Journal of Mathematical Research with Applications
Mar., 2024, Vol. 44, No. 2, pp. 225–238
DOI:10.3770/j.issn:2095-2651.2024.02.009
Http://jmre.dlut.edu.cn

Existence and Finite Time Stability of Nonlinear Riemann-Liouville Fractional Delay Differential Equations

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Abstract This article explores the existence results and finite time stability of nonlinear Riemann-Liouville fractional oscillatory differential equations of order $1 < \varrho < 2$ with pure delay. The approaches we adopted to explore the existence results are fixed point theorems. What's more, based on some important inequalities, we explore the finite time stability of the system. In the end, the rationality of our conclusion is verified by a case.

Keywords fractional oscillatory differential equations; delay differential equation; finite time stability

MR(2020) Subject Classification 34A12; 34A34; 34D20

1. Introduction

Fractional calculus, an important branch of mathematics, has a long historical standing which can depict some phenomena in natural science and engineering applications. For example, in control systems, biological tissues, statistical and stochastic processes, viscoelasticity and other fields. Oscillatory equation, also known as wave equation, is used to describe various wave phenomena in nature. Fractional oscillatory differential equation is the unity of fractional differential equation and oscillatory equation. It is worth reiterating that this kind of equation is of great significance for the study of different fields. We can see the previous literature [1–11].

In particular, the study of the stability is of great significance, from the control system to the social system, all of which are affected to a greater or lesser extent. This article explores the finite time stability (FTS) of nonlinear Riemann-Liouville fractional differential system. If for the initial conditions of a given range, the state will not exceed a predetermined bound in a limited time period. We can consult the previous works [12–21].

In recent past, Khusainov and Shuklin [22] proposed the delayed exponential function to explore a delay differential equation of first order. This function solved some delay differential equations encountered in dealing with practical problems. This idea has aroused the attention of mathematicians and engineers. For example, Mahmudov [23] adopted iterative method to construct basic solution. On the basis of the basic solution, the explicit solutions of Caputo

Received March 8, 2023; Accepted November 15, 2023
Supported by the National Natural Science Foundation of China (Grant No. 11871064).
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type fractional differential equations with delay were given. Li and Wang [24] constructed a new delayed matrix function, and applied the method of constant variation to get the explicit solutions of Riemann-Liouville type fractional delay differential equations.

In the light of the delayed exponential function, Pan and Fan [25] made a further promotion. The fractional oscillatory differential system they explored is as follows

$$\begin{cases} {}^{R}\mathsf{D}^{\varrho}_{-\varsigma^+}\Upsilon(\nu) = \Omega\Upsilon(\nu-\varsigma) + f(\nu), & \nu \in (0,\mathsf{T}], \varsigma > 0, \\ \Upsilon(\nu) = \varphi(\nu), & -\varsigma < \nu \le 0, \\ \mathsf{J}^{2-\varrho}_{-\varsigma^+}\Upsilon(-\varsigma^+) = \mathbf{a}, \quad {}^{R}\mathsf{D}^{\varrho-1}_{-\varsigma^+}\Upsilon(-\varsigma^+) = \mathbf{b}, \end{cases} \tag{1.1}$$

where ${}^{R}\mathsf{D}^{\varrho}_{-\varsigma^+}$ and $\mathsf{J}^{2-\varrho}_{-\varsigma^+}$ denote Riemann-Liouville fractional derivative and intergal, respectively. $f \in C((0,\mathsf{T}],\mathbb{R}^n)$, $\mathsf{T} = l\varsigma < \infty$, l is a determinate number and $\Omega \in \mathbb{R}^{n\times n}$ is a constant matrix. Two new fundamental solutions $\mathcal{P}^{\varsigma}_{\varrho}(\nu)$ and $\mathcal{H}^{\varsigma}_{\varrho}(\nu)$ were defined. In order to get the accurate solution of system (1.1), they adopted the method of constant variation.

According to [25], the accurate solution of system (1.1) can be written in the following structure

$$\Upsilon(\nu) = \mathcal{P}^{\varsigma}_{\varrho}(\nu)\mathbf{b}+\mathcal{H}^{\varsigma}_{\varrho}(\nu)\mathbf{a}+\int_{-\varsigma}^{0}\mathcal{P}^{\varsigma}_{\varrho}(\nu-\varsigma-\zeta)({}^{R}\mathsf{D}^{\varrho}_{-\varsigma^+}\varphi)(\zeta)\mathrm{d}\zeta+\int_{0}^{\nu}\mathcal{P}^{\varsigma}_{\varrho}(\nu-\varsigma-\zeta)f(\zeta)\mathrm{d}\zeta, \ \ \nu \in [0,\mathsf{T}].$$

Different from other articles, they made some requirements for functions φ and f, and introduced two spaces Φ and Ψ in [25].

With the development of science and technology, almost all fields of natural science involve nonlinear problems. Therefore, we explore the nonlinear fractional oscillatory system as below

$$\begin{cases} {}^{R}\mathsf{D}^{\varrho}_{-\varsigma^+}\Upsilon(\nu) = \Omega\Upsilon(\nu-\varsigma) + f(\nu,\Upsilon(\nu)), & \nu \in (0,\mathsf{T}], \varsigma > 0, \\ \Upsilon(\nu) = \varphi(\nu), & -\varsigma < \nu \le 0, \\ \mathsf{J}^{2-\varrho}_{-\varsigma^+}\Upsilon(-\varsigma^+) = \mathbf{a}, \quad {}^{R}\mathsf{D}^{\varrho-1}_{-\varsigma^+}\Upsilon(-\varsigma^+) = \mathbf{b}. \end{cases} \tag{1.2}$$

Existence and FTS results for solutions of system (1.2) are given by using the two fundamental solutions, fixed point theorems and estimations of some important inequalities including Hölder inequality and Gronwall inequality.

The rest of this article is as follows: Section 2 includes some preliminary knowledge. Section 3 contains the existence of solution of system (1.2). We apply two fixed point theorems. Section 4 presents some sufficient conditions to ensure the FTS of system (1.2). In Section 5, the rationality of the results is proved by an instance.

2. Preliminaries

For the convenience of narration, in this section, two new delayed Mittag-Leffler type matrix functions and other preparatory knowledge are given.

Definition 2.1 ([25]) *The ϱ order Riemann-Liouville derivative of function $\omega : [-\varsigma, +\infty) \to \mathbb{R}^n$ can be defined as follows*

$${}^{R}\mathsf{D}^{\varrho}_{-\varsigma^+}\omega(\nu) = \frac{1}{\Gamma(2-\varrho)}\frac{\mathrm{d}^2}{\mathrm{d}\nu^2}\int_{-\varsigma}^{\nu}(\nu-\zeta)^{1-\varrho}\omega(\zeta)\mathrm{d}\zeta, \quad \nu \in (-\varsigma, +\infty),$$

where $\varrho \in (1,2)$ and $\Gamma(\cdot)$ expresses the Gamma function.

Here, two basic solutions which we use are introduced.

Definition 2.2 ([25]) *The delayed matrix function $\mathcal{P}_\varrho^\varsigma(\nu) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is characterized by*

$$\mathcal{P}_\varrho^\varsigma(\nu) = \begin{cases} \theta, & -\infty < \nu \le -\varsigma, \\ I\frac{(\nu+\varsigma)^{\varrho-1}}{\Gamma(\varrho)}, & -\varsigma < \nu \le 0, \\ I\frac{(\nu+\varsigma)^{\varrho-1}}{\Gamma(\varrho)} + \Omega\frac{\nu^{2\varrho-1}}{\Gamma(2\varrho)}, & 0 < \nu \le \varsigma, \\ I\frac{(\nu+\varsigma)^{\varrho-1}}{\Gamma(\varrho)} + \Omega\frac{\nu^{2\varrho-1}}{\Gamma(2\varrho)} + \Omega^2\frac{(\nu-\varsigma)^{3\varrho-1}}{\Gamma(3\varrho)} + \cdots + & \\ \Omega^{\mathsf{k}}\frac{(\nu-(\mathsf{k}-1)\varsigma)^{(\mathsf{k}+1)\varrho-1}}{\Gamma((\mathsf{k}+1)\varrho)}, & (\mathsf{k}-1)\varsigma < \nu \le \mathsf{k}\varsigma, \mathsf{k} \in \mathbb{N}, \end{cases}$$

where θ and I stand for zero and identity matrices, respectively.

