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A Note on Pm-Factorizable Topological Groups

Chunjie MA, Liangxue PENG*
Department of Mathematics, School of Mathematics, Statistics and Mechanics, Beijing University of Technology, Beijing 100124*, P. R. China*

Abstract In this paper, we define a new class of Pm-factorizable topological groups. A topological group G is called Pm-factorizable if, for every continuous function $f : G \to M$ to a metrizable space M, one can find a perfect homomorphism $\pi : G \to K$ onto a second-countable topological group K and a continuous function $g : K \to M$ such that $f = g \circ \pi$. We show that a topological group G is Pm-factorizable if and only if it is $P\mathbb{R}$-factorizable. And we get that if G is a Pm-factorizable topological group and K is any compact topological group, then the group $G \times K$ is Pm-factorizable.

Keywords $\mathbb{R}$-factorizable; $P\mathbb{R}$-factorizable; m-factorizable; Pm-factorizable; $\mathcal{M}$-factorizable

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1. Introduction

Recall that a paratopological group is a group with a topology such that multiplication on the group is jointly continuous. A topological group G is a paratopological group with a topology such that the inverse mapping of G into itself associating x^{-1} with $x \in G$ is continuous. All topological groups in this paper are assumed to be Hausdorff. A semitopological group is a group with a topology in which the left and right translations are continuous [1]. Let $\mathbb{R}$ be the set of real numbers with the usual topology. Recall that a topological group G is $\mathbb{R}$-factorizable [2,3] if, for every continuous real-valued function $f : G \to \mathbb{R}$, one can find a continuous homomorphism $p : G \to H$ onto a second-countable topological group H and a continuous function $g : H \to \mathbb{R}$ such that $f = g \circ p$. Thus every compact topological group is $\mathbb{R}$-factorizable [4, Example 37].

Similarly to the case of $\mathbb{R}$-factorizable topological groups, Sanchis and Tkachenko introduced the notions of $\mathbb{R}_i$-factorizable paratopological groups, for $i \in \{1, 2, 3, 3.5\}$ (see [5, Definition 3.1]). It is proved that all concepts of $\mathbb{R}_i$-factorizability in paratopological groups coincide [6, Theorem 3.8]. Therefore, a paratopological group G is $\mathbb{R}$-factorizable if for every continuous function $f : G \to \mathbb{R}$, one can find a continuous homomorphism $p : G \to H$ onto a separable metrizable paratopological group H and a continuous function $g : H \to \mathbb{R}$ such that $f = g \circ p$. It is proved that every paratopological group with a countable network is $\mathbb{R}$-factorizable [6, Corollary 3.16].

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* Corresponding author
E-mail address: 18848976938@163.com (Chunjie MA); pengliangxue@bjut.edu.cn (Liangxue PENG)

Recall that a semitopological group G is ω-narrow if, for every open neighborhood V of the identity e in G, there exists a countable subset A of G such that $AV = VA = G$ (see [1, p. 117]).

In [7], Peng and Liu introduced the notion of $P\mathbb{R}$-factorizable topological groups. A topological group G is called $P\mathbb{R}$-factorizable if, for every continuous function $f : G \to \mathbb{R}$, one can find a perfect homomorphism $p : G \to H$ onto a second-countable topological group H and a continuous function $g : H \to \mathbb{R}$ such that $f = g \circ p$ (see [7]). Recall that a topological group G is feathered if it contains a non-empty compact set K of countable character in G (see [1, p. 235]). It is proved that a topological group G is $P\mathbb{R}$-factorizable if and only if G is Lindelöf feathered [7, Theorem 2.5]. They also got some other equivalent conditions for $P\mathbb{R}$-factorizable topological groups.

A topological group G is called $\mathcal{M}$-factorizable if for every continuous real-valued function $f : G \to \mathbb{R}$, there exist a continuous homomorphism φ of G onto a metrizable group H and a continuous real-valued function g on H such that $f = g \circ \varphi$ (see [8]). It is proved that a topological group G is $\mathbb{R}$-factorizable if and only if it is $\mathcal{M}$-factorizable and ω-narrow [8, Theorem 3.2].

Similarly to $P\mathbb{R}$-factorizable topological groups, Xie and Yan defined a new class of $P\mathcal{M}$-factorizable topological groups [9]. A topological group G is $P\mathcal{M}$-factorizable if, for every continuous function $f : G \to \mathbb{R}$, one can find a perfect homomorphism $p : G \to L$ onto a metrizable topological group L and a continuous real-valued function h on L such that $f = h \circ p$ (see [9, Definition 1.3]). The relations between $P\mathbb{R}$-factorizable, $P\mathcal{M}$-factorizable and $\mathcal{M}$-factorizable topological groups were studied in [9]. It is proved that a topological group G is $P\mathbb{R}$-factorizable if and only if G is $P\mathcal{M}$-factorizable and ω-narrow [9, Theorem 2.1].

A topological group G is m-factorizable if for every continuous mapping $f : G \to M$ to a metrizable space M, there exists a continuous homomorphism $\pi : G \to K$ onto a second-countable topological group K and a continuous function $g : K \to M$ such that $f = g \circ \pi$ [1, p. 539]. Recall that a space X is said to be pseudo-$\aleph_1$-compact if every discrete family of open sets in X is countable. A topological group G is m-factorizable if and only if G is $\mathbb{R}$-factorizable and pseudo-$\aleph_1$-compact [1, Theorem 8.5.2].

In this paper, we define a new class of Pm-factorizable topological groups. A topological group G is Pm-factorizable if, for every continuous function $f : G \to M$ to a metrizable space M, one can find a perfect homomorphism $\pi : G \to K$ onto a second-countable topological group K and a continuous function $g : K \to M$ such that $f = g \circ \pi$. We show that a topological group G is Pm-factorizable if and only if it is $P\mathbb{R}$-factorizable. And we get that if G is a Pm-factorizable topological group and K is any compact topological group, then the group $G \times K$ is Pm-factorizable.

