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New Upper Bounds for the Inverse of H-Matrices Including S-SDD Matrices and Linear Complementarity Problems

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Abstract A partition reduction method is used to obtain new upper bounds for the inverses of H-matrices and S-strictly diagonally dominant (S-SDD) matrices. The estimates are expressed via the determinants of third order matrices. Numerical experiments with various random matrices show that they are stable and better than the estimates presented in literatures. We use these upper bounds to improve known error estimates for linear complementarity problems with H-matrices and S-SDD matrices.

Keywords linear complementarity problem; error bound; upper bound; S-SDD matrices; H-matrices

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1. Introduction

Linear complementarity problem LCP(M, q) consists in finding a vector $x \geq 0$ such that

$$Mx + q \geq 0,\ x^{\mathrm{T}}(Mx + q) = 0, \tag{1.1}$$

or in proving that the problem has no solution. Linear complementarity problems are used in Nash equilibrium point of bimatrix games, contact and free boundary problems for journal bearing [1–3]. It is well known that an H-matrix with positive diagonals is a P-matrix, whose principle submatrices are all positive. Moreover, the LCP(M, q) has a unique solution for any $q \in \mathbb{R}^n$ if and only if M is a P-matrix [4]. Later on, Chen and Xiang [5] gave a practical error bounds for LCP(M, q) with a P-matrix M:

$$\|x - x^*\|_\infty \leq \max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \|r(x)\|_\infty, \tag{1.2}$$

where x^* is the exact solution of the LCP(M, q), $r(x) = \min\{x, Mx + q\}$, $D = \mathrm{diag}(d_i)$ with $0 \leq d_i \leq 1$. Since $\|r(x)\|_\infty$ can be easily estimated, we focus on the bounds for

$$\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty. \tag{1.3}$$

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Note that various estimation of norm bounds for inverse matrices can be found in [6–12]. The error bounds of LCPs can be derived by using such estimates. Then one tried to present computable upper bounds of (1.3) for matrix M belonging to various subclasses of P-matrices, such as B-matrices [13–15], doubly B-matrices [16], B^S-matrices [17], B-Nekrasov matrices [18], SB-matrices [19, 20], MB-matrices [21], Nekrasov matrices [22], S-Nekrasov and BS-Nekrasov matrices [23], Σ-SDD matrix [24, 25], H-matrices [26], QN-matrices [27], S-QN matrices [28].

As H-matrices with positive diagonals are P-matrices, then the related LCP(M, q) has a unique solution with error bounds (1.2). Though García-Esnaola and Peña [26] has presented a comparison error bounds of LCPs for H-matrices, it is not accurate for the H-matrix M with positive diagonal matrix D such that the diagonal dominance degree is very small. In 2013, García-Esnaola and Peña [24] considered the LCP(M, q) with the matrix M being a Σ-SDD with positive diagonals, and provided upper bounds of (1.3). After that in 2019, Wang and Li [25] improved the results for S-SDD matrices in [24]. Actually, Σ-SDD matrix is just the S-SDD matrix, which belongs to H-matrix [29, 30], defined as follows:

Definition 1.1 ([30]) *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. If there is a $\emptyset \neq S \subset \mathbb{N} := \{1, 2, \dots, n\}$ such that for any $i \in S$ and $j \in S^c$,*

$$\begin{cases} |a_{ii}| > r_i^S(A), \\ (|a_{ii}| - r_i^S(A))(|a_{jj}| - r_j^{S^c}(A)) > r_i^{S^c}(A) r_j^S(A), \end{cases}$$

where S^c is the complement set of S via $\mathbb{N}$, and

$$r_i(A) := \sum_{i\neq j\in\mathbb{N}} |a_{ij}|, \;\; r_i^S(A) := \sum_{i\neq j\in S} |a_{ij}|, \;\; r_i^{S^c}(A) := \sum_{i\neq j\in S^c} |a_{ij}|,$$

then A is an S-SDD matrix.

The diagonal dominance degree [31] is widely used in bounding the inverse of matrices and the eigenvalues [32,33]. The known upper bounds for H-matrices [26] and S-SDD matrices [24,25] are obtained by the fact that H-matrices and S-SDD matrix can be related to a strictly diagonally dominant (SDD) matrix via a positive diagonal matrix [29, 30]. However, the existing upper bounds are not sharp when the diagonal dominance degree of the related SDD matrix are very small. While if the partition reduction method in [15] is used to estimating $\|A^{-1}\|_\infty$ for S-SDD matrix A, it is more accurate and stable.

In this paper, we would first recall the partition reduction method for estimating the SDD M-matrices. While for any H-matrix A, there exists a positive diagonal matrix D such that AD is an SDD matrix, then $\|(AD)^{-1}\|_\infty$ can be well estimated, and we present the upper bounds $\|A^{-1}\|_\infty$ for H-matrices through the estimation of $\|(AD)^{-1}\|_\infty$. Furthermore, we use this upper bound to obtain a new error bound of LCPs for H-matrix which seems to be more efficient in random numerical experiments. In addition, S-SDD matrix is an H-matrix with explicit expressed positive diagonal matrix, when our estimations are used for S-SDD matrix, new upper bound and error bound for S-SDD matrices are proposed.

The rest of the paper are organized as follows: In Section 2, some basic definitions, lemmas and associated theories are presented. In Section 3, we use the partition reduction method to obtain new upper bounds for H-matrix and S-SDD matrix and some random numerical examples are presented to show the efficiency and accuracy of our results. In Section 4, we apply the new obtained upper bounds for new error bounds for H-matrix including S-SDD matrices and some random examples are presented.

2. Preliminaries

In this section, let us introduce notations, definitions and auxiliary results. If $A = (a_{ij})$, $\mathbb{N} = \{1, 2, \ldots, n\}$, then the notation $A \geq 0$ ($A > 0$) means that $a_{ij} \geq 0$ ($a_{ij} > 0$) for all i, j. Consequently, we write $A \geq B$ ($A > 0$) if $A - B \geq 0$ ($A - B > 0$). Let $\mathbb{R}$ and $\mathbb{C}$, denote the sets of the real and complex numbers, respectively. Denote $e_n = (1, 1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$.

Definition 2.1 ([34]) *A matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is called a diagonally dominant (DD) matrix, if*

$$|a_{ii}| \geq r_i(A), \quad i \in \mathbb{N}, \tag{2.1}$$

and strictly diagonally dominant (SDD) matrix if the inequalities in (2.1) all hold strictly. Further more, if there is a positive diagonal matrix D such that AD is an SDD matrix, then A is called a nonsingular H-matrix.

