Journal of Mathematical Research with Applications
Mar., 2024, Vol. 44, No. 2, pp. 213–224
DOI:10.3770/j.issn:2095-2651.2024.02.008
Http://jmre.dlut.edu.cn

Non-Existence of Entire Solution of a Type of System of Equations

Zhiwei ZHOU[1], **Ying ZHANG**[2], **Zhigang HUANG**[1,*]
1. *School of Mathematics, Suzhou University of Science and Technology, Jiangsu* 215009*, P. R. China;*
2. *Information Construction and Management Center, Suzhou University of Science and Technology, Jiangsu* 215009*, P. R. China*

Abstract In this paper, we will prove that the system of differential-difference equations

$$\begin{cases}(f(z)f'(z))^n + p_1^2(z)g^m(z+\eta) = Q_1(z),\\ (g(z)g'(z))^n + p_2^2(z)f^m(z+\eta) = Q_2(z),\end{cases}$$

has no transcendental entire solution $(f(z), g(z))$ with $\rho(f,g) < \infty$ such that $\lambda(f) < \rho(f)$ and $\lambda(g) < \rho(g)$, where $P_1(z), Q_1(z), P_2(z)$ and $Q_2(z)$ are non-vanishing polynomials.

Keywords transcendental entire function; finite order; system of differential-difference equations

MR(2020) Subject Classification 30D35; 39A45

1. Introduction and main results

We use the standard notations of the Nevanlinna theory, i.e., $m(r,f)$, $N(r,f)$ and $T(r,f)$ to denote the proximity function, the counting function and the characteristic function of a meromorphic function $f(z)$, respectively. Define the order and exponents of convergence of zero sequence of f by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \quad \lambda(f) = \limsup_{r\to\infty} \frac{\log^+ N(r,\frac{1}{f})}{\log r}.$$

Moreover, we say that a meromorphic function g is a small function with respect to f if $T(r,g) = S(r,f)$, where $S(r,f) = o(T(r,f))$ outside a possible exceptional set of finite linear measure.

Fermat's Last Theorem says that there do not exist nonzero rational numbers x and y and an integer $n \geq 3$, for which $x^n + y^n = 1$. Analogous to the Fermat's Last Theorem, there have been similar function theory investigations, that is, do there exist meromorphic solutions to Fermat type functional equation $f^n + g^n = 1$. In 1966, Gross [1] and Baker [2] proved that the equation does not admit any nonconstant meromorphic solutions in the complex plane $\mathbb{C}$ if $n > 3$ and

Received March 29, 2023; Accepted August 13, 2023
Supported by the National Natural Science Foundation of China (Grant No. 11971344).
* Corresponding author
E-mail address: 469850863@qq.com (Ying ZHANG); alexehuang@sina.com (Zhigang HUANG)

does not admit any entire solutions if $n > 2$. Since then, this question has aroused the interest of many mathematicians, such as [3–10] and so on.

Consider the equation

$$f^m(z) + g^n(z) = 1, \tag{1.1}$$

which can be regarded as the analogy of function theory to Fermat diophantine equation $x^n + y^m = 1$ over the complex plane $\mathbb{C}$, where $m, n \geq 2$ are positive integers. In genearl, Eq. (1.1) has no non-trivial entire solution provided $m + n < mn$ (see [11]). In 1970, Yang [12] further proved

Theorem 1.1 *Let m, n be positive integers satisfying $\frac{1}{n} + \frac{1}{m} < 1$. Then*

$$a(z)f(z)^n + b(z)g(z)^m = 1 \tag{1.2}$$

does not admit nonconstant entire solutions $f(z)$ and $g(z)$, where a, b are small function with respect to f.

Under the assumption $m > 2, n > 2$, Yang's result shows that Eq. (1.2) has no non-constant entire solutions. The remain cases, however, are still open. Recently, some researchers began to discuss equations in particular where $g(z)$ has some special relationship with $f(z)$ in Eq. (1.2). Tang and Liao [13] extended a study work of the open problem due to Yang and Li [14] through replacing g by $f^{(k)}$ to investigate entire solutions of the following equation $f(z)^2 + P(z)f^{(k)}(z)^2 = Q(z)$, where P, Q are non-zero polynomials. Liu et al. [15] in 2012 took into consideration a type of Fermat type differential-difference equation by changing $g(z)$ to $f(z + c)$,.

Theorem 1.2 ([15]) *The equation*

$$f'(z)^n + f(z + c)^m = 1, \tag{1.3}$$

has no transcendental entire solutions with finite order, provided that $m \neq n$, where n, m are positive integers.

Further, Chen and Lin [3] investigated the non-existence of finite order transcendental entire solutions of Fermat-type differential-difference equation

$$\big(f(z)f'(z)\big)^n + P^2(z)f^m(z + \eta) = Q(z), \tag{1.4}$$

where $P(z)$ and $Q(z)$ are non-zero polynomials, and proved the following result.

Theorem 1.3 *If $m = n$, then the Eq. (1.4) has no finite order transcendental entire solutions, where m and n are positive integers, and $\eta \in \mathbb{C} - \{0\}$.*

For more results related to differential or differential-difference of entire functions, we refer the reader to the review article [9]. We know that the existence of solutions of a differential equation is different from the existence of solutions of systems of differential equations. Thus, the question: What is possible for the system of functional equations?

Inspired by Theorem 1.2, Gao et al. [16] considered the nonexistence of entire solutions of a type of system of differential-difference equations of the form

$$\begin{cases} (\omega_1')^{n_1} + \omega_2(z + c)^{m_1} = Q_1(z), \\ (\omega_2')^{n_2} + \omega_1(z + c)^{m_2} = Q_2(z), \end{cases} \tag{1.5}$$

where $Q_i(z)$ $(i = 1, 2)$ are non-zero polynomials.

The order of growth of meromorphic solutions $(\omega_1(z), \omega_2(z))$ of system (1.5) is defined by

$$\rho = \rho(\omega_1, \omega_2) = \max\{\rho(\omega_1), \rho(\omega_2)\}.$$

Their result can be stated as follows.

Theorem 1.4 *System (1.5) has no meromorphic solutions $(\omega_1(z), \omega_2(z))$ with $\rho(\omega_1, \omega_2) < \infty$ if one of the following conditions is satisfied:*

(i) $m_1 m_2 > n_1 n_2$;

(ii) $m_i > \frac{n_i}{n_i - 1}$.

