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Finite Groups with Some Self-Conjugate-Permutable Subgroups

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Abstract A subgroup H of a group G is said to be self-conjugate-permutable if $HH^x = H^xH$ implies $H^x = H$ for any x of G. A finite group G is called an SC-group (PSC-group, respectively) if all cyclic subgroups of G of order 2 or order 4 (prime order or order 4, respectively) are self-conjugate-permutable in G. In this paper, we first investigate the structure of finite non-solvable groups all of whose second maximal subgroups are SC-groups; then we mainly investigate the structure of finite groups in which all of maximal subgroups of even order are PSC-groups. In fact, we describe the structure of finite groups which are not PSC-groups but all of whose maximal subgroups of even order are PSC-groups.

Keywords SC-group; PSC-group; self-conjugate-permutable subgroup; maximal subgroup

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1. Introduction

In this paper, only finite groups are considered and our notation is standard.

One of the important topics in group theory is to characterize the structure of groups, and many classical results about this topic have been obtained, for example, minimal non-nilpotent groups [1], minimal non-supersolvable groups [2], etc.

In 1991, Foguel [3] introduced conjugate permutable subgroup, i.e., a subgroup H of a group G is said to be conjugate permutable if $HH^x = H^xH$ for any x of G. And he showed that conjugate permutable subgroups must be subnormal. Later, Shen et al. [4] introduced the dual concept of conjugate permutable subgroup, self-conjugate-permutable subgroup. A subgroup H of a group G is said to be self-conjugate-permutable if $HH^x = H^xH$ implies $H^x = H$ for any x of G. It is obvious that maximal subgroups and Hall subgroups of a group G are examples of self-conjugate-permutable subgroups. Shen et al. [4] called a group G a PSC-group if every cyclic subgroup of prime order or order 4 is self-conjugate-permutable in G, and investigated the structure of those groups which are not PSC-groups but all of whose proper subgroups are PSC-groups. Shen et al. [5] characterized PSC-groups and proved that a group G is a solvable T-group (A group G is called a T-group if normality is transitive in G) if and only if all prime power order subgroups of G are self-conjugate-permutable in G.

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The aim of this paper is to investigate the structure of groups in which some subgroups are self-conjugate-permutable. We call a group G an SC-group if all cyclic subgroups of G of order 2 or order 4 are self-conjugate-permutable in G. In Section 3, we investigate the structure of groups all of whose maximal subgroups are SC-groups and determine the non-solvable groups all of whose second maximal subgroups are SC-groups. In Section 4, we mainly investigate the structure of finite groups in which all of maximal subgroups of even order are PSC-groups, and describe the structure of finite groups which are not PSC-groups but all of whose maximal subgroups of even order are PSC-groups.

2. Preliminary results

We collect some lemmas which will be frequently used in the sequel.

Lemma 2.1 ([4, Lemma 2.1]) *Let G be a group. Suppose that H is self-conjugate-permutable in G, $K \le G$ and $N \trianglelefteq G$. Then*

(i) If $H \le K$, then H is self-conjugate-permutable in K;

(ii) Let $N \le K$. Then K/N is self-conjugate-permutable in G/N if and only if K is self-conjugate-permutable in G.

Lemma 2.2 ([4, Lemma 2.2]) *A subgroup H of G is normal if and only if H is subnormal as well as self-conjugate-permutable in G.*

Lemma 2.3 ([4, Lemma 2.3]) *Let G be a group. Suppose that $G = AB$, $A \le G$, $B \le G$. If H is self-conjugate-permutable in B and H is normalized by A, then H is self-conjugate-permutable in G.*

Lemma 2.4 ([4, Corollary 3.2]) *If G is a PSC-group, then G is supersolvable.*

Lemma 2.5 ([5, Theorem 4.2]) *If G is a minimal non-PSC-group, then G is solvable and $|\pi(G)| \le 2$.*

Lemma 2.6 ([5, Lemma 2.10]) *Let G be a PSC-group. If X is a subgroup of G of order q, where q is the largest prime divisor of $|G|$, then X is normal in G.*