Definition 2.2 ([34]) *A is called a nonsingular M-matrix if it can be expressed as $A = sI - B$, where $B \geq 0$ and $s > \rho(B)$, where $\rho(B)$ is the spectral radius of B.*

Definition 2.3 ([34]) *A matrix $A = (a_{ij})$ is called an H-matrix if its comparison matrix $\mu(A) = (\tilde{a}_{ij})$ is an M-matrix, where*

$$\tilde{a}_{ij} := \begin{cases} |a_{ij}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases}$$

Lemma 2.4 ([35]) *Suppose A is an H-matrix, then $|A^{-1}| \leq \{\mu(A)\}^{-1}$.*

Lemma 2.5 ([15]) *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a nonsingular M-matrix such that $Ax \leq b$, then*

$$x \leq (\frac{\det A^{(1,b)}}{\det A}, \ldots, \frac{\det A^{(n,b)}}{\det A})^{\mathrm{T}}.$$

Lemma 2.6 ([24]) *Let A be an S-SDD matrix. Suppose $W = \mathrm{diag}(w_1, \ldots, w_n)$ is a positive diagonal matrix with*

$$w_i = \begin{cases} \gamma, & i \in S \\ 1, & i \in S^c \end{cases}, \tag{2.2}$$

where $0 < \gamma \in I_S$, and

$$I_S := (\max_{i\in S} \frac{r_i^{S^c}(A)}{|a_{ii}| - r_i^S(A)}, \min_{j\in S^c} \frac{|a_{jj}| - r_j^{S^c}(A)}{r_j^S(A)}), \tag{2.3}$$

assuming that if $r_j^S(A)=0$, *then* $\frac{|a_{jj}|-r_j^{S^c}(A)}{r_j^S(A)}:=+\infty$. *Then AW is a strictly diagonally dominant matrix.*

Remark 2.7 In general, the positive diagonal matrix W is not unique. For instance, for any $k\in\mathbb{R}$, kW is also such a positive diagonal matrix.

3. Upper bounds for the inverse of H-matrices including S-SDD matrices

In this section, we would present the upper bound of S-SDD matrices by partition reduction method. Firstly, let us give some notations: Let $\mathbb{N}(\alpha_1,\alpha_2,\dots,\alpha_k)$ be a partition of the set $\mathbb{N}$ such that $\alpha_1,\alpha_2,\dots,\alpha_k$ are nonempty subsets separated from each other. In particular, for $k=3$ we have $\alpha_1\cup\alpha_2\cup\alpha_3=\mathbb{N}$ and $\alpha_1\cap\alpha_2=\emptyset$, $\alpha_1\cap\alpha_3=\emptyset$, $\alpha_2\cap\alpha_3=\emptyset$. In general, we denote by $\mathfrak{P}(\mathbb{N}_k)$ the set of all partitions separating $\mathbb{N}$ into k parts. In addition, let $A^{(j,b)}$ ($j\in\mathbb{N},\ b\in\mathbb{R}^n$) be the matrix obtained from $A\in\mathbb{R}^{n\times n}$ by replacing its j-th column by b.

Lemma 3.1 ([15]) *If $A=(a_{ij})\in\mathbb{R}^{n\times n}$, $n\geq 3$, is an SDD M-matrix, then*

$$\|A^{-1}\|_\infty\leq\min_{\mathbb{N}(\alpha_1,\alpha_2,\alpha_3)\in\mathfrak{P}(\mathbb{N}_3)}\max_{\substack{p\in\alpha_1\\ q\in\alpha_2\\ r\in\alpha_3}}\Big\{\frac{\det S_{(p,q,r)}^{(1,e_3)}}{\det S_{(p,q,r)}},\frac{\det S_{(p,q,r)}^{(2,e_3)}}{\det S_{(p,q,r)}},\frac{\det S_{(p,q,r)}^{(3,e_3)}}{\det S_{(p,q,r)}}\Big\},\tag{3.1}$$

where

$$S_{(p,q,r)}=\begin{pmatrix}\sum\limits_{j\in\alpha_1}a_{pj} & \sum\limits_{j\in\alpha_2}a_{pj} & \sum\limits_{j\in\alpha_3}a_{pj}\\ \sum\limits_{j\in\alpha_1}a_{qj} & \sum\limits_{j\in\alpha_2}a_{qj} & \sum\limits_{j\in\alpha_3}a_{qj}\\ \sum\limits_{j\in\alpha_1}a_{rj} & \sum\limits_{j\in\alpha_2}a_{rj} & \sum\limits_{j\in\alpha_3}a_{rj}\end{pmatrix}.\tag{3.2}$$

Theorem 3.2 *Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ be an M-matrix, and $T=\operatorname{diag}(t_1,\dots,t_n)$ be a positive diagonal matrix such that AT is an SDD matrix. Then*

$$\|A^{-1}\|_\infty\leq \min_{\mathbb{N}(\alpha_1,\alpha_2,\alpha_3)\in\mathfrak{P}(\mathbb{N}_3)}\max_{\substack{p\in\alpha_1\\ q\in\alpha_2\\ r\in\alpha_3}}\Big\{\max_{i\in\alpha_1}t_i\frac{\det ST_{(p,q,r)}^{(1,e_3)}}{\det ST_{(p,q,r)}},\max_{i\in\alpha_2}t_i\frac{\det ST_{(p,q,r)}^{(2,e_3)}}{\det ST_{(p,q,r)}},\max_{i\in\alpha_3}t_i\frac{\det ST_{(p,q,r)}^{(3,e_3)}}{\det ST_{(p,q,r)}}\Big\},\tag{3.3}$$

where

$$ST_{(p,q,r)}=\begin{pmatrix}|a_{pp}|t_p-\sum\limits_{p\neq j\in\alpha_1}|a_{pj}|t_j & -\sum\limits_{j\in\alpha_2}|a_{pj}|t_j & -\sum\limits_{j\in\alpha_3}|a_{pj}|t_j\\ -\sum\limits_{j\in\alpha_1}|a_{qj}|t_j & |a_{qq}|t_q-\sum\limits_{q\neq j\in\alpha_2}|a_{qj}|t_j & -\sum\limits_{j\in\alpha_3}|a_{qj}|t_j\\ -\sum\limits_{j\in\alpha_1}|a_{rj}|t_j & -\sum\limits_{j\in\alpha_2}|a_{rj}|t_j & |a_{rr}|t_r-\sum\limits_{r\neq j\in\alpha_3}|a_{rj}|t_j\end{pmatrix}.\tag{3.4}$$