Regarding Theorems 1.3 and 1.4, the purpose of this paper is to investigate the existence of entire solutions of system of equations of the form

$$\begin{cases} (f(z)f'(z))^n + p_1^2(z)g^m(z+\eta) = Q_1(z), \\ (g(z)g'(z))^n + p_2^2(z)f^m(z+\eta) = Q_2(z), \end{cases} \tag{1.6}$$

where $P_1(z), Q_1(z), P_2(z), Q_2(z)$ are non-zero polynomials.

Theorem 1.5 *The system of equations*

$$\begin{cases} (f(z)f'(z))^n + p_1^2(z)g^m(z+\eta) = Q_1(z), \\ (g(z)g'(z))^n + p_2^2(z)f^m(z+\eta) = Q_2(z), \end{cases} \tag{1.7}$$

has no non-trivial transcendental entire solution $(f(z), g(z))$ with $\rho(f, g) < \infty$ such that $\lambda(f) < \sigma(f)$ and $\lambda(g) < \sigma(g)$, where $p_1(z), Q_1(z), p_2(z)$ and $Q_2(z)$ are non-vanishing polynomials.

2. Preliminary lemmas

In order to prove our result, we need the following lemmas.

Lemma 2.1 ([12]) *Let m, n be positive integers satisfying $\frac{1}{m} + \frac{1}{n} < 1$. Then there are no non-constant entire solutions $f(z)$ and $g(z)$ that satisfy*

$$a(z)f^n(z) + b(z)g^m(z) = 1.$$

Lemma 2.2 ([17]) *If meromorphic functions $f_j(z)$ $(j = 1, 2, \ldots n)$ $(n \geq 2)$ and entire functions $g_j(z)$ $(j = 1, 2, \ldots n)$ $(n \geq 2)$ satisfy the following conditions:*

(1) $\Sigma_{j=1}^n f_j e^{g_j} \equiv 0$;

(2) $g_i - g_j$ *are not constant for* $1 \leq j < i \leq n$;

(3) $T(r, f_j) = o(T(r, e^{g_h - g_l}))$ $(r \to \infty, r \notin E)$ *for* $1 \leq j \leq n$, $1 \leq h < l \leq n$, *where* $E \subset (1, \infty)$ *is of finite linear measure or logarithmic measure, then* $f_j \equiv 0$ $(j = 1, 2, \ldots, n)$.

Lemma 2.3 ([17]) *Let $f(z)$ be an entire function of finite order ρ with zeros $\{z_1, z_2, \ldots\} \subset \mathbb{C} - \{0\}$ and a k-fold zero at the origin. Then $f(z) = z^k P(z) e^{Q(z)}$, where $P(z)$ is the canonical product of $f(z)$ formed with the non-null zeros of $f(z)$, and $Q(z)$ is a polynomial of degree at most ρ.*

Lemma 2.4 ([17]) *Let $f(z), g(z)$ be nonconstant meromorphic functions in the complex plane.*

If $\rho(f) < \rho(g)$, then $\rho(fg) = \rho(g)$.

Lemma 2.5 *Let $\alpha(z), \beta(z)$ be non-constant polynomials with $\deg(\alpha(z)) = \deg(\beta(z)) = d, d \in \mathbb{N}$, $A_j(z)$ $(j = 1, 2, \ldots, n+2)$ be meromorphic functions, and $f_j(z) = n\beta(z+\eta) + 2n(n-j)\alpha(z)$, $j = 0, 1, 2, \ldots, n$, $f_{n+1} = \alpha(z+2\eta)$, $f_{n+2} = 0$, where $n \geq 2$, η is a nonzero constant, which satisfy the following conditions:*

(1) $\sum_{j=0}^{n+2} A_j e^{f_j} = 0$;

(2) $T(r, A_j) = o(T(r, e^{\alpha}))(r \to \infty, r \notin E)$ for $1 \leq j \leq n+2$.

Then $A_{n+1} \equiv 0$ or $A_{n+2} \equiv 0$.

Proof of Lemma 2.5 Suppose that $\alpha(z) = az^d + \cdots$, $\beta(z) = bz^d + \cdots$, $a \neq 0$, $b \neq 0$.

Clearly, we see that $\deg(f_i - f_j) = d$ for $i \neq j$, where $i, j \in \{0, 1, \ldots, n\}$. Further, if $\deg(f_i - f_j) = d$ for $i \neq j$, $i, j \in \{0, 1, \ldots, n+2\}$, then it follows from Lemma 2.2 that $A_{n+1} = A_{n+2} \equiv 0$. In the following, we consider two cases:

Case 1. If there exists some $j_1 \in \{0, 1, 2, \ldots, n\}$ such that $\deg(f_{j_1} - f_{n+2}) < d$, then we have $nb + 2n(n - j_1)a = 0$, and hence $b = -2(n - j_1)a$. We claim that there does not exist $j_2 \in \{0, 1, 2, \ldots, n\}, j_2 \neq j_1$ such that $\deg(f_{j_2} - f_{n+1}) < d$. Otherwise, we have $nb + 2n(n - j_2)a = a$ such that $a = 2n(j_2 - j_1)a$, which is a contradiction. Thus, we have $\sum_{j=0}^{j_1-1} A_j e^{f_j} + \sum_{j_1+1}^{n+1} A_j e^{f_j} + (A_{j_1} e^{f_{j_1}} + A_{n+2}) = 0$. From Lemma 2.2, we have $A_{n+1} \equiv 0$.

Case 2. If there exists some $j_1 \in \{0, 1, 2, \ldots, n\}$ such that $\deg(f_{j_1} - f_{n+1}) < d$, then $nb + 2n(n - j_1)a = a$. Thus, we have $nb = (1 - 2n(n - j_1))a$. We claim that there does not exist f_{j_2} such that $\deg(f_{j_2} - f_{n+2}) < d$. Otherwise, we have $(1 - 2n(n - j_1))a = -2n(n - j_2)a$, which is a contradiction. Therefore, it follows that $\sum_{j=0}^{j_1-1} A_j e^{f_j} + \sum_{j_1+1}^{n} A_j e^{f_j} + (A_{j_1} + A_{n+1} e^{f_{n+1} - f_{j_1}}) e^{f_{j_1}} + A_{n+2} = 0$. Then by Lemma 2.2, we have $A_{n+2} \equiv 0$. $\square$

Lemma 2.6 *Let $\alpha(z), \beta(z)$ be non-constant polynomials with $\deg(\alpha(z)) = \deg(\beta(z)) = d, d \in \mathbb{N}$, $A_j(z)$ $(j = 1, 2, \ldots, m+2)$ be meromorphic functions and $f_j(z) = mj\alpha(z)$, $j = 0, 1, 2, \ldots, m$, $f_{m+1} = m\beta(z)$, $f_{m+2} = m\beta(z) + 2\alpha(z+\eta)$ where $m \geq 3$, η is a nonzero constant which satisfy the following conditions:*

(1) $\sum_{j=0}^{m+2} A_j e^{f_j} = 0$;

(2) $T(r, A_j) = o(T(r, e^{\alpha}))$ $(r \to \infty, r \notin E)$ for $1 \leq j \leq m+2$.

