $\delta(G - A) \geq 2$. By the minimality of G, the graph $G - A$ has an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition. We can extend the partition of $G - A$ to G as follows. First, we put the non-free vertices into the part that its neighbor in $G - A$ is not in. For $10 \leq n(x) \leq 14$, if there are i non-free vertices in V_1, then we choose $\lceil \frac{n(x)}{2} \rceil - i$ vertices in $A - N$ arbitrarily into V_1, where $i \in \{0, 1, 2, 3, 4, 5\}$. Then put the other vertices in A into V_2. For $15 \leq n(x) \leq 16$, if there are i non-free vertices in V_1, then we can choose $\lceil \frac{n(x)}{2} \rceil - i$ vertices in $A - N$ into V_1, where $i \in \{0, 1, 2, 3, 4, 5\}$, then put the other vertices in A into V_2 such that A has an equitable

$(\mathcal{O}_7, \mathcal{O}_7)$-partition with $|V_1| \geq |V_2|$. In this way, we get an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition of G, this leads to a contradiction. □

Lemma 2.6 *Let x be a bad 3-vertex with $T(x) = (0, 1, 2, 0)$ or $T(x) = (1, 0, 1, 1)$, and let y be a 3^+-vertex that is loosely 1-adjacent to x. Then*

(i) $d(y) = 3$ with $t(y) \leq 2$ or

(ii) $d(y) = 4$ with $t(y) \leq 4$ or

(iii) $d(y) = 5$ with $t(y) \leq 6$ or $T(y) = (2, 0, 1, 2)$, or

(iv) $d(y) \geq 6$.

Proof Let x be a bad 3-vertex with $T(x) = (0, 1, 2, 0)$ or $T(x) = (1, 0, 1, 1)$, and let y be a 3^+-vertex that is loosely 1-adjacent to x. Suppose to the contrary that $d(y) = 3$ with $t(y) \geq 3$ or $d(y) = 4$ with $t(y) \geq 5$ or $d(y) = 5$ with $t(y) \geq 7$ and $T(y) \neq (2, 0, 1, 2)$. By Lemmas 2.2–2.5, if $d(y) = 3$, then $3 \leq t(y) \leq 4$; if $d(y) = 4$, then $5 \leq t(y) \leq 6$; if $d(y) = 5$, then $7 \leq t(y) \leq 8$ and $T(y) \neq (2, 0, 1, 2)$. Let $B = A(x) \cup A(y) \cup \{x, y\}$, $|B| = t(x) + t(y) + 1 = t(y) + 5$. Let N be the subset of B composed of all non-free vertices in B. $|N| = d(x) + d(y) - 2 = d(y) + 1$ and each vertex in N has exactly one outer neighbor in $G - B$. Since $g(G) \geq 12$, the chains do not share endvertices other than x and y. So $\delta(G - B) \geq 2$. By the minimality of G, the graph $G - B$ has an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition. First, we put the non-free vertices into the part that its neighbor in $G - B$ is not in. If there are i non-free vertices in N that are put into V_1, then we put $\lceil \frac{|B|}{2} \rceil - i$ vertices in $B - N$ into V_1 and put the other vertices in B into V_2 such that B has an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition with $|V_1| \geq |V_2|$, where $i \in \{0, 1, \ldots, 6\}$. This can be done because $|N| = d(y) + 1 \leq \frac{1}{2}(t(y) + 5) = \frac{1}{2}|B|$. If $|N| = 4$, then $|B| \in \{8, 9\}$; if $|N| = 5$, then $|B| \in \{10, 11\}$; if $|N| = 6$, then $|B| \in \{12, 13\}$. So there are at most seven vertices that are put in the same part. In this way, we obtain an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition of G, this leads to a contradiction. □

Lemma 2.7 *Every 3-vertex y with $T(y) = (0, 0, 2, 1)$ is loosely 1-adjacent to at most one bad 3-vertex.*