Proof Since A is an M-matrix, $A^{-1} \geq 0$ and there exists a positive diagonal matrix T such that AT is an SDD M-matrix. At first, let us consider the linear equation

$$ATx = e_n, \tag{3.5}$$

then when $0 \leq x = (AT)^{-1}e_n \leq \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, one immediately gets:

$$A^{-1}e_n \leq T \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} t_1 y_1 \\ \vdots \\ t_n y_n \end{pmatrix}.$$

Moreover,

$$\|A^{-1}\|_\infty = \|A^{-1}e_n\|_\infty \leq \left\| \begin{pmatrix} t_1 y_1 \\ \vdots \\ t_n y_n \end{pmatrix} \right\|_\infty. \tag{3.6}$$

Recall the partition reduction method for SDD M-matrix. For any partition $\mathbb{N}(\alpha_1, \alpha_2, \alpha_3)$, considering the equation (3.5), let $x_p = \max_{i\in\alpha_1}\{x_i\}$, $x_q = \max_{i\in\alpha_2}\{x_i\}$, and $x_r = \max_{i\in\alpha_3}\{x_i\}$. Then

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \geq \begin{pmatrix} |a_{pp}|t_p - \sum\limits_{p\neq j\in\alpha_1} |a_{pj}|t_j & -\sum\limits_{j\in\alpha_2} |a_{pj}|t_j & -\sum\limits_{j\in\alpha_3} |a_{pj}|t_j \\ -\sum\limits_{j\in\alpha_1} |a_{qj}|t_j & |a_{qq}|t_q - \sum\limits_{q\neq j\in\alpha_2} |a_{qj}|t_j & -\sum\limits_{j\in\alpha_3} |a_{qj}|t_j \\ -\sum\limits_{j\in\alpha_1} |a_{rj}|t_j & -\sum\limits_{j\in\alpha_2} |a_{rj}|t_j & |a_{rr}|t_r - \sum\limits_{r\neq j\in\alpha_3} |a_{rj}|t_j \end{pmatrix} \begin{pmatrix} x_p \\ x_q \\ x_r \end{pmatrix}, \tag{3.7}$$

which by Lemma 2.5 illustrates that

$$\begin{pmatrix} x_p \\ x_q \\ x_r \end{pmatrix} \leq \begin{pmatrix} \frac{\det ST^{(1,e_3)}_{(p,q,r)}}{\det ST_{(p,q,r)}} \\ \frac{\det ST^{(2,e_3)}_{(p,q,r)}}{\det ST_{(p,q,r)}} \\ \frac{\det ST^{(3,e_3)}_{(p,q,r)}}{\det ST_{(p,q,r)}} \end{pmatrix}. \tag{3.8}$$

Hence

$$x_i \leq \begin{cases} x_p \leq \dfrac{\det ST^{(1,e_3)}_{(p,q,r)}}{\det ST_{(p,q,r)}}, & i \in \alpha_1, \\ x_q \leq \dfrac{\det ST^{(2,e_3)}_{(p,q,r)}}{\det ST_{(p,q,r)}}, & i \in \alpha_2, \\ x_r \leq \dfrac{\det ST^{(3,e_3)}_{(p,q,r)}}{\det ST_{(p,q,r)}}, & i \in \alpha_3. \end{cases} \tag{3.9}$$

Then according to (3.6),

$$\|A^{-1}\|_\infty \leq \max\Big\{\max_{i\in\alpha_1} t_i \frac{\det ST^{(1,e_3)}_{(p,q,r)}}{\det ST_{(p,q,r)}}, \max_{i\in\alpha_2} t_i \frac{\det ST^{(2,e_3)}_{(p,q,r)}}{\det ST_{(p,q,r)}}, \max_{i\in\alpha_3} t_i \frac{\det ST^{(3,e_3)}_{(p,q,r)}}{\det ST_{(p,q,r)}}\Big\}. \tag{3.10}$$

At last, applying this method to all partitions of $\mathfrak{P}(\mathbb{N}_3)$, we obtain (3.3). $\square$

Remark 3.3 In particular, if A is an SDD matrix, then $T = I$ is an identity matrix, and the bound reduces to the result in Lemma 3.1. However, for any M-matrix, it is not easy to give an explicit expression of the positive diagonal matrix, lots of the iterative algorithm for H-matrix present the numerical positive diagonal matrix. Nevertheless, as long as we obtain the positive diagonal matrix, we could apply the estimation in (3.3) to measure $\|A^{-1}\|_\infty$ for M-matrix.

By Lemma 2.6, the explicit expression of positive matrix W is proposed for S-SDD matrix A such that AW is an SDD matrix. Then a new upper bound for S-SDD matrix is obtained.

Corollary 3.4 *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an S-SDD matrix, and $W = \mathrm{diag}(w_1, \ldots, w_n)$ be a positive diagonal matrix defined as that in (2.2). Then*

$$\begin{aligned}&\|A^{-1}\|_\infty \\ &\leq \min_{\mathbb{N}(\alpha_1,\alpha_2,\alpha_3)\in\mathfrak{P}(\mathbb{N}_3)} \max_{\substack{p\in\alpha_1\\ q\in\alpha_2\\ r\in\alpha_3}}\Big\{\max_{i\in\alpha_1} w_i \frac{\det SW^{(1,e_3)}_{(p,q,r)}}{\det SW_{(p,q,r)}}, \max_{i\in\alpha_2} w_i \frac{\det SW^{(2,e_3)}_{(p,q,r)}}{\det SW_{(p,q,r)}}, \max_{i\in\alpha_3} w_i \frac{\det SW^{(3,e_3)}_{(p,q,r)}}{\det SW_{(p,q,r)}}\Big\},\end{aligned} \tag{3.11}$$

where

$$SW_{(p,q,r)} = \begin{pmatrix} |a_{pp}|w_p - \sum\limits_{p\neq j\in\alpha_1} |a_{pj}|w_j & -\sum\limits_{j\in\alpha_2} |a_{pj}|w_j & -\sum\limits_{j\in\alpha_3} |a_{pj}|w_j \\ -\sum\limits_{j\in\alpha_1} |a_{qj}|w_j & |a_{qq}|w_q - \sum\limits_{q\neq j\in\alpha_2} |a_{qj}|w_j & -\sum\limits_{j\in\alpha_3} |a_{qj}|w_j \\ -\sum\limits_{j\in\alpha_1} |a_{rj}|w_j & -\sum\limits_{j\in\alpha_2} |a_{rj}|w_j & |a_{rr}|w_r - \sum\limits_{r\neq j\in\alpha_3} |a_{rj}|w_j \end{pmatrix}.$$