Lemma 2.7 *Let G be a non-nilpotent dihedral group of order $2n$ or $4n$, where n is odd. Then G is an SC-group.*

Proof Using similar arguments as the proof in [4, Lemma 2.8], the Lemma is true. □

Lemma 2.8 ([6, Theorem 10.1.4]) *If a group G has a fixed-point-free automorphism of order 2, then G is abelian.*

Lemma 2.9 ([7, Lemma 2.10]) *Suppose that all cyclic subgroups of a group G of order p are normal in G for a fixed prime p. If $|Z(G)|_p \neq 1$, then all elements of order p of G are in $Z(G)$.*

Lemma 2.10 ([8, III, Satz 5.2]) *Let G be a minimal non-nilpotent group. Then*

(i) $G = P \rtimes Q$, *where* $P \in Syl_p(G)$, $Q \in Syl_q(G)$ *with* Q *cyclic;*

(ii) $C_P(Q) = P'$;

(iii) $P/\Phi(P)$ *is a minimal normal subgroup of* $G/\Phi(P)$;

(iv) If P is abelian, then P is elementary abelian. If P is non-abelian, then

$$Z(P) = P' = \Phi(P);$$

(v) If $p > 2$, then the exponent of P is p. If $p = 2$, then the exponent of P is 2 or 4.

Lemma 2.11 ([9, Theorem B]) *Let G be a non-solvable group. Suppose that every solvable subgroup of G is either 2-nilpotent or minimal non-nilpotent. Then G is one of the following groups:*

(i) $\mathrm{PSL}(2, 2^r)$, *where* $2^r - 1$ *is a prime;*

(ii) $\mathrm{PSL}(2, q)$, *where q is a power of a prime with* $q \equiv 3$ *or* $5 \pmod 8$;

(iii) $\mathrm{SL}(2, q)$, *where q is a power of a prime with* $q \equiv 3$ *or* $5 \pmod 8$.

3. Non-solvable groups all of whose second maximal subgroups are SC-groups

In this section, we classify these groups all of whose maximal subgroups are SC-groups and determine the non-solvable groups all of whose second maximal subgroups are SC-groups.

Theorem 3.1 *If a group G is an SC-group, then G is 2-nilpotent.*

Proof Let G be a counterexample of minimal order. It is clear that every maximal subgroup of G satisfies the hypothesis of theorem. Hence, G is a minimal non-2-nilpotent group. By a result in Itǒ [8, IV, Satz 5.4], G is minimal non-nilpotent. It follows from Lemma 2.10 that $G = P \rtimes Q$, where $P \in \mathrm{Syl}_2(G)$ with $\exp(P) \leq 4$.

Burnside Lemma implies that P is non-cyclic. Let x be an arbitrary non-trivial element of P. Then we have that $o(x) = 2$ or $o(x) = 4$. So $\langle x \rangle$ is self-conjugate-permutable in G by hypothesis. It follows from Lemma 2.2 that $\langle x \rangle$ is normal in G. Therefore, $\langle x \rangle Q < G$. By the minimality of G, $\langle x \rangle Q$ is 2-nilpotent, and so $\langle x \rangle Q = \langle x \rangle \times Q$. Thus, $G = P \times Q$ by the arbitrariness of x, a contradiction. □

Theorem 3.2 *If every maximal subgroup of a group G is an SC-group, then one of the following statements holds:*

(I) G is 2-nilpotent;

(II) $G = P \rtimes Q$ *is a minimal non-abelian group, where* $P \in Syl_2(G)$ *with P elementary abelian,* $Q \in Syl_q(G)$ *with Q cyclic;*

(III) $G = Q_8 \rtimes C_{3^n}$ *is a minimal non-nilpotent group, where* Q_8 *is the quaternion group of order 8,* C_{3^n} *is a cyclic 3-group.*