Then $A_{m+1} \equiv 0$ or $A_{m+2} \equiv 0$.

Proof of Lemma 2.6 Suppose that $\alpha(z) = az^d + \cdots$, $\beta(z) = bz^d + \cdots$, $a \neq 0$, $b \neq 0$.

Clearly, we see that $\deg(f_i - f_j) = d$ for $i \neq j$, where $i, j \in \{0, 1, \ldots, m\}$. Further, if $\deg(f_i - f_j) = d$ for $i \neq j$, $i, j \in \{0, 1, \ldots, m+2\}$, then it follows from Lemma 2.2 that $A_{m+1} = A_{m+2} \equiv 0$. In the following, we consider two cases:

Case 1. If there exists some $j_1 \in \{0, 1, 2, \ldots, m\}$ such that $\deg(f_{j_1} - f_{m+2}) < d$, then we have $mj_1a = mb + 2a$. We claim that there does not exist $j_2 \in \{0, 1, 2, \ldots, m\}$, $j_2 \neq j_1$ such that $\deg(f_{j_2} - f_{m+1}) < d$. Otherwise, we have $mj_2a = mb$ such that $m(j_1 - j_2) = 2$, which is a contradiction. Thus, we have $\sum_{j=0}^{j_1-1} A_j e^{f_j} + \sum_{j_1+1}^{m+1} A_j e^{f_j} + (A_{j_1} e^{f_{j_1} - f_{m+2}} + A_{m+2}) e^{f_{m+2}} = 0$. From Lemma 2.2, we have $A_{m+1} \equiv 0$.

Case 2. If there exists some $j_1 \in \{0, 1, 2, \ldots, m\}$ such that $\deg(f_{j_1} - f_{m+1}) < d$, then we have $mj_1 a = mb$. We claim that there does not exist f_{j_2} such that $\deg(f_{j_2} - f_{m+2}) < d$. Otherwise, we have $m(j_2 - j_1) = 2$, which is a contradiction.

Therefore, it follows that

$$\sum_{j=0}^{j_1-1} A_j e^{f_j} + \sum_{j_1+1}^{m} A_j e^{f_j} + (A_{m+1} + A_{j_1} e^{f_{j_1} - f_{m+1}}) e^{f_{m+1}} + A_{m+2} e^{f_{m+2}} = 0.$$

Then by Lemma 2.2, we have $A_{m+2} \equiv 0$. □

3. Proof of Theorem 1.5

Suppose on the contrary that system (1.7) has a transcendental entire solution $(f(z), g(z))$ with $\lambda(f) < \sigma(f)$ and $\lambda(g) < \sigma(g)$, we will deduce a contradiction. From Lemma 2.3, we can set

$$f(z) = \omega_1(z) e^{\alpha_1(z)}, g(z) = \omega_2(z) e^{\alpha_2(z)}, \tag{3.1}$$

where ω_1, ω_2 are the canonical product of f and g, respectively. Therefore, by Lemma 2.1, we only need to consider the following four cases.

Case 1. $n = m = 1$.

Then we can rewrite system of Eq. (1.7) into

$$\begin{cases} f(z)f'(z) + p_1^2(z)g(z+\eta) = Q_1(z) \\ g(z)g'(z) + p_2^2(z)f(z+\eta) = Q_2(z). \end{cases} \tag{3.2}$$

By calculation, we can get a new system of equations

$$\begin{cases} p_1^2(z)g(z+\eta)g'(z+\eta) = g'(z+\eta)\big(Q_1(z) - f(z)f'(z)\big) \\ p_1^2(z)g(z+\eta)g'(z+\eta) = p_1^2(z)\big(Q_2(z+\eta) - p_2^2(z+\eta)f(z+2\eta)\big). \end{cases} \tag{3.3}$$

Thus, Eq. (3.3) leads to

$$g'(z+\eta)\big(f'(z)f(z) - Q_1(z)\big) = p_1^2(z)p_2^2(z+\eta)f(z+2\eta) - p_1^2(z)Q_2(z+\eta). \tag{3.4}$$

Substituting (3.1) into (3.4), we get

$$\begin{aligned} &(\omega_2'(z+\eta) + \omega_2(z+\eta)\alpha_2'(z+\eta))e^{\alpha_2(z+\eta)}(\omega_1(z)e^{\alpha_1(z)}(\omega_1'(z) + \omega_1(z)\alpha_1'(z))e^{\alpha_1(z)} - Q_1(z)) \\ &\quad = p_1^2(z)p_2^2(z+\eta)\omega_1(z+2\eta)e^{\alpha_1(z+2\eta)} - p_1^2(z)Q_2(z+\eta). \end{aligned} \tag{3.5}$$

For convenience, rewrite Eq. (3.5) into

$$A(z)B(z)e^{2\alpha_1(z)+\alpha_2(z+\eta)} - A(z)Q_1(z)e^{\alpha_2(z+\eta)} = C(z)e^{\alpha_1(z+2\eta)} + N(z), \tag{3.6}$$

where

$$A(z) = \omega_2'(z+\eta) + \omega_2(z+\eta)\alpha_2'(z+\eta),\ B(z) = \omega_1(z)\big(\omega_1'(z) + \omega_1(z)\alpha_1'(z)\big),$$

$$C(z) = p_1^2(z)p_2^2(z+\eta)\omega_1(z+2\eta),\ N(z) = -p_1^2(z)Q_2(z+\eta).$$

Next we discuss the following three subcases.

Subcase 1.1. $\deg(\alpha_1(z)) > \deg(\alpha_2(z))$.