For large H-matrices and S-SDD matrices A, we do not compute the bounds for different partitions. Instead, we choose two partitions and compute the corresponding bounds in (3.3) and (3.11). Such experiments are able to show the effectiveness and superiority of our method. Besides, in numerical experiments, we will sort the row indexes from smallest to largest in order of diagonal dominance degree for getting $i_1, i_2, \ldots, i_n$, and then present two partitions as follows,

- partition 1: $\alpha_1 = \{i_1, i_2, \ldots, i_{\lfloor\frac{n}{3}\rfloor}\}$, $\alpha_2 = \{i_{\lfloor\frac{n}{3}\rfloor+1}, \ldots, i_{\lfloor\frac{2n}{3}\rfloor}\}$, $\alpha_3 = \{i_{\lfloor\frac{2n}{3}\rfloor+1}, \ldots, i_n\}$
- partition 2: $\widetilde{\alpha}_1 = \{i_1, i_4, \ldots, i_{3k-2}\}$, $\widetilde{\alpha}_2 = \{i_2, i_5, \ldots, i_{3k-1}\}$, $\widetilde{\alpha}_3 = \{i_3, i_6, \ldots, i_{3k}\}$.

In addition, for the positive diagonal matrix $W = \mathrm{diag}(w_1, \ldots, w_n)$ related to S-SDD matrix, we just make the parameter

$$\gamma = \frac{1}{2}\Big(\max_{i\in S} \frac{r_i^{S^c}(A)}{|a_{ii}| - r_i^S(A)} + \min_{j\in S^c} \frac{|a_{jj}| - r_j^{S^c}(A)}{r_j^S(A)}\Big). \tag{3.12}$$

For comparison, we recall some upper bound of $\|A^{-1}\|_\infty$ for H-matrices and S-SDD matrices [36], respectively:

• If $A=(a_{ij})\in\mathbb{R}^{n\times n}$ is an H-matrix with positive diagonal matrix $D=\mathrm{diag}(d_1,\dots,d_n)$ such that AD is an SDD matrix, then

$$\|A^{-1}\|_\infty\le\max_i\frac{\max\limits_k d_k}{|a_{ii}|d_i-\sum\limits_{j\neq i}|a_{ij}|d_j}.\tag{3.13}$$

• If $A=(a_{ij})\in\mathbb{R}^{n\times n}$ is an S-SDD matrix, then

$$\|A^{-1}\|_\infty\le\min_{S\in\mathfrak{S}(A)}\max_{i\in S,j\in S^c}\frac{|a_{ii}|-r_i^S(A)+r_j^{S^c}(A)}{\big(|a_{ii}|-r_i^S(A)\big)\big(|a_{jj}|-r_j^{S^c}(A)\big)-r_i^{S^c}(A)r_j^S(A)}.\tag{3.14}$$

Next, let us see some random numerical experiments for H-matrices.

Example 3.5 Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ $n\ge4$ be a random H-matrix generated by the following matlab codes:

$$A=-\mathrm{randi}([1,100],n,n),\ \gamma=0.2+\mathrm{exprnd}(3,n,1),\ R=\mathrm{sum}(A'),\ d=0.2+\mathrm{rand}(n,1),$$

$$\begin{aligned}&\text{for } i=1:n\\ &\quad A(i,i)=R(i)-A(i,i)+\gamma(i);\\ &\text{end}\\ &A=A*\mathrm{diag}(d).\end{aligned}$$

Then A is an H-matrix with positive diagonal matrix $\widetilde{D}=\mathrm{diag}(\frac{1}{d_1},\dots,\frac{1}{d_n})$ such that $A\widetilde{D}$ is an SDD matrix. In this example, we increase n from 4 to 100, and randomly generate H-matrices with positive diagonal matrix. For each random H-matrix, we compute the upper bounds (3.13) and (3.3), and plot the results in Figure 1.

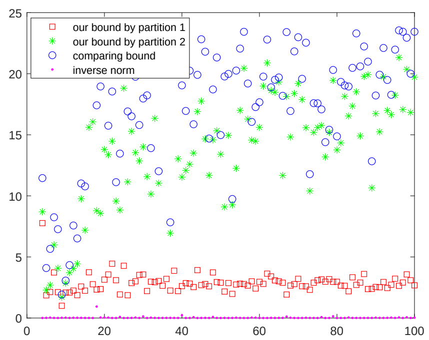

Figure 1 The estimations for $\|A^{-1}\|_\infty$ of some random H-matrices

From the results in the picture, our bounds are sharper and stable, especially when the size of the matrix is large, our results are still accurate and stable.

Next, let us consider the S-SDD matrix. Recall the numerical experiment in [36].

Example 3.6 Suppose

$$A=\begin{pmatrix} 9.2 & -1.3 & -2.1 & -0.5 & -3.3 & -2.5\\ -1.6 & 8.5 & -0.3 & -2.7 & -1.1 & -0.6\\ -0.3 & -4.3 & 9.8 & -1.2 & -0.5 & -2.1\\ -1.7 & -0.9 & -2.5 & 7.8 & -0.3 & -1.4\\ -2.9 & -0.1 & -2.1 & -1.3 & 8.8 & -2.1\\ -3.1 & -1.5 & -0.2 & -1.6 & -0.7 & 7.6 \end{pmatrix},$$

then A is not an SDD matrix but an S-SDD matix with $T(A)=\{1\}$ and $\|A^{-1}\|_\infty=1.4663$. By simple computation, the subset S is in choice of $\{2,3\}$, $\{2,3,4\}$, $\{3,5,6\}$, $\{2,3,4,6\}$, $\{2,3,5,6\}$, $\{3,4,5,6\}$, $\{2,3,4,5,6\}$, and the upper bounds computed by (3.14) and (3.11) are listed in Table 1.

S	Morăca's bound	Our bound by partition 1	Our bound by partition 2
$\{2,3\}$	4.4279	1.8548	2.6876
$\{2,3,4\}$	4.8824	1.7176	2.6914
$\{3,5,6\}$	178.5714	3.0007	36.2781
$\{2,3,4,6\}$	10.1905	1.9425	3.8217
$\{2,3,5,6\}$	26.8627	2.5912	11.1112
$\{3,4,5,6\}$	13.4483	2.5854	9.6983
$\{2,3,4,5,6\}$	9.8473	2.2415	5.6797

Table 1 Morăca's bound and our bound for S-SDD matrix with different S

From the results in Table 1, it is obvious that our upper bounds are more accurate. Thus this example illustrates that our results are sharper in some cases.