Proof Suppose to the contrary that there are two bad 3-vertices that are both loosely 1-adjacent to y. Let y be a 3-vertex with $T(y) = (0, 0, 2, 1)$ and let x_1 and x_2 be bad 3-vertices that are loosely 1-adjacent to y. Let $C = A(x_1) \cup A(x_2) \cup A(y) \cup \{x_1, x_2, y\}$. Let N be the subset of C composed of all non-free vertices in C. Since $g(G) \geq 12$, we can claim that $\delta(G - C) \geq 2$. Otherwise, x_1 and x_2 are bad 3-vertices with $T(x_i) = (1, 0, 1, 1)$ for $i = 1, 2$. Denote the vertices loosely 3-adjacent to x_1 and x_2 as y_1 and y_2, respectively, the vertices y_1 and y_2 are the same vertices and $d(y_1) = 3$, then we have y_1 is a 3-vertex with $t(y_1) = 6$, this contradicts Lemma 2.3. Hence, we always have $\delta(G - C) \geq 2$. By the minimality of G, the graph $G - C$ has an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition. $|N| = d(x_1) + d(x_2) + d(y) - 4 = 5$, $|C| = t(x_1) + t(x_2) + t(y) + 1 = 11$. Every vertex in N has exactly one outer neighbor. We can extend the partition of $G - C$ to G as follows. First, we put the non-free vertices into the part that its neighbor in $G - C$ is not in. If there are i vertices in N that are put into V_1, then put $6 - i$ vertices in $C - N$ into V_1, where

$i \in \{0, 1, \ldots, 5\}$. Last we put the other vertices in C into V_2. In this way, we obtain an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition of G, this leads to a contradiction. □

Lemma 2.8 *There is no* 3*-vertex* y *with* $T(y) = (0, 0, 2, 1)$ *which is loosely 1-adjacent to a bad* 3*-vertex and adjacent to a bad* 3*-vertex simultaneously.*

Proof Suppose to the contrary that there is a 3-vertex y with $T(y) = (0, 0, 2, 1)$ that is loosely 1-adjacent to a bad 3-vertex x_1 and adjacent to a bad 3-vertex x_2 simultaneously. Let $C = A(x_1) \cup A(x_2) \cup A(y) \cup \{x_1, x_2, y\}$. Let N be the subset of C composed of all non-free vertices in C. Since $g(G) \geq 12$, the chains do not share endvertices other than x_1, x_2 and y. So $\delta(G-C) \geq 2$. By the minimality of G, the graph $G - C$ has an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition.

$$|N| = d(x_1) + d(x_2) + d(y) - 4 = 5, \ |C| = t(x_1) + t(x_2) + t(y) + 2 = 12.$$

Every vertex in N has exactly one outer neighbor. First, we put the non-free vertices into the part that its neighbor in $G-C$ is not in. If there are i vertices in N that are put into V_1, then put $6 - i$ vertices in $C - N$ into V_1, where $i \in \{0, 1, \ldots, 5\}$. Last we put the other vertices in C into V_2. In this way, we obtain an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition of G, this leads to a contradiction. □

Lemma 2.9 *Let* x *be a bad* 3*-vertex with* $T(x) = (0, 2, 0, 1)$ *or* $T(x) = (1, 0, 1, 1)$*, and let* y *be the* 3^+*-neighbor of* x*. Then*

(i) $d(y) = 3$ *with* $t(y) \leq 1$ *or*

(ii) $d(y) = 4$ *with* $t(y) \leq 3$ *or*

(iii) $d(y) \geq 5$.