Rewrite (3.6) as

$$A(z)B(z)e^{2\alpha_1(z)+\alpha_2(z+\eta)} = C(z)e^{\alpha_1(z+2\eta)} + (N(z)e^{-\alpha_2(z+\eta)} + A(z)Q_1(z))e^{\alpha_2(z+\eta)}.$$

Set $f_1(z) = 2\alpha_1(z)+\alpha_2(z+\eta)$, $f_2(z) = \alpha_2(z+\eta)$, and $f_3(z) = \alpha_1(z+2\eta)$. Then we have $f_i - f_j \not\equiv$ constant for $1 \le i < j \le 3$, and the coefficients $N(z)e^{-\alpha_2(z+\eta)} + A(z)Q_1(z), A(z)B(z)$, and $C(z)$ are still small functions of $e^{f_i - f_j}$, $1 \le i < j \le 3$. Therefore, by Lemma 2.2, we have $C(z) \equiv 0$, so that $\omega_1(z) \equiv 0$, which contradicts that $f(z)$ is a nontrivial solution.

Subcase 1.2. $\deg(\alpha_1(z)) < \deg(\alpha_2(z))$.

Rewrite (3.6) as

$$(A(z)B(z)e^{2\alpha_1(z)} - A(z)Q_1(z))e^{\alpha_2(z+\eta)} = C(z)e^{\alpha_1(z+2\eta)} + N(z). \tag{3.7}$$

Set $M(z) = A(z)B(z)e^{2\alpha_1(z)} - A(z)Q_1(z)$.

If $M(z) \not\equiv 0$, by Lemma 2.4, we have

$$\rho(M(z)e^{\alpha_2(z+\eta)}) = \rho(e^{\alpha_2(z+\eta)}) > \rho(e^{\alpha_1(z+\eta)}) = \rho(C(z)e^{\alpha_1(z+2\eta)} + N(z))$$

and hence Eq. (3.7) cannot hold.

If $M(z) \equiv 0$, by Lemma 2.2 we get $N(z) \equiv 0, C(z) \equiv 0$, which contradicts that $N(z)$ is a non-zero polynomial.

Subcase 1.3. $\deg(\alpha_1(z)) = \deg(\alpha_2(z))$.

We set $\alpha_1(z) = az^n + \cdots$, $\alpha_2(z) = bz^n + \cdots$, $a \neq 0$, $b \neq 0$, and $f_1(z) = 2\alpha_1(z) + \alpha_2(z+\eta)$, $f_2(z) = \alpha_2(z+\eta)$, $f_3(z) = \alpha_1(z+2\eta)$, $f_4(z) = 0$.

Clearly, we have $f_1(z) = (2a+b)z^n + \cdots$, $f_2(z) = bz^n + \cdots$, $f_3(z) = az^n + \cdots$, and $f_4(z) = 0$. Now we need to treat the following cases.

If $a \neq -\frac{b}{2}$, $a \neq -b$, $a \neq b$, by Lemma 2.2 and Eq. (3.6), we get $N(z) \equiv 0$, which contradicts that $N(z)$ is a non-zero polynomial.

If $a = b$, then we rewrite (3.6) as

$$A(z)B(z)e^{\alpha_2(z+\eta)-\alpha_1(z)}e^{3\alpha_1(z)} - (A(z)Q_1(z)e^{\alpha_2(z+\eta)-\alpha_1(z+2\eta)} + C(z))e^{\alpha_1(z+2\eta)} = N(z). \tag{3.8}$$

We note that $\max\{\deg(\alpha_2(z+\eta) - \alpha_1(z)), \deg(\alpha_2(z+\eta) - \alpha_1(z+2\eta))\} < n$. Thus by Lemma 2.2 and Eq. (3.8), we get $N(z) \equiv 0$, which contradicts that $N(z)$ is a non-zero polynomial.

If $a = -b$, then we rewrite (3.6) as

$$\left(A(z)B(z)e^{\alpha_2(z+\eta)+\alpha_1(z)} - C(z)e^{\alpha_1(z+2\eta)-\alpha_1(z)}\right)e^{\alpha_1(z)} - A(z)Q_1(z)e^{\alpha_2(z+\eta)} = N(z). \tag{3.9}$$

By Lemma 2.2, we get $N(z) \equiv 0$, which contradicts that $N(z)$ is a non-zero polynomial.

If $a = -\frac{b}{2}$, then we rewrite (3.6) as

$$-C(z)e^{\alpha_1(z+2\eta)} - A(z)Q_1(z)e^{\alpha_2(z+\eta)} = N(z) - A(z)B(z)e^{\alpha_3(z)}, \tag{3.10}$$

where $\alpha_3(z) = 2\alpha_1(z) + \alpha_2(z+\eta)$, and $\deg(\alpha_3(z)) < \deg(\alpha_1(z))$. By Lemma 2.2, we have $A(z)Q_1(z) \equiv 0$, $C(z) \equiv 0$, which contradicts that $C(z)$ is a non-zero polynomial.

Case 2. $m = 1$, $n \ge 2$.

Thus, by (1.7) we have

$$\begin{cases}(f(z)f'(z))^n + p_1^2(z)g(z+\eta) = Q_1(z)\\ (g(z)g'(z))^n + p_2^2(z)f(z+\eta) = Q_2(z).\end{cases} \tag{3.11}$$

From system (3.11), we get a new system of equations

$$\begin{cases}p_1^{2n}(z)\big(g(z+\eta)g'(z+\eta)\big)^n = (g'(z+\eta))^n\big(Q_1(z) - (f(z)f'(z))^n\big)^n\\ p_1^{2n}(z)\big(g(z+\eta)g'(z+\eta)\big)^n = p_1^{2n}(z)Q_2(z+\eta) - p_1^{2n}(z)p_2^2(z+\eta)f(z+2\eta).\end{cases} \tag{3.12}$$

Then Eq. (3.12) leads to

$$p_1^{2n}(z)Q_2(z+\eta) - p_1^{2n}(z)p_2^2(z+\eta)f(z+2\eta) = (g'(z+\eta))^n\big(Q_1(z) - (f(z)f'(z))^n\big)^n$$

and hence we get

$$A(z) + B(z)f(z+2\eta) = (g'(z+\eta))^n(N(z) - (f(z)f'(z))^n)^n,$$

where $A(z) = p_1^{2n}Q_2(z+\eta)$, $B(z) = -p_1^{2n}p_2^2(z+\eta)$, $N(z) = Q_1(z)$ are polynomials.