The set of all positive integers is denoted by $\mathbb{N}$ and ω is $\mathbb{N} \cup \{0\}$. In notation and terminology we will follow [1] and [10].

2. Main results

In this section, we will study Pm-factorizable topological groups.

Lemma 2.1 ([7, Theorem 2.6]) *Let G be a topological group. Then the following conditions* *are equivalent:*

(1) G is $P\mathbb{R}$-factorizable;

(2) G is Lindelöf feathered;

(3) G is ω-narrow feathered;

(4) G is a Lindelöf p-space;

(5) There exists a compact invariant subgroup H of G such that the quotient group G/H is separable metrizable.

Thus we have the following result.

Theorem 2.2 *Let G be a topological group. Then G is Pm-factorizable if and only if G is $P\mathbb{R}$-factorizable.*

Proof Suppose that G is a Pm-factorizable topological group, then G is $P\mathbb{R}$-factorizable according to the definitions.

Now we prove the converse. Suppose that G is a $P\mathbb{R}$-factorizable topological group. Then by Lemma 2.1 G is Lindelöf feathered. Let $f : G \to M$ be any continuous mapping of G to a metrizable space M. Since G is Lindelöf, the continuous image $f(G) \subset M$ is Lindelöf. Since the space M is metrizable, $f(G)$ is second-countable. Therefore, we can identify $f(G)$ with a subspace of $\mathbb{R}^\omega$. For every $n < \omega$, denote by p_n the projection of $\mathbb{R}^\omega$ to the nth factor. Then $p_n \circ f : G \to \mathbb{R}$ is a continuous mapping. Since G is $P\mathbb{R}$-factorizable, there exists a perfect homomorphism $\pi_n : G \to K_n$ onto a second-countable topological group K_n and a continuous real-valued function g_n on K_n such that $p_n \circ f = g_n \circ \pi_n$. Denote by π the diagonal product of the family $\{\pi_n : n \in \omega\}$ of perfect homomorphisms. Since the diagonal of any family of perfect mappings is a perfect mapping [10, Theorem 3.7.10], it follows that the mapping $\pi : G \to \pi(G)$ is a perfect homomorphism. For every $n < \omega$, the topological group K_n is second-countable, then the topological group $\prod = \prod_{n<\omega} K_n$ is second-countable. Therefore, the image $K = \pi(G)$ is second-countable as a subgroup of the group $\prod = \prod_{n<\omega} K_n$.

For every $n < \omega$, let $q_n : \prod \to K_n$ be the projection of $\prod$ to the n-th factor. Then $\pi_n = q_n \circ \pi$ for every $n < \omega$. Finally, denote by g the mapping of K onto $f(G)$ satisfying $g(z) = \langle g_n(q_n(z)) : n \in \omega \rangle$ for every $z \in K$. Then the mapping $g : K \to f(G)$ is continuous and the equality $p_n \circ g \circ \pi = g_n \circ q_n \circ \pi = g_n \circ \pi_n = p_n \circ f$ holds for each $n < \omega$. Therefore, $\pi : G \to K$ is a perfect homomorphism and the continuous mapping g satisfies the equality $f = g \circ \pi$. Hence, G is a Pm-factorizable topological group. □

Let $f : X \to Y$ be a mapping. Then f is said to be d-open if for any open set O of X there exists an open set V of $f(X)$ such that $f(O)$ is dense subset of V. Clearly, every continuous open mapping is continuous d-open.

According to [7, Propositions 2.7 and 2.8, Theorem 2.9], the product of countably many $P\mathbb{R}$-factorizable topological groups is $P\mathbb{R}$-factorizable, every closed subgroup of a $P\mathbb{R}$-factorizable topological group is $P\mathbb{R}$-factorizable, and the open continuous homomorphism image of a $P\mathbb{R}$-factorizable topological group is $P\mathbb{R}$-factorizable. Then we have the following result.

Corollary 2.3 *The class of Pm-factorizable topological groups are preserved by countable products and closed subgroups.*

Corollary 2.4 *If $p : G \to H$ is a continuous d-open homomorphism of a Pm-factorizable topological group G onto a topological group H, then H is Pm-factorizable.*

Proof By Theorem 2.2, the topological group G is Pm-factorizable if and only if G is $P\mathbb{R}$-factorizable. And $P\mathbb{R}$-factorizable topological groups are preserved by continuous d-open homomorphisms [9, Theorem 2.6]. Then H is Pm-factorizable. $\square$

In [1, Theorem 8.5.5 (a)], it is proved that if G is an m-factorizable topological group and K is any compact topological group, then the group $G \times K$ is m-factorizable.

Proposition 2.5 *If G is a Pm-factorizable topological group and K is any compact topological group, then the group $G \times K$ is Pm-factorizable.*

Proof Since every compact topological group is $P\mathbb{R}$-factorizable [7, Corollary 2.3], every compact topological group is Pm-factorizable. By Corollary 2.3, the product of countably many Pm-factorizable topological groups is Pm-factorizable. Thus the group $G \times K$ is Pm-factorizable. $\square$

Lemma 2.6 ([9, Theorem 2.1]) *A topological group G is $P\mathbb{R}$-factorizable if and only if G is $P\mathcal{M}$-factorizable and ω-narrow.*

By Theorem 2.2 and Lemma 2.6, we have the following result.

Proposition 2.7 *A topological group G is Pm-factorizable if and only if G is $P\mathcal{M}$-factorizable and ω-narrow.*

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