Proof Let x be a bad 3-vertex with $T(x) = (0, 2, 0, 1)$ or $T(x) = (1, 0, 1, 1)$, and let y be the 3^+-neighbor of x. Suppose to the contrary that $d(y) = 3$ with $t(y) \geq 2$ or $d(y) = 4$ with $t(y) \geq 4$. By Lemmas 2.2–2.4, if $d(y) = 3$, then $2 \leq t(y) \leq 4$; If $d(y) = 4$, then $4 \leq t(y) \leq 6$. Let

$$B = A(x) \cup A(y) \cup \{x, y\}, \ |B| = t(x) + t(y) + 2 = t(y) + 6.$$

Let N be the subset of B composed of all non-free vertices in B. $|N| = d(x) + d(y) - 2 = d(y) + 1$. Each vertex in N has exactly one outer neighbor. Since $g(G) \geq 12$, the chains do not share endvertices other than x and y. So $\delta(G - B) \geq 2$. By the minimality of G, the graph $G - B$ has an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition. First, we put the non-free vertices into the part that its neighbor in $G - B$ is not in. If there are i non-free vertices in N that are put into V_1, then we put $\lceil \frac{|B|}{2} \rceil - i$ vertices in $B - N$ into V_1 and put the other vertices in B into V_2 such that B has an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition with $|V_1| \geq |V_2|$, where $i \in \{0, 1, \ldots, 5\}$. This can be done because $|N| = d(y) + 1 \leq \frac{1}{2}(t(y) + 6) = \frac{1}{2}|B|$. If $|N| = 4$, then $|B| \in \{8, 9, 10\}$; if $|N| = 5$, then $|B| \in \{10, 11, 12\}$. So at most six vertices are put in the same part. In this way, we obtain an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition of G, this leads to a contradiction. □

Lemma 2.10 *Every* 3*-vertex* y *with* $T(y) = (0, 0, 1, 2)$ *is adjacent to at most one bad* 3*-vertex.*

Proof Suppose to the contrary that there is a 3-vertex y with $T(y) = (0, 0, 1, 2)$ that is adjacent to two bad 3-vertices x_1 and x_2 . Let $C = A(x_1) \cup A(x_2) \cup A(y) \cup \{x_1, x_2, y\}$. Let N be the subset of C composed of all non-free vertices in C. Since $g(G) \geq 12$, the chains do not share endvertices other than x_1, x_2 and y. So $\delta(G - C) \geq 2$. By the minimality of G, the graph $G - C$ has an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition. $|N| = d(x_1) + d(x_2) + d(y) - 4 = 5$, $|C| = t(x_1) + t(x_2) + t(y) + 2 = 12$. Every vertex in N has exactly one outer neighbor. First, we put the non-free vertices into the part that its neighbor in $G - C$ is not in. If there are i vertices in N that are put into V_1, then put $6 - i$ vertices in $C - N$ into V_1 , where $i \in \{0, 1, \ldots, 5\}$. Last, we put the other vertices in C into V_2. In this way, we obtain an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition of G, this leads to a contradiction. □

Lemma 2.11 *Every* 3*-vertex* y *with* $T(y) = (0, 1, 0, 2)$ *is adjacent to at most one bad* 3*-vertex.*

Proof Suppose to the contrary that there is a 3-vertex y with $T(y) = (0, 1, 0, 2)$ that is adjacent to two bad 3-vertices x_1 and x_2. Let $C = A(x_1) \cup A(x_2) \cup A(y) \cup \{x_1, x_2, y\}$. Let N be the subset of C composed of all non-free vertices in C. Since $g(G) \geq 12$, the chains do not share endvertices other than x_1, x_2 and y. So $\delta(G - C) \geq 2$. By the minimality of G, the graph $G - C$ has an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition. $|N| = d(x_1) + d(x_2) + d(y) - 4 = 5$, $|C| = t(x_1) + t(x_2) + t(y) + 3 = 13$. Every vertex in N has exactly one outer neighbor. First, we put the non-free vertices into the part that its neighbor in $G - C$ is not in. If there are i vertices in N that are put into V_1, then put $7 - i$ vertices in $C - N$ into V_1, where $i \in \{0, 1, \ldots, 5\}$. Last, we put the other vertices in C into V_2. In this way, we obtain an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition of G, this leads to a contradiction. □

Lemma 2.12 *Every* 4*-vertex* y *with* $T(y) = (1, 0, 3, 0)$ *is loosely* 1*-adjacent to at most one bad* 3*-vertex.*