4. Error bounds of linear complementarity problem for H-matrices including S-SDD matrices

In this section, we would apply the new upper bound for H-matrices and S-SDD matrices to the error bound of linear complementarity problem for H-matrices and S-SDD matrix, respectively.

Before presenting our error bounds, we recall a useful lemma.

Lemma 4.1 ([15]) *Considering an SDD M-matrix*

$$A=\begin{pmatrix} a_{11} & -a_{12} & -a_{13}\\ -a_{21} & a_{22} & -a_{23}\\ -a_{31} & -a_{32} & a_{33} \end{pmatrix},$$

we set

$$h_1(x,y,z):=\frac{\det(I-D+DA)^{(1,e_3)}}{\det(I-D+DA)},$$

$$h_2(x,y,z) := \frac{\det(I-D+DA)^{(3,e_3)}}{\det(I-D+DA)},$$
$$h_3(x,y,z) := \frac{\det(I-D+DA)^{(2,e_3)}}{\det(I-D+DA)}, \tag{4.1}$$

where $D = \operatorname{diag}(x,y,z)$ *and* $0 \le x,y,z \le 1$. *Then*

$$h_1(x,y,z) \le \max\{h_1(0,0,0), h_1(1,0,0), h_1(1,0,1), h_1(1,1,0), h_1(1,1,1)\},$$
$$h_2(x,y,z) \le \max\{h_2(0,0,0), h_2(0,1,0), h_2(1,1,0), h_2(0,1,1), h_2(1,1,1)\},$$
$$h_3(x,y,z) \le \max\{h_3(0,0,0), h_3(0,0,1), h_3(1,0,1), h_3(0,1,1), h_3(1,1,1)\}.$$

As introduced in Section 1, we need to consider (1.3) when M is an H-matrix or S-SDD matrix. At first, let us see some conclusions for $I - D + DM$ if M is an H-matrix.

Theorem 4.2 *Let* $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ *with positive diagonals be an* H*-matrix. Then* $\widetilde{M} = I - D + DM := (\widetilde{m}_{ij})$ *is also an* H*-matrix, where* $D = \operatorname{diag}(d_1,\dots,d_n)$, $0 \le d_i \le 1$, $1 \le i \le n$.

Proof As $M = (m_{ij})$ is an H-matrix, then there exists a positive diagonal matrix $P = \operatorname{diag}(p_1,\dots,p_n)$ such that MP is an SDD matrix,

$$|m_{ii}|p_i > \sum_{j\ne i} |m_{ij}|p_j, \quad i = 1,2,\dots,n. \tag{4.2}$$

While considering $\widetilde{M} = I - D + DM := (\widetilde{m}_{ij})$, for any $i \in \mathbb{N}$,

$$\begin{aligned}
|\widetilde{m}_{ii}|p_i - \sum_{j\ne i} |\widetilde{m}_{ij}|p_j =& |1 - d_i + m_{ii}d_i|p_i - \sum_{j\ne i} |m_{ij}d_i|p_j \\
\ge& |m_{ii}|d_ip_i - \sum_{j\ne i} |m_{ij}|d_ip_j \qquad (4.3)\\
=& d_i(|m_{ii}|p_i - \sum_{j\ne i} |m_{ij}|p_j) \\
\ge& 0. \qquad (4.4)
\end{aligned}$$

It is worthy to point out that the equivalence in (4.3) holds if and only $d_i = 1$, and the equivalence in (4.4) holds if and only if $d_i = 0$, thus

$$|\widetilde{m}_{ii}|p_i - \sum_{j\ne i} |\widetilde{m}_{ij}|p_j > 0, \quad i \in \mathbb{N}, \tag{4.5}$$

which implies $\widetilde{M} = I - D + DM$ is still an H-matrix. □

Lemma 4.3 *Suppose* A *is a nonsingular* M*-matrix with a positive diagonal matrix* $T = \operatorname{diag}(t_1,t_2,t_3)$ *such that* AT *is an SDD matrix, where* $t_i \ge 1$, $i = 1,2,3$ *and*

$$A = \begin{pmatrix} |a_{11}| & -|a_{12}| & -|a_{13}| \\ -|a_{21}| & |a_{22}| & -|a_{23}| \\ -|a_{31}| & -|a_{32}| & |a_{33}| \end{pmatrix}, \tag{4.6}$$

then

$$\frac{\det((I-D+DA)T)^{(1,e_3)}}{\det(I-D+DA)T} \le \frac{\det(I-D+DAT)^{(1,e_3)}}{\det(I-D+DAT)},$$
$$\frac{\det((I-D+DA)T)^{(2,e_3)}}{\det(I-D+DA)T} \le \frac{\det(I-D+DAT)^{(1,e_3)}}{\det(I-D+DAT)},$$
$$\frac{\det((I-D+DA)T)^{(3,e_3)}}{\det(I-D+DA)T} \le \frac{\det(I-D+DAT)^{(1,e_3)}}{\det(I-D+DAT)},$$

where $D = \operatorname{diag}(d_1, \ldots, d_n)$ *with* $0 \le d_i \le 1$, $i = 1, 2, \ldots, n$.

Proof Consider the linear equation

$$(I-D+DA)Tx = e_3, \tag{4.7}$$

by Theorem 4.2, $(I-D+DA)T$ is an SDD M-matrix, so there is a unique solution for (4.7) denoted as

$$x_i = \frac{\det((I-D+DA)T)^{(i,e_3)}}{\det(I-D+DA)T} \ge 0, \quad i = 1, 2, 3.$$

Furthermore, consider the i-th row of (4.7), as $t_i \ge 1$ and $x_i \ge 0$, $i = 1, 2, 3$, then

$$1 = (1 - d_i + d_i|a_{ii}|)t_i x_i - \sum_{j\ne i} d_i|a_{ij}|t_j x_j \le 1 - d_i + d_i|a_{ii}|t_i x_i - \sum_{j\ne i} d_i|a_{ij}|t_j x_j, \tag{4.8}$$

and (4.8) yields

$$(I-D+DAT)x \le e_3. \tag{4.9}$$

It is easy to know, $I-D+DAT$ is still an SDD M-matrix because AT is an SDD M-matrix. Thus by Lemma 2.5,

$$\frac{\det((I-D+DA)T)^{(i,e_3)}}{\det(I-D+DA)T} = x_i \le \frac{\det(I-D+DAT)^{(i,e_3)}}{\det(I-D+DAT)}, \quad i = 1, 2, 3. \tag{4.10}$$

Therefore, the proof is completed. □

Remark 4.4 In fact, for an H-matrix A, if there is positive diagonal matrix T such that AT is an SDD matrix, then a scalar matrix kT $(k \ne 0)$ is also a positive diagonal matrix T such that $A(kT)$ is an SDD matrix. Thus, it is attainable to make $T \ge I$.