Combining this and (3.1), we have

$$\begin{aligned}A(z) + B(z)\omega_1(z+2\eta)e^{\alpha_1(z+2\eta)} =&((\omega_2'(z+\eta) + \omega_2(z+\eta)\alpha_2'(z+\eta))e^{\alpha_2(z+\eta)})^n\\ &(N(z) - (\omega_1(z)(\omega_1'(z) + \omega_1(z)\alpha_1'(z))e^{2\alpha_1(z)})^n)^n.\end{aligned} \tag{3.13}$$

We can rewrite Eq. (3.13) as

$$A(z) + B(z)\omega_1(z+2\eta)e^{\alpha_1(z+2\eta)} = M(z)e^{n\alpha_2(z+\eta)}\big(N(z) + C(z)e^{2n\alpha_1(z)}\big)^n, \tag{3.14}$$

where $M(z) = (\omega_2'(z+\eta) + \omega_2(z+\eta)\alpha_2'(z+\eta))^n$, $C(z) = -(\omega_1(z)(\omega_1'(z) + \omega_1(z)\alpha_1'(z)))^n$.

Next we discuss the following three subcases.

Subcase 2.1. $\deg(\alpha_1(z)) > \deg(\alpha_2(z))$.

Based on the binomial decomposition, we can rewrite Eq. (3.14) as

$$\begin{aligned}B(z)\omega_1(z+2\eta)e^{\alpha_1(z+2\eta)} =&M(z)e^{n\alpha_2(z+\eta)}\sum_{j=0}^{n-1}(C_n^j(N(z))^j(C(z))^{n-j}e^{2n(n-j)\alpha_1(z)})+\\ &(-A(z)e^{-n\alpha_2(z+\eta)} + M(z)N^n(z))e^{n\alpha_2(z+\eta)}.\end{aligned} \tag{3.15}$$

Set $f_{n+1} = \alpha_1(z+2\eta)$, $f_j = n\alpha_2(z+\eta)+2n(n-j)\alpha_1(z)$, $j = 0, 1, 2, \ldots, n-1$, $f_n = n\alpha_2(z+\eta)$. So we have $f_i - f_j \not\equiv$ constant for $0 \le i < j \le n+1$ and $-A(z)e^{-n\alpha_2(z+\eta)} + M(z)N^n(z), B(z)\omega_1(z+2\eta), (N(z))^j(C(z))^{n-j}M(z)$ are still small functions of $e^{f_i - f_j}$, $1 \le i < j \le n+1$. By Lemma 2.2, we get $B(z) \equiv 0$, which contradicts that $B(z)$ is a non-zero polynomial.

Subcase 2.2. $\deg(\alpha_1(z)) < \deg(\alpha_2(z))$.

Set $f_1 = \alpha_1(z+2\eta)$, $f_2 = n\alpha_2(z+\eta)$, $f_3 = 0$, $H(z) = (N(z)+C(z)e^{2n\alpha_1(z)})^n$. Then Eq. (3.14) becomes

$$A(z) + B(z)\omega_1(z+2\eta)e^{f_1(z)} = M(z)H(z)e^{f_2(z)}.$$

We have $f_i - f_j \not\equiv$ constant, $1 \le i < j \le 2$, and $A(z)e^{-f_1(z)} + B(z)\omega_1(z+2\eta), H(z)M(z)$ are still small functions of $e^{f_i - f_j}$, $1 \le i < j \le 2$. By Lemma 2.2, we get $B(z) \equiv 0$, which contradicts that $B(z)$ is a non-zero polynomial.

Subcase 2.3. $\deg(\alpha_1(z)) = \deg(\alpha_2(z))$.

Set

$$f_{n+1} = \alpha_1(z+2\eta),\ f_{n+2} = 0,\ f_j = n\alpha_2(z+\eta) + 2n(n-j)\alpha_1(z),\quad j = 0,1,2,\ldots,n$$

and $\alpha_1(z) = az^n + \cdots$, $\alpha_2(z) = bz^n + \cdots$, $a \neq 0$, $b \neq 0$.

Based on the binomial decomposition, we rewrite Eq. (3.15) as

$$B(z)\omega_1(z+2\eta)e^{f_{n+1}(z)} + A(z)e^{f_{n+2}(z)} = \Sigma_{j=0}^{n} B_j(z)e^{f_j(z)}, \tag{3.16}$$

where $B_j(z) = C_n^j(N(z))^j(C(z))^{n-j}$.

By Lemma 2.5, we have $A(z) \equiv 0$ or $B(z)\omega_1(z+2\eta) \equiv 0$, which contradicts that $A(z), B(z)$ are non-zero polynomials.

Case 3. $n = 1$, $m \geq 2$.

System of equations can be rewritten as

$$\begin{cases} f(z)f'(z) + p_1^2(z)g^m(z+\eta) = Q_1(z) \\ g(z)g'(z) + p_2^2(z)f^m(z+\eta) = Q_2(z). \end{cases} \tag{3.17}$$

From (3.17), we get a new system of equations

$$\begin{cases} p_1^2(z)\big(g(z+\eta)g'(z+\eta)\big)^m = \big(g'(z+\eta)\big)^m\big(Q_1(z) - f(z)f'(z)\big) \\ p_1^2(z)\big(g(z+\eta)g'(z+\eta)\big)^m = p_1^2(z)\big(Q_2(z+\eta) - p_2^2(z+\eta)f^m(z+2\eta)\big)^m. \end{cases} \tag{3.18}$$

By calculation, we get

$$p_1^2(z)\big(Q_2(z+\eta) - p_2^2(z+\eta)f^m(z+2\eta)\big)^m = (g'(z+\eta))^m(Q_1(z) - f(z)f'(z)). \tag{3.19}$$

Substituting (3.1) into Eq. (3.19), we have

$$\begin{aligned} &((\omega_2'(z+\eta) + \omega_2(z+\eta)\alpha_2'(z+\eta))e^{\alpha_2(z+\eta)})^m(Q_1(z) - \omega_1(z)e^{\alpha_1(z)}(\omega_1'(z) + \omega_1(z)\alpha_1'(z))e^{\alpha_1(z)}) \\ &= p_1^2(z)\big(Q_2(z+\eta) - p_2^2(z+\eta)\omega_1^m(z+2\eta)e^{m\alpha_1(z+2\eta)}\big)^m. \end{aligned} \tag{3.20}$$

We rewrite Eq. (3.20) as

$$A^m(z)e^{m\alpha_2(z+\eta)}\big(B(z)e^{2\alpha_1(z)} - Q_1(z)\big) = p_1^2(z)\big(Q_2(z+\eta) - C(z)e^{m\alpha_1(z+2\eta)}\big)^m, \tag{3.21}$$

where $A(z) = \omega_2'(z+\eta) + \omega_2(z+\eta)\alpha_2'(z+\eta)$, $B(z) = \omega_1(z)(\omega_1'(z) + \omega_1(z)\alpha_1'(z))$, $C(z) = p_2^2(z+\eta)\omega_1^m(z+2\eta)$.