Proof Suppose to the contrary that there is a 4-vertex y with $T(y) = (0, 1, 3, 0)$ that is loosely 1-adjacent to at least two bad 3-vertices x_1 and x_2. Let $C = A(x_1) \cup A(x_2) \cup A(y) \cup \{x_1, x_2, y\}$. Let N be the subset of C composed of all non-free vertices in C. Since $g(G) \geq 12$, we can claim that $\delta(G - C) \geq 2$. Otherwise, x_1 and x_2 are bad 3-vertices with $T(x_i) = (1, 0, 1, 1)$ at the same time, $i = 1, 2$. Denote the vertices loosely 3-adjacent to x_1 and x_2 as y_1 and y_2, respectively, the vertices y_1 and y_2 are the same vertices and $d(y_1) = 3$, then we have y_1 is a 3-vertex with $t(y_1) = 6$, this contradicts Lemma 2.3. Hence, we always have $\delta(G - C) \geq 2$. By the minimality of G, the graph $G - C$ has an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition. $|N| = d(x_1) + d(x_2) + d(y) - 4 = 6$, $|C| = t(x_1) + t(x_2) + t(y) + 1 = 15$. Every vertex in N has exactly one outer neighbor. First, we put the non-free vertices into the part that its neighbor in $G - C$ is not in. If there are i vertices in N that are put into V_1, then put $8 - i$ vertices in $C - N$ into V_1, where $i \in \{0, 1, \ldots, 6\}$. Last, we put the other vertices in C into V_2. In this way, we obtain an equitable $(\mathcal{O}_7, \mathcal{O}_7)$-partition of G, this leads to a contradiction. □

3. Discharging

The maximum average degree of a graph G is

$$\mathrm{mad}(G) = \max\{\frac{2|E(H)|}{|V(H)|} | H \subseteq G\}.$$

By Euler's formula, a planar graph G with girth g satisfies $\mathrm{mad}(G) < \frac{2g}{g-2}$ (see [8]). Consider the minimal counterexample G. Since $g(G) \geq 12$, we have $\mathrm{mad}(G) < \frac{12}{5}$. For any $x \in V(G)$, let $\mu(x) = d(x) - \frac{12}{5}$ be the initial charge. We have

$$\sum_{x \in V(G)} \mu(x) = \sum_{x \in V(G)} (d(x) - \frac{12}{5}) < 0.$$

Next, we redistribute the charges among vertices according to the following rules:

(R1) Every 3^+-vertex gives $\frac{1}{5}$ to each 2-vertex in its incident chains.

(R2) Every 3^+-vertex y gives $\frac{1}{5}$ to each bad 3-vertex x that is loosely 1-adjacent to y, where $d(x) = 3$, $T(x) = (0, 1, 2, 0)$ or $T(x) = (1, 0, 1, 1)$.

(R3) Every 3^+-vertex y gives $\frac{1}{5}$ to each bad 3-vertex x that is adjacent to y, where $d(x) = 3$, $T(x) = (0, 2, 0, 1)$ or $T(x) = (1, 0, 1, 1)$.

Let $\mu'(x)$ be the final charge of x after applying rules (R1)–(R3). Next, we prove $\mu'(x) \geq 0$ for all $x \in V(G)$.

Let $x \in V(G)$. If $d(x) = 2$, then $\mu'(x) = (2 - \frac{12}{5}) + \frac{1}{5} \times 2 = 0$ by (R1).