Therefore, by Theorems 4.2, 3.2 and Lemma 4.3, our error bound of LCPs for H-matrix is proposed.

Theorem 4.5 *Let* $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ *be an* H*-matrix with positive diagonals with positive diagonal matrix* $T = \operatorname{diag}(t_1, \ldots, t_n)$ *such that* MT *is an SDD matrix. Then*

$$\begin{aligned}&\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_\infty \\ &\le \min_{\mathbb{N}(\alpha_1,\alpha_2,\alpha_3)\in\mathfrak{P}(\mathcal{N}_3)} \Big\{\max_{\substack{p\in\alpha_1\\ q\in\alpha_2\\ r\in\alpha_3}}\Big\{ \max_{k\in\{1,2,3\}} \Big\{ \max_{i\in\alpha_k} t_i \cdot \max_{x,y,z\in\{0,1\}} h_k^{(p,q,r)}(x,y,z)\Big\}\Big\}\Big\},\end{aligned} \tag{4.11}$$

where

$$h_k^{(p,q,r)}(x,y,z)=\frac{\det(I-D+DMT)^{(k,e_3)}}{\det(I-D+DMT)},\quad k=1,2,3$$

with the matrix

$$MT=\begin{pmatrix} |m_{pp}|t_p-\sum\limits_{p\neq j\in\alpha_1}|m_{pj}|t_j & -\sum\limits_{j\in\alpha_2}|m_{pj}|t_j & -\sum\limits_{j\in\alpha_3}|m_{pj}|t_j \\ -\sum\limits_{j\in\alpha_1}|m_{qj}|t_j & |m_{qq}|t_q-\sum\limits_{q\neq j\in\alpha_2}|m_{qj}|t_j & -\sum\limits_{j\in\alpha_3}|m_{qj}|t_j \\ -\sum\limits_{j\in\alpha_1}|m_{rj}|t_j & -\sum\limits_{j\in\alpha_2}|m_{rj}|t_j & |m_{rr}|t_r-\sum\limits_{r\neq j\in\alpha_3}|m_{rj}|t_j \end{pmatrix}. \tag{4.12}$$

Proof By the proof of Theorem 4.2, $I-D+DM$ is an H-matrix with the positive diagonal matrix T such that $(I-D+DM)T$ is an SDD matrix, then by Theorem 3.2,

$$\begin{aligned}&\|(I-D+DM)^{-1}\|_\infty\\ &\le \min_{\mathbb{N}(\alpha_1,\alpha_2,\alpha_3)\in\mathfrak{P}(\mathbb{N}_3)}\max_{\substack{p\in\alpha_1\\ q\in\alpha_2\\ r\in\alpha_3}}\Big\{\max_{k\in\{1,2,3\}}\Big\{\max_{i\in\alpha_k}t_i\frac{\det((I-D+DM)T)^{(k,e_3)}}{\det(I-D+DM)T}\Big\}\Big\}\\ &\le \min_{\mathbb{N}(\alpha_1,\alpha_2,\alpha_3)\in\mathfrak{P}(\mathbb{N}_3)}\max_{\substack{p\in\alpha_1\\ q\in\alpha_2\\ r\in\alpha_3}}\Big\{\max_{k\in\{1,2,3\}}\Big\{\max_{i\in\alpha_k}t_i\frac{\det(I-D+DMT)^{(k,e_3)}}{\det(I-D+DMT)}\Big\}\Big\}\end{aligned}$$

with MT in (4.12). Then by Lemma 4.3, the proof is completed. □

Next, let us consider the case that M is an S-SDD matrix.

Theorem 4.6 *Let $M=(m_{ij})\in\mathbb{R}^{n\times n}$ with positive diagonals be an S-SDD matrix. Then $\widetilde{M}=I-D+DM:=(\widetilde{m}_{ij})$ is also an S-SDD matrix, where $D=\operatorname{diag}(d_1,\dots,d_n)$, $0\le d_i\le 1$, $1\le i\le n$.*

Proof As for

$$\widetilde{m}_{ij}=\begin{cases}1-d_i+d_im_{ii}, & i=j;\\ d_im_{ij}, & i\neq j,\end{cases}$$

then for any $i\in S$,

$$|\widetilde{m}_{ii}|-r_i^S(\widetilde{M})=1-d_i+d_im_{ii}-d_ir_i^S(M)\ge d_i(m_{ii}-r_i^S(M))\ge 0. \tag{4.13}$$

And for any $i\in S$, $j\in S^c$,

$$\begin{aligned}&(|\widetilde{m}_{ii}|-r_i^S(\widetilde{M}))(\widetilde{m}_{jj}|-r_j^{S^c}(\widetilde{M}))=(1-d_i+d_im_{ii}-d_ir_i^S(M))(1-d_j+d_jm_{jj}-d_jr_j^{S^c}(M))\\ &\ge d_i(m_{ii}-r_i^S(M))d_j(m_{jj}-r_j^{S^c}(M))\ge d_id_jr_i^{S^c}(M)r_j^S(M)\\ &=r_i^{S^c}(\widetilde{M})r_j^S(\widetilde{M}).\end{aligned} \tag{4.14}$$

It is worth pointing out that the equality of two inequalities in (4.13) and (4.18) cannot hold at the same time. Then for any $i\in S$, $j\in S^c$,

$$\begin{cases}|\widetilde{m}_{ii}|-r_i^S(\widetilde{M})>0,\\ (|\widetilde{m}_{ii}|-r_i^S(\widetilde{M}))(\widetilde{m}_{jj}|-r_j^{S^c}(\widetilde{M}))>r_i^{S^c}(\widetilde{M})r_j^S(\widetilde{M}).\end{cases}$$

Thus by Definition 1.1, $\widetilde{M}$ is also an S-SDD matrix. □

By Theorem 4.2, for any S-SDD matrix A, if the positive diagonal matrix W defined in (2.2) makes AW an SDD matrix, then $(I-D+DM)W$ is also an SDD matrix, where $D = \text{diag}(d_1,\ldots,d_n)$ with $0 \le d_i \le 1$, $i = 1,2,\ldots,n$. In addition, the similar results in Lemma 4.3 are also established for S-SDD matrix. Thus we obtain a new error bound of LCPs for S-SDD matrices.