Definition 2.3 ([25]) *The delayed matrix function $\mathcal{H}_\varrho^\varsigma(\nu) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is characterized by*

$$\mathcal{H}_\varrho^\varsigma(\nu) = \begin{cases} \theta, & -\infty < \nu \le -\varsigma, \\ I\frac{(\nu+\varsigma)^{\varrho-2}}{\Gamma(\varrho-1)}, & -\varsigma < \nu \le 0, \\ I\frac{(\nu+\varsigma)^{\varrho-2}}{\Gamma(\varrho-1)} + \Omega\frac{\nu^{2\varrho-2}}{\Gamma(2\varrho-1)}, & 0 < \nu \le \varsigma, \\ I\frac{(\nu+\varsigma)^{\varrho-2}}{\Gamma(\varrho-1)} + \Omega\frac{\nu^{2\varrho-2}}{\Gamma(2\varrho-1)} + \Omega^2\frac{(\nu-\varsigma)^{3\varrho-2}}{\Gamma(3\varrho-1)} + \cdots + & \\ \Omega^{\mathsf{k}}\frac{(\nu-(\mathsf{k}-1)\varsigma)^{(\mathsf{k}+1)\varrho-2}}{\Gamma((\mathsf{k}+1)\varrho-1)}, & (\mathsf{k}-1)\varsigma < \nu \le \mathsf{k}\varsigma, \mathsf{k} \in \mathbb{N}, \end{cases}$$

where θ and I stand for zero and identity matrices, respectively.

Remark 2.4 Functions $\mathcal{P}_\varrho^\varsigma(\nu)$ and $\mathcal{H}_\varrho^\varsigma(\nu)$ are continuous on $\mathbb{R}$ and $(-\varsigma, \infty)$, respectively. We find that the right limit of function $\mathcal{H}_\varrho^\varsigma(\nu)$ at the point $\nu = -\varsigma$ does not exist. As a result, the point $\nu = -\varsigma$ is called a discontinuity of second kind for the function $\mathcal{H}_\varrho^\varsigma(\nu)$.

Theorem 2.5 *Suppose X is a complete metric space and $\Xi : X \to X$ is a contractive mapping. Then Ξ has a precisely one fixed point.*

Theorem 2.6 *Suppose X is a Banach space, operator $\Xi : X \to X$ is a continuous compact mapping, and there is a nonempty bounded convex closed set $O \subseteq X$ which makes $\Xi O \subseteq O$ established. Then Ξ has at least one fixed point $w \in O$, that is, $\Xi w = w$.*

Definition 2.7 ([26]) *The nonhomogeneous system (1.2) is finite time stable concerning $\{0, J, \varsigma, \delta, \eta\}$, $\delta < \eta$, iff $\|\varphi\| < \delta$, $\|\mathbf{a}\| < \delta$ and $\|\mathbf{b}\| < \delta$, signify the solution Υ meeting $\|\Upsilon\| = \sup_{\nu\in J} \|\Upsilon(\nu)\| < \eta$, where $J = [0, \mathsf{T}]$.*

Lemma 2.8 ([27]) *Suppose $x(\nu)$ is nonnegative and locally integrable on $[0, \mathsf{T})$ $(\mathsf{T} \le +\infty)$, and $g(\nu)$, $a(\nu)$ are nonnegative, nondecreasing continuous functions defined on $[0, \mathsf{T})$, $g(\nu) \le M$, $\varrho > 0$ with*

$$x(\nu) \le a(\nu) + g(\nu)\int_0^\nu (\nu-\zeta)^{\varrho-1}x(\zeta)\mathrm{d}\zeta$$

on this interval. Then,

$$x(\nu) \le a(\nu)E_\varrho(g(\nu)\Gamma(\varrho)\nu^\varrho), \quad 0 \le \nu < \mathsf{T},$$

where the function $E_\varrho(\cdot)$ is defined as $E_\varrho(\zeta) = \sum_{\mu=0}^{\infty} \frac{\zeta^\mu}{\Gamma(\mu\varrho+1)}$.

In order to study the existence and finite time stability results, the following lemmas are needed.

Lemma 2.9 *For $\nu \in ((\mathsf{k}-1)\varsigma, \mathsf{k}\varsigma]$, $\mathsf{k} \in \mathbb{N}^+$ and $1 < \varrho < 2$, we obtain*

(i) $\|\mathcal{P}_\varrho^\varsigma(\nu)\| \le (\nu+\varsigma)^{\varrho-1} E_{\varrho,\varrho}(\|\Omega\|(\nu+\varsigma)^\varrho)$.

(ii) $\|\mathcal{H}_\varrho^\varsigma(\nu)\| \le (\nu+\varsigma)^{\varrho-2} E_{\varrho,\varrho-1}(\|\Omega\|(\nu+\varsigma)^\varrho)$.

Here, the function $E_{\lambda,\mu}(\cdot)$ is defined as $E_{\lambda,\mu}(\zeta) = \sum_{j=0}^{\infty} \frac{\zeta^j}{\Gamma(j\lambda+\mu)}$, where $\lambda, \mu > 0$ and $\zeta \in \mathbb{R}$.

Proof For $\nu \in ((\mathsf{k}-1)\varsigma, \mathsf{k}\varsigma]$, $\mathsf{k} \in \mathbb{N}^+$, on the basis of Definitions 2.2 and 2.3, we obtain

(i)

$$\begin{aligned}
&\|\mathcal{P}_\varrho^\varsigma(\nu)\| \\
&\le [\frac{(\nu+\varsigma)^{\varrho-1}}{\Gamma(\varrho)} + \|\Omega\|\frac{\nu^{2\varrho-1}}{\Gamma(\varrho+\varrho)} + \|\Omega\|^2\frac{(\nu-\varsigma)^{3\varrho-1}}{\Gamma(2\varrho+\varrho)} + \cdots + \|\Omega\|^{\mathsf{k}}\frac{(\nu-(\mathsf{k}-1)\varsigma)^{(\mathsf{k}+1)\varrho-1}}{\Gamma(\mathsf{k}\varrho+\varrho)}] \\
&\le [\frac{(\nu+\varsigma)^{\varrho-1}}{\Gamma(\varrho)} + \|\Omega\|\frac{(\nu+\varsigma)^{2\varrho-1}}{\Gamma(\varrho+\varrho)} + \|\Omega\|^2\frac{(\nu+\varsigma)^{3\varrho-1}}{\Gamma(2\varrho+\varrho)} + \cdots + \|\Omega\|^{\mathsf{k}}\frac{(\nu+\varsigma)^{(\mathsf{k}+1)\varrho-1}}{\Gamma(\mathsf{k}\varrho+\varrho)}] \\
&\le (\nu+\varsigma)^{\varrho-1} E_{\varrho,\varrho}(\|\Omega\|(\nu+\varsigma)^\varrho).
\end{aligned}$$