We need to treat three subcases:

Subcase 3.1. $\deg(\alpha_1(z)) < \deg(\alpha_2(z))$.

If $A^m(z)(B(z)e^{2\alpha_1(z)} - Q_1(z)) \equiv 0$, then $(Q_2(z+\eta) - C(z)e^{m\alpha_1(z+2\eta)})^m \equiv 0$. Based on the binomial decomposition and Lemma 2.2, we get $Q_2^m(z+\eta) \equiv 0$, which contradicts that $Q_2(z+\eta)$ is a non-zero polynomial.

If $A^m(z)\big(B(z)e^{2\alpha_1(z)} - Q_1(z)\big) \not\equiv 0$, then by Lemma 2.4, we have

$$\begin{aligned} &\rho(A^m(z)e^{m\alpha_2(z+\eta)}\big(B(z)e^{2\alpha_1(z)} - Q_1(z)\big)) = \rho(e^{\alpha_2(z)}) \\ &> \rho(e^{\alpha_1(z)}) = \rho(p_1^2(z)\big(Q_2(z+\eta) - C(z)e^{m\alpha_1(z+2\eta)}\big)^m), \end{aligned} \tag{3.22}$$

which is a contradiction.

Subcase 3.2. $\deg(\alpha_1(z)) > \deg(\alpha_2(z))$.

Based on the binomial decomposition, Eq. (3.21) can be written as

$$\begin{aligned}&A^m(z)B(z)e^{m\alpha_2(z+\eta)+2\alpha_1(z)} - A^m(z)Q_1(z)e^{m\alpha_2(z+\eta)}\\&= p_1^2(z)\sum_{r=0}^{m} C_m^r Q_2^{m-r}(z+\eta)C^r(z)e^{mr\alpha_1(z+\eta)}.\end{aligned} \tag{3.23}$$

Set

$$f_j = mj\alpha_1(z+\eta),\ j=0,1,2,\ldots,m,\ f_{m+1} = m\alpha_2(z+\eta),\ f_{m+2} = m\alpha_2(z+\eta)+2\alpha_1(z).$$

If $m > 2$, then for $i \neq j$ we have $\deg(f_i - f_j) = \deg\alpha_1$, where $i, j \in \{0, 1, \ldots, m+2\}$. Clearly, $A^m(z)B(z)$, $p_1^2(z)Q_2^{m-r}(z+\eta)C^r(z)$, $A^m(z)Q_1(z)$ are still small functions of $e^{f_i - f_j}$. Therefore, by Lemma 2.2 we get $Q_1(z) \equiv 0$. Since $Q_1(z)$ is a non-zero polynomial, we obtain a contradiction.

If $m = 2$, then from Eq. (3.23) we obtain

$$\begin{aligned}&A^2(z)B(z)e^{2\alpha_2(z+\eta)+2\alpha_1(z)} - A^2(z)Q_1(z)e^{2\alpha_2(z+\eta)}\\&= 2p_1^2(z)Q_2(z+\eta)C(z)e^{2\alpha_1(z+\eta)} + p_1^2(z)C^2(z)e^{4\alpha_1(z+\eta)} + p_1^2(z)Q_2^2(z+\eta).\end{aligned}$$

Hence, we get

$$\begin{aligned}&p_1^2(z)C^2(z)e^{4\alpha_1(z+\eta)} + (p_1^2(z)Q_2^2(z+\eta)e^{-2\alpha_2(z+\eta)} + A^2(z)Q_1(z))e^{2\alpha_2(z+\eta)}+\\&(2p_1^2(z)Q_2(z+\eta)C(z) - A^2(z)B(z)e^{2\alpha_2(z+\eta)+2\alpha_1(z)-2\alpha_1(z+\eta)})e^{2\alpha_1(z+\eta)} = 0.\end{aligned} \tag{3.24}$$

Set $f_1 = 2\alpha_2(z+\eta)$, $f_2 = 2\alpha_1(z+\eta)$ and $f_3 = 4\alpha_1(z+\eta)$. Clearly, $\deg(f_i - f_j) = \deg\alpha_1$ for $i \neq j$, and $C^2(z)$,

$$p_1^2(z)Q_2^2(z+\eta)e^{-2\alpha_2(z+\eta)} + A^2(z)Q_1(z),$$

$$2p_1^2(z)Q_2(z+\eta)C(z) - A^2(z)B(z)e^{2\alpha_2(z+\eta)+2\alpha_1(z)-2\alpha_1(z+\eta)}$$

are still small functions of $e^{f_i - f_j}$, $1 \le i < j \le 3$. By Lemma 2.2, we get $p_1^2(z)C(z) \equiv 0$, which contradicts the fact that $p_1(z), p_2(z)$ are non-zero polynomials.

Subcase 3.3. $\deg(\alpha_1(z)) = \deg(\alpha_2(z))$.

It follows from Eq. (3.21) that

$$\begin{aligned}&A^m(z)B(z)e^{m\alpha_2(z+\eta)+2\alpha_1(z)} - A^m(z)Q_1(z)e^{m\alpha_2(z+\eta)}\\&= p_1^2(z)\sum_{r=0}^{m} C_m^r Q_2^{m-r}(z+\eta)C^r(z)e^{mr\alpha_1(z+\eta)}.\end{aligned} \tag{3.25}$$

Let

$$f_j = mj\alpha_1(z+\eta),\ j=0,1,2,\ldots,m,\ f_{m+1} = m\alpha_2(z+\eta),\ f_{m+2} = m\alpha_2(z+\eta)+2\alpha_1(z)$$

and suppose that $\alpha_1(z) = az^n + \cdots$, $\alpha_2(z) = bz^n + \cdots$, $a \neq 0$, $b \neq 0$.

In the following, we discuss two cases.

Subcase 3.3.1. $m > 2$.

Rewrite Eq. (3.25) as

$$A^m(z)B(z)e^{f_{m+2}(z)} - A^m(z)Q_1(z)e^{f_{m+1}(z)} = p_1^2(z)\sum_{r=0}^{m} B_r(z)e^{f_r(z)}, \tag{3.26}$$

where $B_r(z) = C_m^r Q_2^{m-r}(z+\eta)C^r(z)$.