Proof Assume that G is non-2-nilpotent. By hypothesis and Theorem 3.1, every maximal subgroup of G is 2-nilpotent. Therefore, by a result in Itǒ [8, IV, Satz 5.4], G is a minimal non-nilpotent group. Lemma 2.10 implies that $G = P \rtimes Q$, where $P \in \mathrm{Syl}_2(G)$ with $\exp(P) \leq 4$, $Q \in \mathrm{Syl}_q(G)$ with Q cyclic. If P is abelian, then P is elementary abelian by Lemma 2.10, and so G is of type (II). Now we consider that P is non-abelian. Let x be an arbitrary non-trivial element of P. Then we have that $o(x) = 2$ or $o(x) = 4$. By hypothesis, $\langle x \rangle$ is self-conjugate-permutable in P. It follows that $\langle x \rangle$ is normal in P from Lemma 2.2. Hence P is a Hamiltonian group. By a result in [8, III, Satz 7.12], we have that $P = Q_8 \times A$, where Q_8 is the quaternion group of order 8, and A is an elementary abelian 2-group. By Lemma 2.10 again,

$$A \leq Z(P) = P' \leq Q_8,$$

which leads to $A = 1$, so $P = Q_8$. Now,

$$G/C_G(Q_8) = N_G(Q_8)/C_G(Q_8) \lesssim \mathrm{Aut}(Q_8).$$

Since $\mathrm{Aut}(Q_8) \cong S_4$, the symmetry group of degree 4, we have $q = 3$. Hence G is of type (III). $\square$

Theorem 3.3 *Let G be a non-solvable group. Then all second maximal subgroups of G are SC-groups if and only if G is isomorphic to one of the following types:*

(I) $\mathrm{PSL}(2, 2^r)$, *where $2^r - 1$ is a prime;*

(II) $\mathrm{PSL}(2, p)$, *where p is a prime with $p > 3$, $p \equiv 3$ or $5 \pmod 8$, $p^2 \not\equiv 1 \pmod 5$;*

(III) $\mathrm{PSL}(2, 3^f)$, *where f is an odd prime with $3^f \equiv 3$ or $5 \pmod 8$;*

(IV) $\mathrm{SL}(2, p)$, *where p is a prime with $p > 3$, $p \equiv 3$ or $5 \pmod 8$, $p^2 \not\equiv 1 \pmod 5$;*

(V) $\mathrm{SL}(2, 3^f)$, *where f is an odd prime with $3^f \equiv 3$ or $5 \pmod 8$.*

Proof Assume that G is a non-solvable group. Let M be any maximal subgroup of G. Then all maximal subgroups of M are SC-groups by hypothesis. It follows that M is 2-nilpotent or a minimal non-nilpotent group from Theorem 3.2. Therefore, M is solvable. By Lemma 2.11, G is one of the following groups:

(i) $\mathrm{PSL}(2, 2^r)$, where $2^r - 1$ is a prime;

(ii) $\mathrm{PSL}(2, q)$, where q is a power of a prime with $q \equiv 3$ or $5 \pmod 8$;

(iii) $\mathrm{SL}(2, q)$, where q is a power of a prime with $q \equiv 3$ or $5 \pmod 8$.

Case 1. $G \cong \mathrm{PSL}(2, 2^r)$, where $2^r - 1$ is a prime.

Suppose $G \cong \mathrm{PSL}(2, 2^r)$, where $2^r - 1$ is a prime. Then by [8, II, Satz 8.27], G has maximal subgroups:

(1) minimal non-abelian group N of order $2^r(2^r - 1)$;

(2) the dihedral groups of order $2(2^r \pm 1)$.

It is clear that every maximal subgroup of N is an SC-group. By Lemma 2.7, the dihedral groups of order $2(2^r \pm 1)$ are SC-groups. Hence G is of type (I).

Case 2. $G \cong \mathrm{PSL}(2, p)$, where p is a prime with $p > 3$, $p \equiv 3$ or $5 \pmod 8$, $p^2 \not\equiv 1 \pmod 5$ or $G \cong \mathrm{PSL}(2, 3^f)$, where f is an odd prime with $3^f \equiv 3$ or $5 \pmod 8$.