(ii)

$$\begin{aligned}
&\|\mathcal{H}_\varrho^\varsigma(\nu)\| \\
&\le [\frac{(\nu+\varsigma)^{\varrho-2}}{\Gamma(\varrho-1)} + \|\Omega\|\frac{\nu^{2\varrho-2}}{\Gamma(2\varrho-1)} + \|\Omega\|^2\frac{(\nu-\varsigma)^{3\varrho-2}}{\Gamma(3\varrho-1)} + \cdots + \|\Omega\|^{\mathsf{k}}\frac{(\nu-(\mathsf{k}-1)\varsigma)^{(\mathsf{k}+1)\varrho-2}}{\Gamma((\mathsf{k}+1)\varrho-1)}] \\
&\le [\frac{(\nu+\varsigma)^{\varrho-2}}{\Gamma(\varrho-1)} + \|\Omega\|\frac{(\nu+\varsigma)^{2\varrho-2}}{\Gamma(2\varrho-1)} + \|\Omega\|^2\frac{(\nu+\varsigma)^{3\varrho-2}}{\Gamma(3\varrho-1)} + \cdots + \|\Omega\|^{\mathsf{k}}\frac{(\nu+\varsigma)^{(\mathsf{k}+1)\varrho-2}}{\Gamma((\mathsf{k}+1)\varrho-1)}] \\
&\le (\nu+\varsigma)^{\varrho-2} E_{\varrho,\varrho-1}(\|\Omega\|(\nu+\varsigma)^\varrho). \quad \square
\end{aligned}$$

Remark 2.10 According to D'Alembert's test, we obtain

$$\begin{aligned}
\lim_{j\to\infty} \frac{\frac{\zeta^{j+1}}{\Gamma((j+1)\lambda+\mu)}}{\frac{\zeta^j}{\Gamma(j\lambda+\mu)}} &= \lim_{j\to\infty} \frac{\Gamma(j\lambda+\mu)}{\Gamma(j\lambda+\mu+\lambda)}\zeta \\
&= \lim_{j\to\infty} \frac{\Gamma(j\lambda+\mu)\zeta}{(j\lambda+\mu-1+\lambda)\cdots(j\lambda+\mu)\Gamma(j\lambda+\mu)} \\
&= \lim_{j\to\infty} \frac{\zeta}{(j\lambda+\mu-1+\lambda)\cdots(j\lambda+\mu)} = 0 < 1.
\end{aligned}$$

So the infinite series $\sum_{j=0}^{\infty} \frac{\zeta^j}{\Gamma(j\lambda+\mu)} < \infty$ holds, which means the existence of $E_{\lambda,\mu}(\zeta)$. On the other hand, it is obvious that the function is a power series, so the function $E_{\lambda,\mu}(\zeta)$ is continuous on its convergence domain $\mathbb{R}$. For more details, we can see [1].

Lemma 2.11 ([25]) *For $\nu \in ((\mathsf{k}-1)\varsigma, \mathsf{k}\varsigma]$, $\mathsf{k} \in \mathbb{N}^+$, we obtain*

$$\begin{aligned}
\int_{-\varsigma}^{0} \|\mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)\| \mathrm{d}\zeta \le & \|\Omega\|^{\mathsf{k}} \frac{(\nu-(\mathsf{k}-1)\varsigma)^{(\mathsf{k}+1)\varrho}}{\Gamma((\mathsf{k}+1)\varrho+1)} + \\
& \sum_{j=1}^{\mathsf{k}} \frac{\|\Omega\|^{j-1}}{\Gamma(j\varrho+1)} \left[(\nu-(j-2)\varsigma)^{j\varrho} - (\nu-(j-1)\varsigma)^{j\varrho}\right].
\end{aligned}$$

Lemma 2.12 ([25]) *For* $\nu \in ((\mathsf{k}-1)\varsigma, \mathsf{k}\varsigma]$, $\mathsf{k} \in \mathbb{N}^{+}$, *we obtain*

$$\int_{0}^{\nu} \|\mathcal{P}_{\varrho}^{\varsigma}(\nu-\varsigma-\zeta)\|\mathrm{d}\zeta \leq \sum_{j=0}^{\mathsf{k}-1} \frac{\|\varOmega\|^{j}}{\Gamma((j+1)\varrho+1)}(\nu-j\varsigma)^{(j+1)\varrho}.$$

Here, we give another method to estimate this integral.

Lemma 2.13 *For* $\nu \in ((\mathsf{k}-1)\varsigma, \mathsf{k}\varsigma]$, $\mathsf{k} \in \mathbb{N}^{+}$, *we obtain*

$$\int_{0}^{\nu} \|\mathcal{P}_{\varrho}^{\varsigma}(\nu-\varsigma-\zeta)g(\zeta)\|\mathrm{d}\zeta \leq E_{\varrho,\varrho}(\|\varOmega\|\nu^{\varrho}) \int_{0}^{\nu} (\nu-\zeta)^{\varrho-1}\|g(\zeta)\|\mathrm{d}\zeta, \quad g \in C([0,\mathsf{T}],\mathbb{R}^{n}).$$

Proof Let $\nu \in ((\mathsf{k}-1)\varsigma, \mathsf{k}\varsigma]$ and $\mathsf{k} \in \mathbb{N}^{+}$, in the light of Definition 2.2, we obtain

$$\begin{aligned}
&\int_{0}^{\nu} \|\mathcal{P}_{\varrho}^{\varsigma}(\nu-\varsigma-\zeta)g(\zeta)\|\mathrm{d}\zeta \\
&\leq \int_{0}^{\nu-(\mathsf{k}-1)\varsigma} [\frac{(\nu-\zeta)^{\varrho-1}}{\Gamma(\varrho)} + \|\varOmega\|\frac{(\nu-\varsigma-\zeta)^{2\varrho-1}}{\Gamma(\varrho+\varrho)} + \cdots + \\
&\quad \|\varOmega\|^{\mathsf{k}-1}\frac{(\nu-(\mathsf{k}-1)\varsigma-\zeta)^{\mathsf{k}\varrho-1}}{\Gamma((\mathsf{k}-1)\varrho+\varrho)}]\|g(\zeta)\|\mathrm{d}\zeta + \int_{\nu-(\mathsf{k}-1)\varsigma}^{\nu-(\mathsf{k}-2)\varsigma} [\frac{(\nu-\zeta)^{\varrho-1}}{\Gamma(\varrho)} + \\
&\quad \|\varOmega\|\frac{(\nu-\varsigma-\zeta)^{2\varrho-1}}{\Gamma(\varrho+\varrho)} + \cdots + \|\varOmega\|^{\mathsf{k}-2}\frac{(\nu-(\mathsf{k}-2)\varsigma-\zeta)^{(\mathsf{k}-1)\varrho-1}}{\Gamma((\mathsf{k}-2)\varrho+\varrho)}]\|g(\zeta)\|\mathrm{d}\zeta + \cdots + \\
&\quad \int_{\nu-2\varsigma}^{\nu-\varsigma} [\frac{(\nu-\zeta)^{\varrho-1}}{\Gamma(\varrho)} + \|\varOmega\|\frac{(\nu-\varsigma-\zeta)^{2\varrho-1}}{\Gamma(\varrho+\varrho)}]\|g(\zeta)\|\mathrm{d}\zeta + \int_{\nu-\varsigma}^{\nu} \frac{(\nu-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\|g(\zeta)\|\mathrm{d}\zeta \\
&\leq \int_{0}^{\nu} \frac{(\nu-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\|g(\zeta)\|\mathrm{d}\zeta + \int_{0}^{\nu-\varsigma} \|\varOmega\|\frac{(\nu-\varsigma-\zeta)^{2\varrho-1}}{\Gamma(2\varrho)}\|g(\zeta)\|\mathrm{d}\zeta + \cdots + \\
&\quad \int_{0}^{\nu-(\mathsf{k}-1)\varsigma} \|\varOmega\|^{\mathsf{k}-1}\frac{(\nu-(\mathsf{k}-1)\varsigma-\zeta)^{\mathsf{k}\varrho-1}}{\Gamma(\mathsf{k}\varrho)}\|g(\zeta)\|\mathrm{d}\zeta \\
&\leq \int_{0}^{\nu} \frac{(\nu-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\|g(\zeta)\|\mathrm{d}\zeta + \int_{0}^{\nu-\varsigma} \|\varOmega\|\frac{(\nu-\zeta)^{2\varrho-1}}{\Gamma(2\varrho)}\|g(\zeta)\|\mathrm{d}\zeta + \cdots + \\
&\quad \int_{0}^{\nu-(\mathsf{k}-1)\varsigma} \|\varOmega\|^{\mathsf{k}-1}\frac{(\nu-\zeta)^{\mathsf{k}\varrho-1}}{\Gamma(\mathsf{k}\varrho)}\|g(\zeta)\|\mathrm{d}\zeta \\
&\leq \int_{0}^{\nu} \sum_{j=0}^{\mathsf{k}-1} \frac{\|\varOmega\|^{j}}{\Gamma(j\varrho+\varrho)}(\nu-\zeta)^{(j+1)\varrho-1}\|g(\zeta)\|\mathrm{d}\zeta \\
&\leq E_{\varrho,\varrho}(\|\varOmega\|\nu^{\varrho}) \int_{0}^{\nu} (\nu-\zeta)^{\varrho-1}\|g(\zeta)\|\mathrm{d}\zeta.
\end{aligned}$$