If $A(z) \equiv 0$, then by Lemma 2.2 we have $C_m^r Q_2^{m-r}(z+\eta)C^r(z) \equiv 0$ for $r = 0, \dots, m$. If $B(z) \equiv 0$, then by Lemma 2.2, we also have for some r, $C_m^r Q_2^{m-r}(z+\eta)C^r(z) \equiv 0$. Thus, we have $Q_2(z) \equiv 0$ or $C(z) \equiv 0$. This is a contradiction. If $A(z) \not\equiv 0, B(z) \not\equiv 0$, then by Lemma 2.6, we still have $A^m(z)B(z) \equiv 0$ or $A^m(z)Q_1(z) \equiv 0$. Clearly, it is impossible.

Subcase 3.3.2. $m = 2$.

Rewrite Eq. (3.25) as

$$\begin{aligned} &A^2(z)B(z)e^{2\alpha_2(z+\eta)+2\alpha_1(z)} - A^2(z)Q_1(z)e^{2\alpha_2(z+\eta)} \\ &\quad = 2p_1^2(z)Q_2(z+\eta)C(z)e^{2\alpha_1(z+\eta)} + p_1^2(z)C^2(z)e^{4\alpha_1(z+\eta)} + p_1^2(z)Q_2^2(z+\eta). \end{aligned} \tag{3.27}$$

We set $\alpha_1(z) = az^n + \cdots$, $\alpha_2(z) = bz^n + \cdots$, $a \neq 0$, $b \neq 0$, and $f_1(z) = 2\alpha_2(z+\eta) + 2\alpha_1(z)$, $f_2(z) = 2\alpha_2(z+\eta)$, $f_3(z) = 2\alpha_1(z+\eta)$, $f_4(z) = 4\alpha_1(z+\eta)$, $f_5(z) = 0$.

If $a \neq -b$, $a \neq b$, by Lemma 2.2, we get $p_1^2(z)Q_2^2(z+\eta) \equiv 0$. It is a contradiction because $p_1(z), Q_2(z+\eta)$ are non-zero polynomials.

If $a = -b$, we rewrite Eq. (3.27) as

$$\begin{aligned} &A^2(z)B(z)e^{\alpha_3(z)} - A^2(z)Q_1(z)e^{2\alpha_2(z+\eta)} \\ &\quad = 2p_1^2(z)Q_2(z+\eta)C(z)e^{2\alpha_1(z+\eta)} + p_1^2(z)C^2(z)e^{4\alpha_1(z+\eta)} + p_1^2(z)Q_2^2(z+\eta), \end{aligned} \tag{3.28}$$

where $\alpha_3(z) = 2\alpha_2(z+\eta) + 2\alpha_1(z)$, and $\deg(\alpha_3(z)) < \deg(\alpha_1(z))$. By Lemma 2.2, we get $Q_2^2(z+\eta) \equiv 0$, which contradicts that $Q_2(z+\eta)$ is a non-zero polynomial.

If $a = b$, we rewrite Eq. (3.27) as

$$\begin{aligned} &-A^2(z)Q_1(z)e^{2\alpha_2(z+\eta)} = (2p_1^2(z)Q_2(z+\eta)C(z) + A^2(z)Q_1(z)e^{2\alpha_2(z+\eta)-2\alpha_1(z+\eta)})e^{2\alpha_1(z+\eta)} + \\ &\quad p_1^2(z)C^2(z)e^{4\alpha_1(z+\eta)} + p_1^2(z)Q_2^2(z+\eta). \end{aligned} \tag{3.29}$$

By Lemma 2.2, we get $p_1^2(z)C^2(z) \equiv 0$, which is a contradiction.

Case 4. $n = m = 2$.

Clearly, from (1.7), we have

$$\begin{cases} (f(z)f'(z))^2 + p_1^2(z)g^2(z+\eta) = Q_1(z) \\ (g(z)g'(z))^2 + p_2^2(z)f^2(z+\eta) = Q_2(z). \end{cases} \tag{3.30}$$

Then it follows from Lemma 2.3 that

$$\begin{cases} f(z)f'(z) + ip_1(z)g(z+\eta) = M_1(z)e^{h_1(z)}, \\ f(z)f'(z) - ip_1(z)g(z+\eta) = M_2(z)e^{-h_1(z)}, \\ g(z)g'(z) + ip_2(z)f(z+\eta) = M_3(z)e^{h_2(z)}, \\ g(z)g'(z) - ip_2(z)f(z+\eta) = M_4(z)e^{-h_2(z)}, \end{cases} \tag{3.31}$$

where $M_1(z)M_2(z) = Q_1(z)$, $M_3(z)M_4(z) = Q_2(z)$, and $M_1(z)$, $M_2(z)$, $M_3(z)$, $M_4(z)$, $h_1(z)$, $h_2(z)$ are nonzero polynomials.

From (3.31), we get

$$g(z+\eta) = \frac{M_1(z)e^{h_1(z)} - M_2(z)e^{-h_1(z)}}{2ip_1(z)}, \tag{3.32}$$

$$f(z)f'(z) = \frac{M_1(z)e^{h_1(z)} + M_2(z)e^{-h_1(z)}}{2}, \tag{3.33}$$

$$f(z+\eta) = \frac{M_3(z)e^{h_2(z)} - M_4(z)e^{-h_2(z)}}{2ip_2(z)}, \tag{3.34}$$

$$g(z)g'(z+\eta) = \frac{M_3(z)e^{h_2(z)} + M_4(z)e^{-h_2(z)}}{2}. \tag{3.35}$$

By (3.34), we get $f(z) = \frac{M_3(z-\eta)e^{h_2(z-\eta)} - M_4(z-\eta)e^{-h_2(z-\eta)}}{2ip_2(z-\eta)}$.

We rewrite $f(z)$ as

$$f(z) = M_7(z)e^{h_2(z-\eta)} + M_8(z)e^{-h_2(z-\eta)}, \tag{3.36}$$

where $M_7(z) = \frac{M_3(z-\eta)}{2ip_2(z-\eta)}$, $M_8(z) = \frac{M_4(z-\eta)}{2ip_2(z-\eta)}$.