Suppose $G \cong \mathrm{PSL}(2, q)$, where q is a power of a prime with $q \equiv 3$ or $5 \pmod 8$. Let $q = p^n$ with p a prime. We first consider $p > 3$. If $n > 1$, then $\mathrm{PSL}(2, p^n)$ contains a non-solvable proper subgroup $\mathrm{PSL}(2,p)$, a contradiction. Hence $n = 1$. Note that $p^2 \not\equiv 1 \pmod 5$. Otherwise, by [8, II, Satz 8.27], $\mathrm{PSL}(2,p)$ contains a proper subgroup which is isomorphic to a non-solvable alternating group A_5 of degree 5, a contradiction. By [8, II, Satz 8.27] again, G has maximal subgroups:

(1) the alternating group A_4 of degree 4;

(2) the dihedral groups of order $p \pm 1$;

(3) Frobenius group F with a cyclic complement H of order $(p-1)/2$ and kernel K of order p.

Clearly, all maximal subgroups of A_4 are SC-groups. Since $p = q \equiv 3$ or $5 \pmod 8$, we have that $p - 1 = 2s$ or $4s$, and $p + 1 = 2t$ or $4t$, where s and t are odd integers. Then by Lemma 2.7, the dihedral groups of order $p \pm 1$ are SC-groups. The order of Sylow 2-subgroups of F is at most 2, so all maximal subgroups of F are SC-groups. Therefore, G is of type (II). We next consider $p = 3$. If n is even, then

$$\mathrm{PSL}(2,9) \leq \mathrm{PSL}(2,p^n),$$

and so $\mathrm{PSL}(2,p^n)$ contains a non-solvable proper subgroup A_5, the alternating group of degree 5, a contradiction. Thus n is odd. If n is an odd composite, then let $n = uv$, where u is a prime with $u < n$. By [8, II, Satz 8.27], we get that $\mathrm{PSL}(2,3^n)$ contains a non-solvable proper subgroup $\mathrm{PSL}(2,3^u)$, a contradiction. Therefore, n is an odd prime. Using similar arguments as mentioned earlier, G has only three kinds of maximal subgroups, and all of them satisfy the condition. Hence G is of type (III).

Case 3. $G \cong \mathrm{SL}(2,p)$, where p is a prime with $p > 3$, $p \equiv 3$ or $5 \pmod 8$, $p^2 \not\equiv 1 \pmod 5$ or $G \cong \mathrm{SL}(2,3^f)$, where f is an odd prime with $3^f \equiv 3$ or $5 \pmod 8$.

Suppose $G \cong \mathrm{SL}(2,q)$, where q is a power of a prime with $q \equiv 3$ or $5 \pmod 8$. Note that $\mathrm{SL}(2,q)$ possesses a unique element of order 2, and the Sylow 2-subgroups of $\mathrm{SL}(2,q)$ are isomorphic to Q_8, where Q_8 is the quaternion group of order 8. Let x be the unique element of $\mathrm{SL}(2,q)$ of order 2. Then $\langle x\rangle \trianglelefteq \mathrm{SL}(2,q)$, and so $\langle x\rangle$ is self-conjugate-permutable in $\mathrm{SL}(2,q)$. Furthermore,

$$\mathrm{SL}(2,q)/\langle x\rangle \cong \mathrm{PSL}(2,q).$$

Let $q = p^f$ with p a prime. By similar arguments as the proof in Case 2, we conclude that $f = 1$ and $p^2 \not\equiv 1 \pmod 5$ for $p > 3$, f is an odd prime for $p = 3$, and all second maximal subgroups of $\mathrm{SL}(2,q)/\langle x\rangle$ are SC-groups. Let C be any cyclic subgroup of $\mathrm{SL}(2,q)$ of order 4. It is clear that $\langle x\rangle < C$. Let M_1 be any second maximal subgroup of $\mathrm{SL}(2,q)$ and contains C. Then we get that $C/\langle x\rangle$ is self-conjugate-permutable in $M_1/\langle x\rangle$. By Lemma 2.1, C is self-conjugate-permutable in M_1. Therefore, all second maximal subgroups of $\mathrm{SL}(2,q)$ are SC-groups. Now we have that G is of type (IV) when $p > 3$, G is of type (V) when $p = 3$.