This proof is completed. □

3. Existence results

In this paper, the finite time stability is studied by explicit solution, so the existence of the solution is the prerequisite for this study. In this section, Banach fixed point theorem and Schauder fixed point theorem are used to prove it. In the first place, we introduce a space $\varPhi = \{\varphi \in C((-\varsigma,0],\mathbb{R}^{n}) : \mathsf{J}_{-\varsigma^{+}}^{2-\varrho}\varphi \in AC^{2}((-\varsigma,0],\mathbb{R}^{n})\}$. It is an absolutely continuous function after taking the first derivative of $\mathsf{J}_{-\varsigma^{+}}^{2-\varrho}\varphi$.

Definition 3.1 *Function $\Upsilon \in C((-\varsigma, \mathsf{T}], \mathbb{R}^n)$ is referred to as the solution of system (1.2) if it can be represented by the subsequent form:*

$$\Upsilon(\nu) = \begin{cases} \mathcal{P}_\varrho^\varsigma(\nu)\mathbf{b} + \mathcal{H}_\varrho^\varsigma(\nu)\mathbf{a} + \int_{-\varsigma}^0 \mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)({}^R\mathsf{D}_{-\varsigma^+}^\varrho\varphi)(\zeta)\mathrm{d}\zeta + \\ \int_0^\nu \mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta, \quad \nu \in [0,\mathsf{T}], \\ \varphi(\nu), \quad \nu \in (-\varsigma, 0]. \end{cases}$$

Here, we introduce some assumptions:

$[E_1]$ $f : [0,\mathsf{T}] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. There exists $K > 0$ such that $\|f(\nu,y) - f(\nu,z)\| \le K\|y-z\|$, $\nu \in [0,\mathsf{T}]$ and $y, z \in \mathbb{R}^n$.

$[E_2]$ $\rho = K\sum_{j=0}^{l-1} \frac{\|\Omega\|^j}{\Gamma((j+1)\varrho+1)}(\mathsf{T}-j\varsigma)^{(j+1)\varrho} < 1$.

$[E_3]$ ${}^R\mathsf{D}_{-\varsigma^+}^\varrho\varphi \in C((-\varsigma,0],\mathbb{R}^n)$ and $M = \sup_{-\varsigma<\zeta\le 0}\|({}^R\mathsf{D}_{-\varsigma^+}^\varrho\varphi)(\zeta)\| < \infty$.

$[E_4]$ $f : [0,\mathsf{T}] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. There exists $L > 0$ such that

$$\|f(\zeta,\Upsilon)\| \le L\|\Upsilon\|, \ \zeta \in [0,\mathsf{T}] \text{ and } \Upsilon \in \mathbb{R}^n.$$

In the next, we demonstrate the existence results through fixed point theorems. For convenience of representation, we define

$$\psi_1(\nu) = (\nu+\varsigma)^{\varrho-1}E_{\varrho,\varrho}(\|\Omega\|(\nu+\varsigma)^\varrho),$$

$$\psi_2(\nu) = (\nu+\varsigma)^{\varrho-2}E_{\varrho,\varrho-1}(\|\Omega\|(\nu+\varsigma)^\varrho),$$

$$\psi_3^{\mathsf{k}}(\nu) = \sum_{j=1}^{\mathsf{k}} \frac{\|\Omega\|^{j-1}}{\Gamma(j\varrho+1)}\left[(\nu-(j-2)\varsigma)^{j\varrho} - (\nu-(j-1)\varsigma)^{j\varrho}\right] + \|\Omega\|^{\mathsf{k}}\frac{(\nu-(\mathsf{k}-1)\varsigma)^{(\mathsf{k}+1)\varrho}}{\Gamma((\mathsf{k}+1)\varrho+1)},$$

$$\psi_4^{\mathsf{k}}(\nu) = \sum_{j=0}^{\mathsf{k}-1}\frac{\|\Omega\|^j}{\Gamma((j+1)\varrho+1)}(\nu-j\varsigma)^{(j+1)\varrho},$$

where $1 < \varrho < 2$, $\nu \in ((\mathsf{k}-1)\varsigma, \mathsf{k}\varsigma]$ and $\mathsf{k} \in \{1,2,\dots,l\}$.

Remark 3.2 According to the continuity of function $E_{\lambda,\mu}(\zeta)$ in Remark 2.10, it can be known that these three composite functions $E_{\varrho,\varrho}(\|\Omega\|(\nu+\varsigma)^\varrho)$, $E_{\varrho,\varrho-1}(\|\Omega\|(\nu+\varsigma)^\varrho)$ and $E_{\varrho,\varrho}(\|\Omega\|\nu^\varrho)$ are continuous for $\nu \in \mathbb{R}$, and the composite function $\psi_2(\nu) = (\nu+\varsigma)^{\varrho-2}E_{\varrho,\varrho-1}(\|\Omega\|(\nu+\varsigma)^\varrho)$ is continuous for $\nu \in [0,\mathsf{T}]$. Clearly, these four continuous functions are bounded on $[0,\mathsf{T}]$.

Remark 3.3 It is easy to prove that $\psi_1(\nu)$ is monotonically increasing function by the method of differentiation. Through the same way, for fixed k, the functions $\psi_3^{\mathsf{k}}(\nu)$ and $\psi_4^{\mathsf{k}}(\nu)$ are monotonically increasing functions when $\nu \in ((\mathsf{k}-1)\varsigma, \mathsf{k}\varsigma]$, $\mathsf{k} \in \{1,2,\dots,l\}$. On the other hand, by calculating we find that $\psi_3^{\mathsf{k}-1}((\mathsf{k}-1)\nu) = \psi_3^{\mathsf{k}}((\mathsf{k}-1)\nu)$, $\psi_4^{\mathsf{k}-1}((\mathsf{k}-1)\nu) = \psi_4^{\mathsf{k}}((\mathsf{k}-1)\nu)$. Therefore, $\psi_3^{\mathsf{k}}(\nu) \le \psi_3^l(\mathsf{T})$, $\psi_4^{\mathsf{k}}(\nu) \le \psi_4^l(\mathsf{T})$, where $\nu \in ((\mathsf{k}-1)\varsigma, \mathsf{k}\varsigma]$, $\mathsf{k} \in \{1,2,\dots,l\}$.

Theorem 3.4 *Let $\varphi \in \Phi$. Assume that $[E_1]$ and $[E_2]$ are met. System (1.2) has a single solution Υ.*

Proof Define an operator $\Xi : C([0,\mathsf{T}],\mathbb{R}^n) \to C([0,\mathsf{T}],\mathbb{R}^n)$ by

$$\Xi\Upsilon(\nu) = \mathcal{P}_\varrho^\varsigma(\nu)\mathbf{b} + \mathcal{H}_\varrho^\varsigma(\nu)\mathbf{a} + \int_{-\varsigma}^0 \mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)({}^R\mathsf{D}_{-\varsigma^+}^\varrho\varphi)(\zeta)\mathrm{d}\zeta + \int_0^\nu \mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta.$$

Obviously, Ξ is well defined due to $[E_1]$ and Remark 2.4. Then, we shall verify that Ξ is a contractive mapping.