Differentiating (3.36), we get

$$f'(z) = M_5(z)e^{h_2(z-\eta)} + M_6(z)e^{-h_2(z-\eta)}, \tag{3.37}$$

where

$$M_5(z) = \frac{M_3'(z-\eta)p_2(z) + M_3(z-\eta)p_2(z-\eta)h_2'(z-\eta) - p_2'(z-\eta)M_3(z-\eta)}{2ip_2^2(z-\eta)}$$

and

$$M_6(z) = \frac{M_4(z-\eta)h_2'(z-\eta)p_2(z-\eta) - M_4'(z-\eta)p_2(z-\eta) + M_4(z-\eta)p_2'(z-\eta)}{2ip_2^2(z-\eta)}.$$

Since $M_3(z), q(z), h_2(z-\eta)$ are nonzero polynomials, we have $\deg(M_3(z-\eta)p_2(z-\eta)h_2'(z-\eta)) > \deg(M_3'(z-\eta)p_2(z))$ and $\deg(M_3(z-\eta)p_2(z-\eta)h_2'(z-\eta)) > \deg(p_2'(z-\eta)M_3(z-\eta))$. Clearly, $M_5(z) \not\equiv 0$. Similarly, we have $M_6(z) \not\equiv 0$.

Combining (3.36) and (3.37), we get

$$f(z)f'(z) = M_5(z)M_7(z)e^{2h_2(z-\eta)} + M_6(z)M_8(z)e^{-2h_2(z-\eta)} + M_9(z), \tag{3.38}$$

where $M_9(z) = M_5(z)M_8(z) + M_6(z)M_7(z)$.

From (3.33) and (3.38), we have

$$M_5(z)M_7(z)e^{2h_2(z-\eta)} + M_6(z)M_8(z)e^{-2h_2(z-\eta)} + M_9(z) = \frac{M_1(z)e^{h_1(z)} + M_2(z)e^{-h_1(z)}}{2}. \tag{3.39}$$

Now let $f_1(z) = 2h_2(z-\eta)$, $f_2(z) = -2h_2(z-\eta)$, $f_3(z) = h_1(z)$, $f_4(z) = -h_1(z)$, $f_5(z) = 0$. Then Eq. (3.39) can be rewritten as

$$M_5(z)M_7(z)e^{f_1(z)} + M_6(z)M_8(z)e^{f_2(z)} + M_9(z) = \frac{M_1(z)e^{f_3(z)} + M_2(z)e^{f_4(z)}}{2}. \tag{3.40}$$

Now we need to treat two cases:

Subcase 4.1. $\deg(h_1(z)) > \deg(h_2(z))$ or $\deg(h_1(z)) < \deg(h_2(z))$. By Lemma 2.2, we get $M_5(z)M_7(z) \equiv 0$, which is a contradiction.

Subcase 4.2. $\deg(h_1(z)) = \deg(h_2(z))$. We set $h_2(z) = az^n + \cdots$, $h_1(z) = bz^n + \cdots$, $a \neq 0$, $b \neq 0$.

If $2a \neq b$, $2a \neq -b$, then by Lemma 2.2, we get $M_5(z)M_7(z) \equiv 0$. Clearly, it is a contradiction.

If $2a = b$ or $2a = -b$, by Lemma 2.2, we get $M_9(z) \equiv 0$, and hence $M_5(z)M_8(z) + M_6(z)M_7(z) \equiv 0$.

Thus, $(M_3'p_2 + M_3p_2h_2' - p_2'M_3)(-\frac{M_4}{p_2}) = (M_4h_2'p_2 - M_4'p_2 + M_4p_2')(\frac{M_3}{p_2})$. It follows that $2M_3M_4h_2' = M_3M_4' - M_3'M_4$, which is impossible. The proof of Theorem 1.5 is completed. □

Acknowledgements We thank the referees for their time and comments.

References

[1] F. GROSS. *On the equation $f^n + g^n = 1$*. Bull. Amer. Math. Soc., 1966, **72**(1): 86–88.

[2] I. N. BAKER. *On a class of meromorphic functions*. Proc. Amer. Math. Soc., 1966, **17**: 819–822.

[3] Junfan CHEN, Shuqing LIN. *On the existence of solutions of Fermat-Type differential-difference equations*. Bull. Korean Math. Soc., 2021, **58**(4): 83–102.

[4] Minfeng CHEN, Zongsheng GAO, Yunfei DU. *Existence of entire solutions of some non-linear differential-difference equations*. J. Inequal. Appl., 2017, Paper No. 90, 17 pp.

[5] Feng LÜ, Qi HAN. *On the Fermat-type equation $f^3(z) + f^3(z + c) = 1$*. Aequationes Math., 2017, **91**(1): 129–136.

[6] Peichu HU, Wenbo WANG, Linlin WU. *Entire solutions of differential-difference equations of Fermat type*. Bull. Korean. Math. Soc., 2022, **59**(1): 83–99.

[7] Kai LIU, Lei MA, Xiaoyang ZHAI. *The generalized Fermat type difference equations*. Bull. Korean Math. Soc., 2018, **55**(6): 1845–1858.

[8] Hua WANG, Hongyan XU, Jin TU. *The existence and forms of solutions for some Fermat differential-difference equations*. AIMS Math., 2020, **5**(1): 685–700.

[9] Peichu HU, Linlin WU. *Topics in Fermat-type functional equations*. J. Shandong Univ. Natural Sci., 2021, **56**(10): 23–37.

[10] Qiongyan WANG. *Admissible meromorphic solutions of algebraic differential-difference equations*. Math. Meth. Appl. Sci., 2019, **42**(9): 3044–3053.

[11] N. TODA. *On the functional equation $\sum_{i=0}^{p} a_i f^{n_i} = 1$*. Tohoku Math. J., 1971, **23**: 289–299.

[12] Chungchun YANG. *A generalization of a theorem of P. Montel on entire functions*. Proc. Amer. Math. Soc., 1970, **26**: 332–334.

[13] Jiafeng TANG, Liangwen LIAO. *The transcendental meromorphic solutions of a certain type of nonlinear differential equations*. J. Math. Anal. Appl., 2007, **334**(1): 517–527.

[14] Chungchun YANG, Ping LI. *On the transcendental solutions of a certain differential equations*. Arch. Math. (Basel), 2004, **82**(5): 442–448.

[15] Kai LIU, Tingbin CAO, Hongze CAO. *Entire solutions of Fermat Type differential-difference equations*. Arch. Math. (Basel), 2012, **99**(2): 147–155.

[16] Lingyun GAO. *On entire solutions of two types of systems of complex differential-difference equations*. Acta Math. Sci., 2017, **37**(1): 187–194.

[17] Chungchun YANG, Hongxun YI. *Uniqueness Theory of Meromorphic Functions*. Kluwer Academic Publishers Group, Dordrecht, 2003.