Conversely, it is easy to examine that a group of one of types (I)–(V) is the group whose all second maximal subgroups are SC-groups. $\square$

4. Groups all of whose maximal subgroups of even order are PSC-groups

In this section, we study groups with the property that all of whose maximal subgroups of even order are PSC-groups.

Theorem 4.1 *Let G be a group of even order. Suppose that all maximal subgroups of G of even order are PSC-groups. Then G is solvable.*

Proof If G is 2-nilpotent, then G has the normal 2-complement M. By Feit-Thompson theorem [10] on the solvability of group of odd order, M is solvable, and so G is solvable. Now assume that G is not 2-nilpotent and let M be any maximal subgroup of G. If M is of odd order, then M is 2-nilpotent. If M is of even order, then M is a PSC-group by hypothesis. By Lemma 2.4, M is 2-nilpotent. So G is minimal non-2-nilpotent, which implies that G is solvable. □

Theorem 4.2 *Let G be a non-PSC-group of even order. If all maximal subgroups of G of even order are PSC-groups, then $|\pi(G)| \leq 3$.*

Proof By Theorem 4.1, G is solvable. Let $\pi(G) = \{p_1, p_2, \ldots, p_s\}$ with $2 = p_1 < p_2 < \cdots < p_s$ and $\{P_1, P_2, \ldots, P_s\}$ be a Sylow basis of G. If G is a minimal non-PSC-group, then $|\pi(G)| \leq 2$ by Lemma 2.5. So the conclusion holds.

Now we assume that G is not a minimal non-PSC-group. By hypothesis, G possesses a maximal subgroup M of odd order which is not a PSC-group. Without loss of generality, let $M = P_2 \cdots P_s$.

Since M is not a PSC-group, there exists a minimal subgroup X of M such that X is not self-conjugate-permutable in M. Suppose that $s \geq 4$. Then for each $i \in \{2, 3, \ldots, s-1\}$, $P_1P_iP_s$ is a proper subgroup of G of even order, thus $P_1P_iP_s$ is a PSC-group by hypothesis. If $X \leq P_s$, then X is normal in $P_1P_iP_s$ by Lemma 2.6, and hence X is normal in M, a contradiction. Therefore, we may assume that $X \leq P_t$ for some fixed $t \in \{2, 3, \ldots, s-1\}$. Suppose that $t > 2$. Set

$$H = \prod_{r=2}^{t} P_r, \quad K = \prod_{k=t}^{s} P_k.$$

We have $M = HK$ with $X \leq K < M$. Since both P_1H and P_1K are proper subgroups of G of even order, they are PSC-groups by hypothesis. Therefore, X is self-conjugate-permutable in K and H normalizes X by Lemmas 2.1 and 2.6. Consequently, X is self-conjugate-permutable in M by Lemma 2.3, a contradiction. Hence $t = 2$.

By hypothesis, $P_1P_2P_j$ is a PSC-group for each $j \in \{3, \ldots, s\}$. Therefore, $P_1P_2P_j$ is supersolvable by Lemma 2.4, which implies that X normalizes P_j for each $j \in \{3, \ldots, s\}$. Hence, $B = X\prod_{j=3}^{s} P_j$ is a proper subgroup of G. If $X = P_2$, then X is a Sylow subgroup of G, and so X is self-conjugate-permutable in G, a contradiction. If $X < P_2$, then $B < M$ and $M = P_2B$. Since X is a Sylow subgroup of B, X is self-conjugate-permutable in B. By hypothesis, P_1P_2 is a PSC-group. It follows from Lemma 2.6 that P_2 normalizes X. Therefore,

$$M = P_2B = N_M(X)B,$$

and we have that X is self-conjugate-permutable in M by Lemma 2.3, a contradiction. Thus $|\pi(G)| \leq 3$. $\square$

Theorem 4.3 *Let G be a non-PSC-group of even order. If all maximal subgroups of G of even order are PSC-groups, then one of the following statements holds:*

(I) G is a minimal non-PSC-group;

(II) $|G|_2 = 2$, i.e., the Sylow 2-subgroups of G are of order 2.