In terms of Lemma 2.12 and for $x(\cdot), y(\cdot) \in C([0,\mathsf{T}],\mathbb{R}^n)$, we obtain

$$\begin{aligned}\|\Xi x(\nu)-\Xi y(\nu)\| &\leq K\int_0^{\nu}\|\mathcal{P}_{\varrho}^{\varsigma}(\nu-\varsigma-\zeta)\|\mathrm{d}\zeta\|x-y\|\\ &\leq K\sum_{j=0}^{\mathsf{k}-1}\frac{\|\Omega\|^j}{\Gamma((j+1)\varrho+1)}(\nu-j\varsigma)^{(j+1)\varrho}\|x-y\|\\ &\leq K\sum_{j=0}^{l-1}\frac{\|\Omega\|^j}{\Gamma((j+1)\varrho+1)}(\mathsf{T}-j\varsigma)^{(j+1)\varrho}\|x-y\|,\end{aligned}$$

which implies that $\|\Xi x-\Xi y\|\leq\rho\|x-y\|$. On the basis of assumption $[E_2]$, Ξ is a contractive mapping. We can apply Theorem 2.5 to deal with the rest proof. □

Theorem 3.5 *Let $\varphi\in\Phi$. Assume that $[E_3]$, $[E_4]$ and the formula $L\psi_4^l(\mathsf{T})<1$ hold. System (1.2) has at least one solution Υ.*

Proof Define an operator $\varXi: C([0,\mathsf{T}],\mathbb{R}^n)\to C([0,\mathsf{T}],\mathbb{R}^n)$ by

$$\varXi\Upsilon(\nu)=\mathcal{P}_{\varrho}^{\varsigma}(\nu)\mathbf{b}+\mathcal{H}_{\varrho}^{\varsigma}(\nu)\mathbf{a}+\int_{-\varsigma}^{0}\mathcal{P}_{\varrho}^{\varsigma}(\nu-\varsigma-\zeta)({}^{R}\mathsf{D}_{-\varsigma^+}^{\varrho}\varphi)(\zeta)\mathrm{d}\zeta+\int_0^{\nu}\mathcal{P}_{\varrho}^{\varsigma}(\nu-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta.$$

Obviously, if the mapping $\varXi$ has a fixed point Υ, $\tilde{\Upsilon}$ is a solution of system (1.2) where

$$\tilde{\Upsilon}(\nu)=\begin{cases}\Upsilon(\nu), & \nu\in[0,\mathsf{T}],\\ \varphi(\nu), & \nu\in(-\varsigma,0].\end{cases}$$

Next, we adopt Theorem 2.6.

(i) $\varXi: C([0,\mathsf{T}],\mathbb{R}^n)\to C([0,\mathsf{T}],\mathbb{R}^n)$ is a continuous mapping. Assume that $\lim_{n\to\infty}\Upsilon_n=\Upsilon$, $\Upsilon_n\in C([0,\mathsf{T}],\mathbb{R}^n)$ and $\Upsilon\in C([0,\mathsf{T}],\mathbb{R}^n)$. For any $\nu\in[0,\mathsf{T}]$, we obtain

$$\begin{aligned}\|\varXi\Upsilon_n-\varXi\Upsilon\| &\leq \max_{\nu\in[0,\mathsf{T}]}\left\|\int_0^{\nu}\mathcal{P}_{\varrho}^{\varsigma}(\nu-\varsigma-\zeta)f(\zeta,\Upsilon_n(\zeta))\mathrm{d}\zeta-\int_0^{\nu}\mathcal{P}_{\varrho}^{\varsigma}(\nu-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta\right\|\\ &\leq \max_{\nu\in[0,\mathsf{T}]}\left\|\int_0^{\nu}\mathcal{P}_{\varrho}^{\varsigma}(\nu-\varsigma-\zeta)[f(\zeta,\Upsilon_n(\zeta))-f(\zeta,\Upsilon(\zeta))]\mathrm{d}\zeta\right\|.\end{aligned}$$

According to the Lemma 2.13, we obtain

$$\begin{aligned}\|\varXi\Upsilon_n-\varXi\Upsilon\| &\leq \max_{\nu\in[0,\mathsf{T}]}\left\|\int_0^{\nu}\mathcal{P}_{\varrho}^{\varsigma}(\nu-\varsigma-\zeta)\,[f(\zeta,\Upsilon_n(\zeta))-f(\zeta,\Upsilon(\zeta))]\,\mathrm{d}\zeta\right\|\\ &\leq \max_{\nu\in[0,\mathsf{T}]}E_{\varrho,\varrho}(\|\Omega\|\nu^{\varrho})\int_0^{\nu}(\nu-\zeta)^{\varrho-1}\,\|f(\zeta,\Upsilon_n(\zeta))-f(\zeta,\Upsilon(\zeta))\|\,\mathrm{d}\zeta.\end{aligned}$$

According to Lebesgue's dominated convergence theorem, we obtain $\|\varXi\Upsilon_n-\varXi\Upsilon\|\to 0$ as $n\to\infty$. Namely, the mapping $\varXi$ is a continuous mapping.

(ii) Denote $C_r=\{\Upsilon(\nu)\in\mathbb{R}^n:\|\Upsilon(\nu)\|\leq r\}$ and $O_r=\{\Upsilon\in C([0,\mathsf{T}],\mathbb{R}^n):\Upsilon(\nu)\in C_r,\nu\in[0,\mathsf{T}]\}$. Next we prove there exists $r>0$, which makes $\varXi O_r\subseteq O_r$, where O_r is a bounded closed convex subset of $C([0,\mathsf{T}],\mathbb{R}^n)$.

For any $\Upsilon\in O_r$, r is sufficiently large, $\nu\in((\mathsf{k}-1)\varsigma,\mathsf{k}\varsigma]$ and $\mathsf{k}\in\{1,2,\ldots,l\}$, on the basis of

Lemmas 2.9, 2.11, 2.12 and the assumption $[E_4]$, we obtain

$$\begin{aligned}\|\Xi\Upsilon(\nu)\| \leq& \|\mathcal{P}_{\varrho}^{\varsigma}(\nu)\mathbf{b}\| + \|H_{\varrho}^{\varsigma}(\nu)\mathbf{a}\| + \int_{-\varsigma}^{0} \|\mathcal{P}_{\varrho}^{\varsigma}(\nu-\varsigma-\zeta)({}^{R}\mathsf{D}_{-\varsigma^{+}}^{\varrho}\varphi)(\zeta)\|\mathrm{d}\zeta + \\ & \int_{0}^{\nu} \|\mathcal{P}_{\varrho}^{\varsigma}(\nu-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\|\mathrm{d}\zeta \\ \leq& \psi_1(\nu)\|\mathbf{b}\| + \psi_2(\nu)\|\mathbf{a}\| + M\psi_3^{\mathsf{k}}(\nu) + L\|\Upsilon(\zeta)\|\psi_4^{\mathsf{k}}(\nu) \\ \leq& \psi_1(\mathsf{T})\|\mathbf{b}\| + M\psi_3^{l}(\mathsf{T}) + Lr\psi_4^{l}(\mathsf{T}) + \sup_{\nu\in[0,\mathsf{T}]}\psi_2(\nu)\|\mathbf{a}\|,\end{aligned}$$

which means

$$\|\Xi\Upsilon\| = \sup_{\nu\in J}\|\Xi\Upsilon(\nu)\| \leq \psi_1(\mathsf{T})\|\mathbf{b}\| + M\psi_3^{l}(\mathsf{T}) + Lr\psi_4^{l}(\mathsf{T}) + \sup_{\nu\in[0,\mathsf{T}]}\psi_2(\nu)\|\mathbf{a}\|.$$

Since the formula $L\psi_4^{l}(\mathsf{T}) < 1$ holds, there exists a sufficiently large r, we obtain $\|\Xi\Upsilon\| \leq r$. In other words, $\Xi O_r \subseteq O_r$.