Theorem 4.7 *Let $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ be an S-SDD matrices with positive diagonals. I_S is defined as that in (2.3) and $0 < \gamma \in I_S$, denote $\overline{W} = \text{diag}(\overline{w}_1,\ldots,\overline{w}_n)$, where*

$$\overline{w}_i = \begin{cases} \gamma^{\text{sgn}(\gamma-1)}, & i \in S; \\ \gamma^{\text{sgn}(\gamma-1)-1}, & i \in S^c, \end{cases} \tag{4.15}$$

then

$$\begin{aligned} &\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_\infty \\ &\le \min_{\mathbb{N}(\alpha_1,\alpha_2,\alpha_3)\in\mathfrak{P}(\mathcal{N}_3)} \Big\{\max_{\substack{p\in\alpha_1\\ q\in\alpha_2\\ r\in\alpha_3}}\Big\{\max_{k\in\{1,2,3\}}\Big\{\max_{i\in\alpha_k} w_i \cdot \max_{x,y,z\in\{0,1\}} g_k^{(p,q,r)}(x,y,z)\Big\}\Big\}\Big\}, \end{aligned} \tag{4.16}$$

where $\text{sgn}(x)$ is the sign function, and

$$g_k^{(p,q,r)}(x,y,z) = \frac{\det(I-D+DMW)^{(k,e_3)}}{\det(I-D+DMW)}, \quad k = 1,2,3 \tag{4.17}$$

with

$$MW = \begin{pmatrix} |m_{pp}|w_p - \sum\limits_{p\ne j\in\alpha_1} |m_{pj}|w_j & -\sum\limits_{j\in\alpha_2}|m_{pj}|w_j & -\sum\limits_{j\in\alpha_3}|m_{pj}|w_j \\ -\sum\limits_{j\in\alpha_1}|m_{qj}|w_j & |m_{qq}|w_q - \sum\limits_{q\ne j\in\alpha_2}|m_{qj}|w_j & -\sum\limits_{j\in\alpha_3}|m_{qj}|w_j \\ -\sum\limits_{j\in\alpha_1}|m_{rj}|w_j & -\sum\limits_{j\in\alpha_2}|m_{rj}|w_j & |m_{rr}|w_r - \sum\limits_{r\ne j\in\alpha_3}|m_{rj}|w_j \end{pmatrix}. \tag{4.18}$$

At last, we will present some numerical experiments to show the efficiency and superiority of our error bounds. When compared with existing results, we recall some error bounds for H-matrices and S-SDD matrices as follows.

• García-Esnaola and Peña [26] presented an error bound of LCPs for H-matrix $M = (m_{ij})$ with positive diagonal matrix $T = \text{diag}(t_i)$ such that MT is an SDD matrix as follows:

$$\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_\infty \le \max\Big\{\frac{\max_i t_i}{\min_i\{\beta_i\}}, \frac{\max_i t_i}{\min_i\{t_i\}}\Big\} \tag{4.19}$$

where $\beta_i = |m_{ii}|t_i - \sum_{j\ne i}|m_{ij}|t_j$.

• García-Esnaola and Peña [24] presented an error bound of LCPs for S-SDD matrices as follows:

$$\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_\infty \le \begin{cases} \max\{\frac{\gamma}{\overline{\beta}}, \gamma\}, & \gamma > 1, \\ \max\{\frac{1}{\overline{\beta}}, \frac{1}{\gamma}\}, & \gamma < 1, \end{cases} \tag{4.20}$$

where $W = \text{diag}(w_1, w_2, \ldots, w_n)$ with $w_i = \begin{cases} \gamma, & i \in S \\ 1, & i \in S^c \end{cases}$, $\gamma \in I_S$ and $\bar{\beta} = \min_{i\in\mathbb{N}} |a_{ii}|w_i - \sum_{j\neq i} |a_{ij}|w_j$.

In the experiments, we just choose two partitions as that in the numerical experiments of Section 3 and make the parameter γ as that in (3.12).

Recalling (1.2), we first compute a numerical solution to LCP(M, q), because it exists in $r(x) = \min\{x, Mx + q\}$. In the experiments, we apply the modulus-based matrix splitting iteration method [1] to obtain the numerical solution.

Algorithm 1 ([1]) (Modulus-Based Matrix Splitting Iteration Method)

1. Let $M = Q_1 - Q_2$ be a splitting of the matrix $M \in \mathbb{R}^{n\times n}$, Λ be a positive diagonal matrix, and η be a positive constant.

2. Given an initial vector $y^{(1)} \in \mathbb{R}^n$, determine $y^{(k+1)} \in \mathbb{R}^n$ from the system

$$(\Lambda + Q_1)y^{(k+1)} = Q_2y^{(k)} + (\Lambda - M)|y^{(k)}| - \eta q, \quad k = 1, 2, \ldots,$$

set

$$x^{(k)} := \frac{1}{\eta}(y^{(k)} + |y^{(k)}|), \quad k = 1, 2, \ldots.$$

3. Go until the sequence $\{x^{(k)}\}$ converges.

Finding a numerical solution x of LCP(M, q) by Algorithm 1, we obtain $r(x) = \min\{x, Mx + q\}$. Combining upper bounds for $\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ leads to the error estimates for LCP(M, q) in (1.2).

Example 4.8 Let $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ be a random H-matrix generated by matlab code:

$$M = -\text{randi}([1, 100], n, n), \; R = \text{sum}(M'), \; \gamma = 0.2 + abs(\text{exprnd}(3, n, 1)), \; d = 0.2 + \text{rand}(n, 1)$$

$$\begin{aligned} &\text{for } i = 1 : n \\ &\quad M(i, i) = M(i, i) - R(i) + \gamma(i) \\ &\text{end} \\ &\text{M=M*diag(d)} \end{aligned}$$

$$q = -M \cdot x, \qquad \text{where} \qquad x = (2, 2, 2, \ldots, 2)^{\mathrm{T}}.$$

Then M is an H-matrix with positive matrix $D = \text{diag}(\frac{1}{d_1}, \ldots, \frac{1}{d_n})$ such that MD is an SDD matrix. In addition, the exact solution of LCP(M, q) is $x = (2, 2, \ldots, 2)$. We will compute the numerical solution x^* of LCP(M, q) by Algorithm 1, and measure the real error as $\|x - x^*\|_\infty$. In addition, we would compute the error bounds by (4.20) and (4.16). Here, we increase the size of matrix n from 5 to 100, and all results are plotted in Figure 2. For better observing difference between our bounds and the existing one, we plot the natural logarithm for all error bounds.