Proof Suppose that G is not a minimal non-PSC-group. Then there exists a maximal subgroup M of G of order odd such that M is not a PSC-group by hypothesis. By Theorem 4.1, G is solvable. Then we can let M be a Hall $2'$-subgroup of G and $G = MR$, where $R \in \mathrm{Syl}_2(G)$. We prove that R is of order 2 and M is the normal 2-complement from two cases as follows.

Case 1. $O_2(G) \neq 1$.

If $O_2(G) < R$, then $M < MO_2(G) < MR = G$, which contradicts that M is a maximal subgroup of G. So $O_2(G) = R$ is the normal Sylow 2-subgroup of G. Note that M is non-abelian. For each cyclic subgroup C of M, CR is a proper subgroup of G of even order, and hence it is a PSC-group by hypothesis. If $|R| > 2$, then we can choose a subgroup R_0 of R of order 2. Since R_0 is self-conjugate-permutable in CR, R_0 is normal in CR by Lemma 2.2, which implies that M normalizes R_0. Therefore, $M < MR_0 < MR = G$, a contradiction. So R is of order 2.

Case 2. $O_2(G) = 1$.

Clearly, $O_{2'}(G) \neq 1$, and so $O_2(\overline{G}) \neq 1$, where $\overline{G} = G/O_{2'}(G)$. As $M < MO_{2',2}(G)$, it follows that $O_2(\overline{G}) = O_{2',2}(G) = \overline{R}$ from the maximality of M. Thus,

$$\overline{G} = N_{\overline{G}}(\overline{R}) = N_G(R)O_{2'}(G)/O_{2'}(G).$$

It follows that $G = N_G(R)O_{2'}(G)$. If $N_G(R) = G$, then $|R| = 2$ by similar arguments as in Case 1. Now we consider that $N_G(R) < G$. Let H be a Hall $2'$-subgroup of $N_G(R)$. Then $G = R(HO_{2'}(G))$ and $M = HO_{2'}(G)$. By hypothesis, R_1 is self-conjugate-permutable in $N_G(R)$ for every subgroup R_1 of R of order 2. Therefore, R_1 is normal in $N_G(R)$ by Lemma 2.2, which implies that R_1H is a subgroup of G. Thus

$$(R_1H)O_{2'}(G) = R_1(HO_{2'}(G)) = R_1M > M.$$

It follows that $G = R_1M$, which leads to $R = R_1$. Hence $|R| = |R_1| = 2$. $\square$

Corollary 4.4 *Let G be a non-PSC-group of even order and suppose that all maximal subgroups of G of even order are PSC-groups. If $4 \mid |G|$, then G is a minimal non-PSC-group.*

Proof It is obvious by Theorem 4.3. $\square$

Theorem 4.5 *Let G be a non-PSC-group and $\pi(G) = \{2, p\}$, where $P \in \mathrm{Syl}_p(G)$, $R \in \mathrm{Syl}_2(G)$. Suppose that all maximal subgroups of G of even order are PSC-groups. Then one of the following statements holds:*

(I) G is a minimal non-PSC-group with $|R| > 2$;

(II) $G = P \times R$, *where* P *is a minimal non-*PSC*-group and* $|R| = 2$;

(III) $G = P \rtimes R$ *is a minimal non-*PSC*-group, where* P *is elementary abelian and* $|R| = 2$.

Proof If $|R| > 2$, then by hypothesis and Corollary 4.4, G is a minimal non-PSC-group, and so G is of type (I).

Now we consider the case $|R| = 2$.

If G is nilpotent, then P must be a non-PSC-group since G is a non-PSC-group. Furthermore, by hypothesis, M_1R is a PSC-group for each maximal subgroup M_1 of P, and so M_1 is a PSC-group. Thus, P is a minimal non-PSC-group, and G is of type (II).