(iii) ΞO_r is a relative compact set in $C([0,\mathsf{T}],\mathbb{R}^n)$.

Firstly, for any $\Upsilon \in O_r$, that is, ΞO_r is a uniformly bounded set in $C([0,\mathsf{T}],\mathbb{R}^n)$.

Next, we prove ΞO_r is an equicontinuous set in $C([0,\mathsf{T}],\mathbb{R}^n)$. In fact, $\forall 0 \leq \nu_1 < \nu_2 \leq \mathsf{T}$ and $\Upsilon \in O_r$, we obtain

$$\begin{aligned}\|(\Xi\Upsilon)(\nu_2) - (\Xi\Upsilon)(\nu_1)\| =& \|\mathcal{P}_{\varrho}^{\varsigma}(\nu_2)\mathbf{b} + \mathcal{H}_{\varrho}^{\varsigma}(\nu_2)\mathbf{a} + \int_{-\varsigma}^{0} \mathcal{P}_{\varrho}^{\varsigma}(\nu_2-\varsigma-\zeta)({}^{R}\mathsf{D}_{-\varsigma^{+}}^{\varrho}\varphi)(\zeta)\mathrm{d}\zeta + \\ & \int_{0}^{\nu_2} \mathcal{P}_{\varrho}^{\varsigma}(\nu_2-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta - \mathcal{P}_{\varrho}^{\varsigma}(\nu_1)\mathbf{b} - \mathcal{H}_{\varrho}^{\varsigma}(\nu_1)\mathbf{a} - \\ & \int_{-\varsigma}^{0} \mathcal{P}_{\varrho}^{\varsigma}(\nu_1-\varsigma-\zeta)({}^{R}\mathsf{D}_{-\varsigma^{+}}^{\varrho}\varphi)(\zeta)\mathrm{d}\zeta - \int_{0}^{\nu_1} \mathcal{P}_{\varrho}^{\varsigma}(\nu_1-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta\| \\ \leq& \|(\mathcal{P}_{\varrho}^{\varsigma}(\nu_2) - \mathcal{P}_{\varrho}^{\varsigma}(\nu_1))\mathbf{b} + (\mathcal{H}_{\varrho}^{\varsigma}(\nu_2) - \mathcal{H}_{\varrho}^{\varsigma}(\nu_1))\mathbf{a} + \\ & \int_{-\varsigma}^{0} (\mathcal{P}_{\varrho}^{\varsigma}(\nu_2-\varsigma-\zeta) - \mathcal{P}_{\varrho}^{\varsigma}(\nu_1-\varsigma-\zeta))({}^{R}\mathsf{D}_{-\varsigma^{+}}^{\varrho}\varphi)(\zeta)\mathrm{d}\zeta + \\ & \int_{0}^{\nu_2} \mathcal{P}_{\varrho}^{\varsigma}(\nu_2-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta - \int_{0}^{\nu_1} \mathcal{P}_{\varrho}^{\varsigma}(\nu_1-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta\| \\ \leq& \|(\mathcal{P}_{\varrho}^{\varsigma}(\nu_2) - \mathcal{P}_{\varrho}^{\varsigma}(\nu_1))\mathbf{b}\| + \|(\mathcal{H}_{\varrho}^{\varsigma}(\nu_2) - \mathcal{H}_{\varrho}^{\varsigma}(\nu_1))\mathbf{a}\| + \\ & \Big\|\int_{-\varsigma}^{0} (\mathcal{P}_{\varrho}^{\varsigma}(\nu_2-\varsigma-\zeta) - \mathcal{P}_{\varrho}^{\varsigma}(\nu_1-\varsigma-\zeta))({}^{R}\mathsf{D}_{-\varsigma^{+}}^{\varrho}\varphi)(\zeta)\mathrm{d}\zeta\Big\| + \\ & \Big\|\int_{0}^{\nu_2} \mathcal{P}_{\varrho}^{\varsigma}(\nu_2-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta - \int_{0}^{\nu_1} \mathcal{P}_{\varrho}^{\varsigma}(\nu_1-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta\Big\|.\end{aligned}$$

Due to $\nu_1 < \nu_2$, one obtains

$$\begin{aligned}&\Big\|\int_{0}^{\nu_2} \mathcal{P}_{\varrho}^{\varsigma}(\nu_2-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta - \int_{0}^{\nu_1} \mathcal{P}_{\varrho}^{\varsigma}(\nu_1-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta\Big\| \\ &\leq \Big\|\int_{0}^{\nu_1} (\mathcal{P}_{\varrho}^{\varsigma}(\nu_2-\varsigma-\zeta) - \mathcal{P}_{\varrho}^{\varsigma}(\nu_1-\varsigma-\zeta))f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta + \int_{\nu_1}^{\nu_2} \mathcal{P}_{\varrho}^{\varsigma}(\nu_2-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta\Big\| \\ &\leq \Big\|\int_{0}^{\nu_1} (\mathcal{P}_{\varrho}^{\varsigma}(\nu_2-\varsigma-\zeta) - \mathcal{P}_{\varrho}^{\varsigma}(\nu_1-\varsigma-\zeta))f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta\Big\| + \Big\|\int_{\nu_1}^{\nu_2} \mathcal{P}_{\varrho}^{\varsigma}(\nu_2-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta\Big\|\end{aligned}$$

$$\leq Lr\int_0^{\nu_1}\|\mathcal{P}_\varrho^\varsigma(\nu_2-\varsigma-\zeta)-\mathcal{P}_\varrho^\varsigma(\nu_1-\varsigma-\zeta)\|\mathrm{d}\zeta+\Big\|\int_{\nu_1}^{\nu_2}\mathcal{P}_\varrho^\varsigma(\nu_2-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\mathrm{d}\zeta\Big\|.$$

In the light of Remark 2.4, we get that the function $\mathcal{P}_\varrho^\varsigma(\nu)$ is uniformly continuous in $[-\varsigma,\mathsf{T}]$, and thus, one acquires $\|\mathcal{P}_\varrho^\varsigma(\nu_2-\varsigma-\zeta)-\mathcal{P}_\varrho^\varsigma(\nu_1-\varsigma-\zeta)\|\to 0$ as $\nu_2\to\nu_1$. Accordingly, we can make a conclusion that $\|(\varXi\Upsilon)(\nu_2)-(\varXi\Upsilon)(\nu_1)\|\to 0$ as $\nu_2\to\nu_1$ for all $\Upsilon\in O_r$, and $\varXi O_r$ is an equicontinuous set in $C([0,\mathsf{T}],\mathbb{R}^n)$. According to Arzelà-Ascoli theorem, $\varXi O_r$ is a relative compact set in $C([0,\mathsf{T}],\mathbb{R}^n)$.

In conclusion, on the basis of Theorem 2.6, $\varXi$ has at least one fixed point Υ. That is to say, $\tilde{\Upsilon}$ is a solution of system (1.2). □

4. FTS results

Based on the explicit solution, we will further study the finite time stability. In this part, we assume that the solution of system (1.2) exists and meets the conditions in Section 3. What's more, Hölder inequality and Gronwall inequality are main tools to study FTS problems.

$[E_5]$ There exists a function $w(\cdot)\in C([0,\mathsf{T}],\mathbb{R}^+)$ such that $\|f(\zeta,\Upsilon)\|\leq w(\zeta)$, for $\zeta\in[0,\mathsf{T}]$ and $\Upsilon\in\mathbb{R}^n$.

$[E_6]$ There exists a $\psi(\cdot)\in L^q([0,\mathsf{T}],\mathbb{R}^+)$, $\frac{1}{q}=1-\frac{1}{p}$, $p>1$ such that $\|f(\zeta,\Upsilon)\|\leq\psi(\zeta)$ for $\zeta\in[0,\mathsf{T}]$ and $\Upsilon\in\mathbb{R}^n$.