Assume $d(x) = 3$, it follows from Lemma 2.3 that $t(x) \leq 4$. If $t(x) = 0$, then x is adjacent to at most three bad 3-vertices, thus $\mu'(x) \geq (3 - \frac{12}{5}) - \frac{1}{5} \times 3 = 0$ by (R3). If $t(x) = 1$, then Lemma 2.10 implies that x is adjacent to at most one bad 3-vertex, thus $\mu'(x) \geq (3 - \frac{12}{5}) - \frac{1}{5} \times 1 - \frac{1}{5} \times 2 = 0$ by (R1), (R2) and (R3). If $t(x) = 2$ with $T(x) = (0, 0, 2, 1)$, then Lemmas 2.7, 2.8 imply that x is loosely 1-adjacent to at most one bad 3-vertex, and it is impossible that x is loosely 1-adjacent to a bad 3-vertex and adjacent to a bad 3-vertex at the same time, thus $\mu'(x) \geq (3 - \frac{12}{5}) - \frac{1}{5} \times 2 - \frac{1}{5} \times 1 = 0$ by (R1), (R2) and (R3). If $t(x) = 2$ with $T(x) = (0, 1, 0, 2)$, then Lemma 2.11 implies that x is adjacent to at most one bad 3-vertex, thus $\mu'(x) \geq (3 - \frac{12}{5}) - \frac{1}{5} \times 2 - \frac{1}{5} \times 1 = 0$ by (R1), (R2) and (R3). If $t(x) = 3$, then Lemma 2.6 implies x is not loosely 1-adjacent to bad 3-vertex, and Lemma 2.9 implies x is not adjacent to bad 3-vertex, thus $\mu'(x) \geq (3 - \frac{12}{5}) - \frac{1}{5} \times 3 = 0$ by (R1). If $t(x) = 4$, then $\mu'(x) \geq (3 - \frac{12}{5}) - \frac{1}{5} \times 4 + \frac{1}{5} \times 1 = 0$ by (R1), (R2) and (R3).

Assume $d(x) = 4$, it follows from Lemma 2.4 that $t(x) \leq 6$. If $t(x) \leq 4$, then x is loosely 1-adjacent to or adjacent to at most four bad 3-vertices, so $\mu'(x) \geq (4 - \frac{12}{5}) - \frac{1}{5} \times 4 - \frac{1}{5} \times 4 = 0$ by (R1), (R2) and (R3). If $t(x) = 5$, then x is incident with at least one 2^+-chain, namely, x is loosely 1-adjacent to or adjacent to at most three bad 3-vertices, hence $\mu'(x) \geq (4 - \frac{12}{5}) - \frac{1}{5} \times 5 - \frac{1}{5} \times 3 = 0$ by (R1), (R2) and (R3). If $t(x) = 6$ with $T(x) = (1, 0, 3, 0)$, then Lemma 2.12 implies that x is loosely 1-adjacent to at most one bad 3-vertex, thus $\mu'(x) \geq (4 - \frac{12}{5}) - \frac{1}{5} \times 6 - \frac{1}{5} \times 1 = \frac{1}{5}$ by (R1), (R2) and (R3). If $t(x) = 6$ with $T(x) \neq (1, 0, 3, 0)$, then x is incident with at least two 2^+-chains, hence $\mu'(x) \geq (4 - \frac{12}{5}) - \frac{1}{5} \times 6 - \frac{1}{5} \times 2 = 0$ by (R1), (R2) and (R3).

Assume $d(x) = 5$, it follows from Lemma 2.5 that $t(x) \leq 8$ or $T(x) = (3, 0, 0, 2)$. If $t(x) \leq 8$, then x is loosely 1-adjacent to or adjacent to at most five bad 3-vertices, so $\mu'(x) \geq (5-\frac{12}{5})-\frac{1}{5}\times 8-\frac{1}{5}\times 5=0$ by (R1), (R2) and (R3). If $T(x)=(3,0,0,2)$, then $\mu'(x)\geq(5-\frac{12}{5})-\frac{1}{5}\times 9-\frac{1}{5}\times 2=\frac{2}{5}$ by (R1), (R2) and (R3).

Assume $d(x)\geq 6$, then $\mu'(x)\geq (d(x)-\frac{12}{5})-\frac{1}{5}\times 3\times d(x)=\frac{2}{5}d(x)-\frac{12}{5}\geq 0$ by (R1), (R2) and (R3).

We have proved that $\mu'(x)\geq 0$ for all $x\in V(G)$, then $\sum_{x\in V(G)}\mu'(x)\geq 0$, this contradicts $\sum_{x\in V(G)}\mu(x)<0$. This completes the proof. □

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