If G is non-nilpotent, then G is supersolvable by a result in [11, I, Corollary 1.10]. By Maschke's theorem, we have that

$$P/\Phi(P) = V_1/\Phi(P) \times V_2/\Phi(P) \times \cdots \times V_d/\Phi(P),$$

each $V_i/\Phi(P)$ is R-invariant and of order p for all $i \in \{1, 2, \ldots, d\}$. Set $P_i = \prod_{j \neq i} V_j$. Then P_i is a maximal subgroup of P and R-invariant. By hypothesis, P_iR is a PSC-group. Lemma 2.2 implies that each subgroup of P_i of order p is normal in P_iR. Suppose that $C_{P_k}(R) \neq 1$ for some $k \in \{1, 2, \ldots, d\}$. By Lemma 2.9, all elements of P_k of order p are in $Z(P_kR)$. By a result in Itô [8, IV, Satz 5.5], R is normal in P_kR. Hence, $P_kR = P_k \times R$ as P_k is R-invariant. In addition, $[R, \Phi(P)] \leq [R, P_k] = 1$. Since $\Phi(P) \leq P_j$ $(j = 1, 2, \ldots, d)$, we have that $\Phi(P) \leq C_{P_j}(R)$ for all $j \in \{1, 2, \ldots, d\}$. Therefore, $P_jR = P_j \times R$ for all $j \in \{1, 2, \ldots, d\}$, which implies that $G = PR$ is nilpotent, a contradiction. Thus, $C_{P_i}(R) = 1$ for all $i \in \{1, 2, \ldots, d\}$.

We first prove that $C_P(R) = 1$. If $C_P(R) \neq 1$, then $C_P(R)$ is of order p since $C_P(R) \cap P_i = 1$. Let $V = C_P(R)\Phi(P)$. Then V is R-invariant and $V < P$, and therefore VR is a PSC-group by hypothesis. Using similar arguments as given earlier, we conclude that $VR = V \times R$. Hence $[R, \Phi(P)] = 1$, which contradicts that $C_{P_i}(R) = 1$ for all $i \in \{1, 2, \ldots, d\}$. This contradiction induces that $C_P(R) = 1$, and hence P is abelian by Lemma 2.8.

We next prove that P is elementary abelian. If $\Omega_1(P) < P$, then $\Omega_1(P)R$ is a PSC-group by hypothesis. Lemma 2.2 implies that each subgroup A of P of order p is normal in $\Omega_1(P)R$, and so A is normal in G. Thus, G is a PSC-group as $|R| = 2$, a contradiction. Therefore, $\Omega_1(P) = P$ and P is elementary abelian.

We now get that the maximal subgroup of G of odd order is elementary abelian, and so it is a PSC-group. Thus, G is a minimal non-PSC-group by hypothesis. So G is of type (III). □

Corollary 4.6 *Let* G *be a non-*PSC*-group with* $\pi(G) = \{2, p\}$ *and* G *be non-nilpotent. Then all maximal subgroups of* G *of even order are* PSC*-groups if and only if* G *is a minimal non-*PSC*-group.*

Proof It is clear by Theorem 4.5. □

Theorem 4.7 *Let* G *be a non-*PSC*-group of even order and* $|\pi(G)| = 3$, *where* $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ *and* $R \in \mathrm{Syl}_2(G)$. *Suppose that all maximal subgroups of* G *of even order are* PSC*-groups. Then one of the following statements holds:*

(I) $G = M \times R$, where M is a minimal non-PSC-group and $|R| = 2$;

(II) $G = M \rtimes R = Q \rtimes (P \times R)$, where M is a minimal non-PSC-group with P cyclic, Q elementary abelian, $C_Q(R) = 1$ and $|R| = 2$.