Theorem 4.1 *Assume that $[E_3]$ and $[E_5]$ are met. System (1.2) is FTS regarding $\{0,J,\varsigma,\delta,\eta\}$ in the event of*

$$\delta\psi_1(\mathsf{T})+\sup_{\nu\in J}\delta\psi_2(\nu)+M\psi_3^l(\mathsf{T})+\|w\|_c\psi_4^l(\mathsf{T})<\eta.$$

Proof In terms of Lemmas 2.9, 2.11 and 2.12, for $\nu\in((\mathsf{k}-1)\varsigma,\mathsf{k}\varsigma]$ and $\mathsf{k}\in\{1,2,\ldots,l\}$, we have

$$\begin{aligned}\|\Upsilon(\nu)\|\leq&\|\mathcal{P}_\varrho^\varsigma(\nu)\mathbf{b}\|+\|\mathcal{H}_\varrho^\varsigma(\nu)\mathbf{a}\|+\int_{-\varsigma}^0\|\mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)({}^R\mathsf{D}_{-\varsigma^+}^\varrho\varphi)(\zeta)\|\mathrm{d}\zeta+\\&\int_0^\nu\|\mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\|\mathrm{d}\zeta\\\leq&\delta\|\mathcal{P}_\varrho^\varsigma(\nu)\|+\delta\|\mathcal{H}_\varrho^\varsigma(\nu)\|+M\int_{-\varsigma}^0\|\mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)\|\mathrm{d}\zeta+\\&\int_0^\nu\|\mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)\|w(\zeta)\mathrm{d}\zeta\\\leq&\delta\psi_1(\mathsf{T})+\sup_{\nu\in J}\delta\psi_2(\nu)+M\psi_3^l(\mathsf{T})+\|w\|_c\psi_4^l(\mathsf{T}),\end{aligned}$$

which implies

$$\|\Upsilon\|=\sup_{\nu\in J}\|\Upsilon(\nu)\|\leq\delta\psi_1(\mathsf{T})+\sup_{\nu\in J}\delta\psi_2(\nu)+M\psi_3^l(\mathsf{T})+\|w\|_c\psi_4^l(\mathsf{T})<\eta.$$

This proof is completed. □

Theorem 4.2 *Assume that $[E_3]$ and $[E_6]$ are met. System (1.2) is FTS in relation to $\{0,J,\varsigma,\delta,\eta\}$*

in the event of

$$\delta\psi_1(\mathsf{T})+\sup_{\nu\in[0,\mathsf{T}]}\delta\psi_2(\nu)+M\psi_3^l(\mathsf{T})+\|\psi\|_{L^q}\sum_{i=1}^{l}\Big(\frac{\|\Omega\|^{i-1}}{\Gamma(i\varrho)}\frac{(\mathsf{T}-(i-1)\varsigma)^{i\varrho-1+\frac{1}{p}}}{(pi\varrho-p+1)^{\frac{1}{p}}}\Big)<\eta.$$

Proof Here, we use mathematical induction to prove the results.

(1) When $\mathsf{k}=1$, $\nu\in(0,\varsigma]$, and on the basis of Lemmas 2.9 and 2.11, we have

$$\begin{aligned}
&\|\Upsilon(\nu)\|\\
&\le\|\mathcal{P}_\varrho^\varsigma(\nu)\mathbf{b}\|+\|\mathcal{H}_\varrho^\varsigma(\nu)\mathbf{a}\|+\int_{-\varsigma}^{0}\|\mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)({}^R\mathsf{D}_{-\varsigma^+}^\varrho\varphi)(\zeta)\|\mathrm{d}\zeta+\\
&\quad\int_0^\nu\|\mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\|\mathrm{d}\zeta\\
&\le\|\frac{(\nu-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\mathbf{b}\|+\|\Omega\frac{\nu^{2\varrho-1}}{\Gamma(2\varrho)}\mathbf{b}\|+\|\frac{(\nu-\zeta)^{\varrho-2}}{\Gamma(\varrho-1)}\mathbf{a}\|+\|\Omega\frac{\nu^{2\varrho-2}}{\Gamma(2\varrho-1)}\mathbf{a}\|+M\int_{-\varsigma}^{0}\frac{(\nu-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\mathrm{d}\zeta+\\
&\quad M\int_{-\varsigma}^{\nu-\varsigma}\|\Omega\|\frac{(\nu-\varsigma-\zeta)^{2\varrho-1}}{\Gamma(2\varrho)}\mathrm{d}\zeta+\|\psi\|_{L^q}\Big(\int_0^\nu\frac{(\nu-\zeta)^{p\varrho-p}}{(\Gamma(\varrho))^p}\mathrm{d}\zeta\Big)^{\frac{1}{p}}\\
&\le\delta\psi_1(\varsigma)+\sup_{\nu\in[0,\varsigma]}\delta\psi_2(\nu)+\Big[\frac{M}{\Gamma(\varrho+1)}[(\varsigma+\varsigma)^\varrho-\varsigma^\varrho]+\frac{M\|\Omega\|}{\Gamma(2\varrho+1)}\varsigma^\varrho\Big]+\frac{\|\psi\|_{L^q}}{\Gamma(\varrho)}\frac{\varsigma^{\varrho-1+\frac{1}{p}}}{(p\varrho-p+1)^{\frac{1}{p}}}\\
&\le\delta\psi_1(\varsigma)+\sup_{\nu\in[0,\varsigma]}\delta\psi_2(\nu)+M\psi_3^1(\varsigma)+\|\psi\|_{L^q}\sum_{i=1}^{1}\Big(\frac{\|\Omega\|^{i-1}}{\Gamma(i\varrho)}\frac{(\varsigma-(i-1)\varsigma)^{i\varrho-1+\frac{1}{p}}}{(pi\varrho-p+1)^{\frac{1}{p}}}\Big)\\
&\le\delta\psi_1(\mathsf{T})+\sup_{\nu\in[0,\mathsf{T}]}\delta\psi_2(\nu)+M\psi_3^l(\mathsf{T})+\|\psi\|_{L^q}\sum_{i=1}^{l}\Big(\frac{\|\Omega\|^{i-1}}{\Gamma(i\varrho)}\frac{(\mathsf{T}-(i-1)\varsigma)^{i\varrho-1+\frac{1}{p}}}{(pi\varrho-p+1)^{\frac{1}{p}}}\Big),
\end{aligned}$$

which implies

$$\sup_{\nu\in[0,\varsigma]}\|\Upsilon(\nu)\|\le\delta\psi_1(\mathsf{T})+\sup_{\nu\in[0,\mathsf{T}]}\delta\psi_2(\nu)+M\psi_3^l(\mathsf{T})+\|\psi\|_{L^q}\sum_{i=1}^{l}\Big(\frac{\|\Omega\|^{i-1}}{\Gamma(i\varrho)}\frac{(\mathsf{T}-(i-1)\varsigma)^{i\varrho-1+\frac{1}{p}}}{(pi\varrho-p+1)^{\frac{1}{p}}}\Big)<\eta.$$