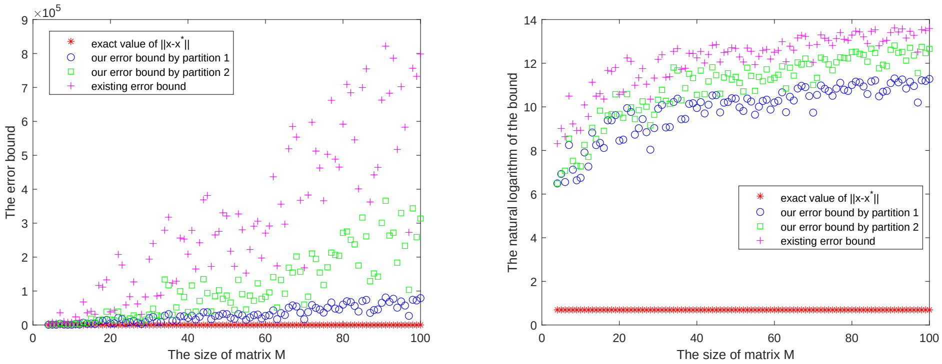

Figure 2 The estimations of $\|(I-D+DM)^{-1}\|_\infty$ for H-matrix M with increasing orders

From Figure 2, our error bounds are efficiency and accuracy, and as long as the size of matrix M increases, our error bounds are also more stable than existing ones. This example illustrates that our error bounds are more accurate and stable in some cases.

In the rest examples, we only consider the upper bounds of $\|(I-D+DM)^{-1}\|_\infty$. Firstly, let us see the case for some random H-matrices M.

Example 4.9 We generate some random H-matrices by the following matlab code:

$$A=-\mathrm{randi}([1,100],n,n),\ \gamma=0.01+\mathrm{exprnd}(3,n,1),\ R=\mathrm{sum}(A'),\ d=0.02+\mathrm{rand}(n,1),$$

$$\begin{aligned}&\text{for } i=1:n\\&\quad A(i,i)=A(i,i)-R(i)+\gamma(i);\\&\text{end}\\&A=A*\mathrm{diag}(d).\end{aligned}$$

Then A is an H-matrix with positive diagonal matrix $\widetilde{D}=\mathrm{diag}(\frac{1}{d_1},\dots,\frac{1}{d_n})$ such that $A\widetilde{D}$ is an SDD matrix. Then the upper bounds for $\|(I-D+DM)^{-1}\|_\infty$ computed by (4.20) and (4.16) are plotted in Figure 3 with parameter γ in (3.12). From the results in Figure 3, it is not hard to see that our results are sharper and more stable, which shows the efficiency and superiority of our result.

Example 4.10 Consider the S-SDD matrix in the Example in [24],

$$M=\begin{pmatrix}3&-1&0&0&-1&0\\-1&5&-1&-1&0&-1\\-2&-1&7&-1&-2&0\\-\frac{1}{2}&-\frac{1}{2}&-\frac{1}{4}&3&-1&-1\\-\frac{2}{5}&-\frac{1}{5}&-\frac{1}{2}&-1&6&-3\\-\frac{1}{3}&-\frac{2}{5}&-\frac{2}{5}&-1&-1&3\end{pmatrix}.$$

Take the set $S=\{1,2,3\}$ and $S^c=\{4,5,6\}$. Then $I_S=(\frac{3}{4},\frac{4}{5})$. We choose $0<\gamma\in I_S$, and compute bounds for $\|(I-D+DM)^{-1}\|_\infty$ by (4.20) and (4.16) with increasing γ, all bounds are plotted in Figure 4. From the results in Figure 4, our bounds are more accurate with the same γ. Furthermore, our results may have not attained the smallest upper bounds. Thus our error bound is more efficient in some cases.

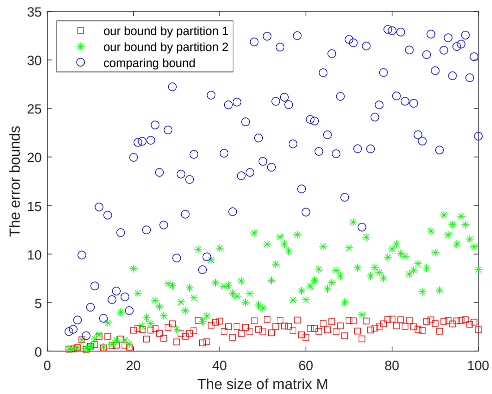

Figure 3 The estimations of $\|(I-D+DM)^{-1}\|_\infty$ for H-matrix M with increasing orders

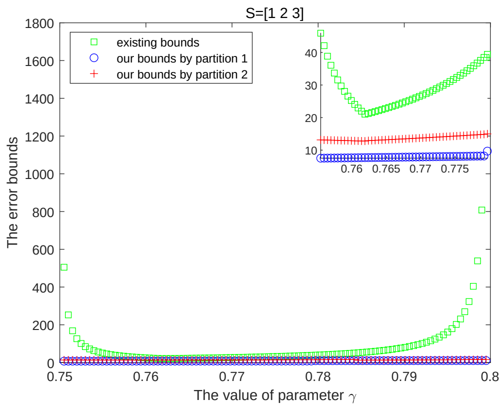

Figure 4 The estimations of $\|(I-D+DM)^{-1}\|_\infty$ for S-SDD matrix M with different parameter γ

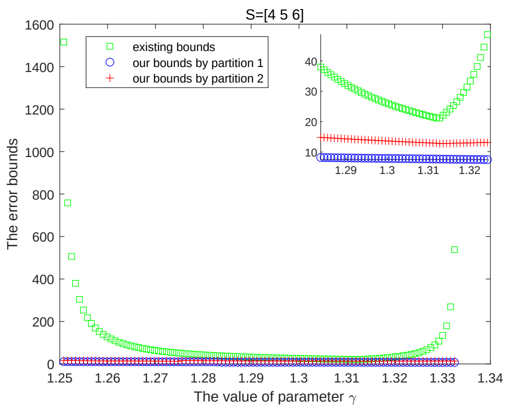

Figure 5 The estimations of $\|(I-D+DM)^{-1}\|_\infty$ for S-SDD matrix M with different parameter γ

Taking $S=\{4,5,6\}$ and $S^c=\{1,2,3\}$, we have $I_S=(\frac{5}{4},\frac{4}{3})$. Similarly, we plot all results in Figure 5. From the results in Figure 5, our error bounds are also of more efficiency and accuracy. It also shows the superiority of our results.

5. Concluding remarks

By partition reduction method, we present new upper bounds of the inverse of H-matrices including S-SDD matrices, which are more accurate in some cases. When we apply the upper bound to error bound of LCPs, new error bounds of LCPs for H-matrices and S-SDD matrices are presented, and they are also sharper than existing ones in some cases.

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