Proof Since $|\pi(G)| = 3$, G is not a minimal non-PSC-group by Lemma 2.5. Therefore, there exists a maximal subgroup M of G of order odd such that M is not a PSC-group by hypothesis. By Theorem 3.1, G is solvable, and hence we can let $\{P, Q, R\}$ be a Sylow basis of G, where $\pi(G) = \{p, q, r\}$ with $r = 2$. Furthermore, we can let M be a Hall $2'$-subgroup of G and $G = MR$, where $M = PQ$. By Theorem 4.3, we have that $|R| = 2$ and M is the normal 2-complement.

Suppose $C_M(R) = 1$. Then an automorphism of R acting on M is both of order 2 and fixed-point-free. Lemma 2.8 implies that M is abelian, and so it is a PSC-group, a contradiction. This contradiction leads to $C_M(R) > 1$.

By hypothesis, PR is a PSC-group, and so each subgroup of P of order p is normal in PR by Lemma 2.6. If $p \mid |C_M(R)|$, then by Lemma 2.9, all elements of P of order p are in $Z(PR)$. It follows that $PR = P \times R$ from Lemma 2.4 and a result in Itô [8, IV, Satz 5.5]. If $q \mid |C_M(R)|$, then $QR = Q \times R$ by the similar arguments as above. Therefore, $MR = M \times R$ when $pq \mid |C_M(R)|$. Then M_1R is a proper subgroup of G of even order for each proper subgroup M_1 of M, and hence M_1R is a PSC-group by hypothesis. So M_1 is a PSC-group, and M is a minimal non-PSC-group. Therefore, G is of type (I).

Without loss of generality, let $C_P(R) = P$ and $C_Q(R) = 1$. Since $|R| = 2$, we have $N_G(R) = C_G(R) = PR$. Set $R^G = \langle R^g | g \in G\rangle$. Then $R^G \leq QR$. By Frattini argument, we get that $G = N_G(R)R^G = PRR^G$, and hence $Q \leq R^G$. Now we conclude that $R^G = QR$, which implies that Q is normal in G.

We first prove that P is cyclic. If not, then both P_1QR and P_2QR are PSC-groups by hypothesis for two different maximal subgroups P_1, P_2 of P. By Lemma 2.2, we get that each subgroup A of Q of order q is normal in P_1QR and P_2QR, and hence A is normal in G. Since PR is a PSC-group by hypothesis, Lemma 2.6 implies that T is normal in P for each subgroup T of P of order p. On the other hand, TQR is a PSC-group by hypothesis. Then by Lemma 2.3, T is self-conjugate-permutable in G since $G = P(TQR)$. Thus, G is a PSC-group, a contradiction.

We next prove that Q is elementary abelian. It follows that Q is abelian from Lemma 2.8. If $\Omega_1(Q) < Q$, then $\Omega_1(Q)PR$ is a PSC-group by hypothesis. Lemma 2.2 implies that each subgroup N of Q of order q is normal in $\Omega_1(Q)PR$, and hence N is normal in G. As M is a non-PSC-group, it is clear that $|P| > p$ and there exists a subgroup T_0 of P of order p such that T_0 is not self-conjugate-permutable in M. Consider the subgroup T_0QR. By hypothesis, T_0 is self-conjugate-permutable in T_0QR. Since $G = P(T_0QR)$, T_0 is self-conjugate-permutable in G by Lemma 2.3, a contradiction. Therefore, $\Omega_1(Q) = Q$ and Q is elementary abelian.

We now prove that M is a minimal non-PSC-group. For any maximal subgroup M_1 of M, we have that $p \mid |M : M_1|$ or $q \mid |M : M_1|$. If $p \mid |M : M_1|$, then $Q \leq M_1$, and hence $M_1 \trianglelefteq G$. Now we consider that $q \mid |M : M_1|$. Since QR is a PSC-group, R normalizes each subgroup of Q of order q by Lemma 2.2. Therefore, R normalizes M_1 as Q is elementary abelian. So M_1R is a proper subgroup of G of even order in both cases. By hypothesis, M_1R is a PSC-group and hence M_1 is a PSC-group. Therefore, M is a minimal non-PSC-group and G is of type (II). □

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