(2) Similarly, we can prove other cases. In fact, let $(\mathsf{k}-1)\varsigma<\nu\le\mathsf{k}\varsigma$, $\mathsf{k}\in\{2,3,\dots,l\}$. According to Lemmas 2.9 and 2.11, one can show

$$\begin{aligned}
&\|\Upsilon(\nu)\|\\
&\le\|\mathcal{P}_\varrho^\varsigma(\nu)\mathbf{b}\|+\|\mathcal{H}_\varrho^\varsigma(\nu)\mathbf{a}\|+\int_{-\varsigma}^{0}\|\mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)({}^R\mathsf{D}_{-\varsigma^+}^\varrho\varphi)(\zeta)\|\mathrm{d}\zeta+\\
&\quad\int_0^\nu\|\mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\|\mathrm{d}\zeta\\
&\le\delta\psi_1(\mathsf{k}\varsigma)+\sup_{\nu\in[0,\mathsf{k}\varsigma]}\delta\psi_2(\nu)+M\int_{-\varsigma}^{0}\Big[\frac{(\nu-\zeta)^{\varrho-1}}{\Gamma(\varrho)}+\|\Omega\|\frac{(\nu-\varsigma-\zeta)^{2\varrho-1}}{\Gamma(\varrho+\varrho)}+\cdots+\\
&\quad\|\Omega\|^{\mathsf{k}-1}\frac{(\nu-(\mathsf{k}-1)\varsigma-\zeta)^{\mathsf{k}\varrho-1}}{\Gamma((\mathsf{k}-1)\varrho+\varrho)}\Big]\mathrm{d}\zeta+M\int_{-\varsigma}^{\nu-\mathsf{k}\varsigma}\|\Omega\|^{\mathsf{k}}\frac{(\nu-\mathsf{k}\varsigma-\zeta)^{(\mathsf{k}+1)\varrho-1}}{\Gamma(\mathsf{k}\varrho+\varrho)}\mathrm{d}\zeta+\\
&\quad\int_0^\nu\frac{(\nu-\zeta)^{\varrho-1}}{\Gamma(\varrho)}\psi(\zeta)\mathrm{d}\zeta+\cdots+\|\Omega\|^{\mathsf{k}-1}\int_0^{\nu-(\mathsf{k}-1)\varsigma}\frac{(\nu-(\mathsf{k}-1)\varsigma-\zeta)^{\mathsf{k}\varrho-1}}{\Gamma(\mathsf{k}\varrho)}\psi(\zeta)\mathrm{d}\zeta\\
&\le\delta\psi_1(\mathsf{k}\varsigma)+\sup_{\nu\in[0,\mathsf{k}\varsigma]}\delta\psi_2(\nu)+M\psi_3^{\mathsf{k}}(\mathsf{k}\varsigma)+
\end{aligned}$$

$$\|\psi\|_{L^q}\Big(\int_0^\nu \frac{(\nu-\zeta)^{p\varrho-p}}{(\Gamma(\varrho))^p}\mathrm{d}\zeta\Big)^{\frac{1}{p}}+\cdots+\|\Omega\|^{\mathsf{k}-1}\|\psi\|_{L^q}\Big(\int_0^\nu \frac{(\nu-(\mathsf{k}-1)\varsigma-\zeta)^{p\mathsf{k}\varrho-p}}{(\Gamma(\mathsf{k}\varrho))^p}\mathrm{d}\zeta\Big)^{\frac{1}{p}}$$

$$\leq \delta\psi_1(\mathsf{k}\varsigma)+\sup_{\nu\in[0,\mathsf{k}\varsigma]}\delta\psi_2(\nu)+M\psi_3^{\mathsf{k}}(\mathsf{k}\varsigma)+\|\psi\|_{L^q}\sum_{i=1}^{\mathsf{k}}\Big(\frac{\|\Omega\|^{i-1}}{\Gamma(i\varrho)}\frac{(\mathsf{k}\varsigma-(i-1)\varsigma)^{i\varrho-1+\frac{1}{p}}}{(pi\varrho-p+1)^{\frac{1}{p}}}\Big)$$

$$\leq \delta\psi_1(\mathsf{T})+\sup_{\nu\in[0,\mathsf{T}]}\delta\psi_2(\nu)+M\psi_3^{l}(\mathsf{T})+\|\psi\|_{L^q}\sum_{i=1}^{l}\Big(\frac{\|\Omega\|^{i-1}}{\Gamma(i\varrho)}\frac{(\mathsf{T}-(i-1)\varsigma)^{i\varrho-1+\frac{1}{p}}}{(pi\varrho-p+1)^{\frac{1}{p}}}\Big),$$

which implies

$$\sup_{\nu\in[(\mathsf{k}-1)\varsigma,\mathsf{k}\varsigma]}\|\Upsilon(\nu)\|\leq \delta\psi_1(\mathsf{T})+\sup_{\nu\in[0,\mathsf{T}]}\delta\psi_2(\nu)+ M\psi_3^{l}(\mathsf{T})+\|\psi\|_{L^q}\sum_{i=1}^{l}\Big(\frac{\|\Omega\|^{i-1}}{\Gamma(i\varrho)}\frac{(\mathsf{T}-(i-1)\varsigma)^{i\varrho-1+\frac{1}{p}}}{(pi\varrho-p+1)^{\frac{1}{p}}}\Big)<\eta.$$

In a word, when $\nu\in[0,\mathsf{T}]$ and $\mathsf{T}=l\varsigma$, we could make a conclusion

$$\|\Upsilon\|=\sup_{\nu\in[0,\mathsf{T}]}\|\Upsilon(\nu)\|\leq \delta\psi_1(\mathsf{T})+\sup_{\nu\in[0,\mathsf{T}]}\delta\psi_2(\nu)+ M\psi_3^{l}(\mathsf{T})+\|\psi\|_{L^q}\sum_{i=1}^{l}\Big(\frac{\|\Omega\|^{i-1}}{\Gamma(i\varrho)}\frac{(\mathsf{T}-(i-1)\varsigma)^{i\varrho-1+\frac{1}{p}}}{(pi\varrho-p+1)^{\frac{1}{p}}}\Big)<\eta.$$

The proof is completed. □

Theorem 4.3 *Assume that* $[E_3]$ *and* $[E_5]$ *are met. System* (1.2) *is FTS in relation to* $\{0,J,\varsigma,\delta,\eta\}$ *in the event of*

$$\delta\psi_1(\mathsf{T})+\sup_{\nu\in J}\delta\psi_2(\nu)+M\psi_3^{l}(\mathsf{T})+\frac{\|w\|_c\mathsf{T}^{\varrho}}{\varrho}E_{\varrho,\varrho}(\|\Omega\|\mathsf{T}^{\varrho})<\eta.$$

Proof By using the Lemmas 2.9, 2.11 and 2.13, for $\nu\in((\mathsf{k}-1)\varsigma,\mathsf{k}\varsigma]$ and $\mathsf{k}\in\{1,2,\dots,l\}$, we have

$$\begin{aligned}\|\Upsilon(\nu)\|\leq&\|\mathcal{P}_{\varrho}^{\varsigma}(\nu)\mathbf{b}\|+\|\mathcal{H}_{\varrho}^{\varsigma}(\nu)\mathbf{a}\|+\int_{-\varsigma}^{0}\|\mathcal{P}_{\varrho}^{\varsigma}(\nu-\varsigma-\zeta)({}^{R}\mathrm{D}_{-\varsigma^+}^{\varrho}\varphi)(\zeta)\|\mathrm{d}\zeta+\\&\int_0^\nu\|\mathcal{P}_{\varrho}^{\varsigma}(\nu-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\|\mathrm{d}\zeta\\\leq&\delta\|\mathcal{P}_{\varrho}^{\varsigma}(\nu)\|+\delta\|\mathcal{H}_{\varrho}^{\varsigma}(\nu)\|+M\int_{-\varsigma}^{0}\|\mathcal{P}_{\varrho}^{\varsigma}(\nu-\varsigma-\zeta)\|\mathrm{d}\zeta+\\&\int_0^\nu(\nu-\zeta)^{\varrho-1}E_{\varrho,\varrho}(\|\Omega\|\nu^{\varrho})w(\zeta)\mathrm{d}\zeta\\\leq&\delta\psi_1(\mathsf{T})+\sup_{\nu\in J}\delta\psi_2(\nu)+M\psi_3^{l}(\mathsf{T})+\frac{\|w\|_c\mathsf{T}^{\varrho}}{\varrho}E_{\varrho,\varrho}(\|\Omega\|\mathsf{T}^{\varrho}),\end{aligned}$$

which implies

$$\|\Upsilon\|=\sup_{\nu\in J}\|\Upsilon(\nu)\|\leq\delta\psi_1(\mathsf{T})+\sup_{\nu\in J}\delta\psi_2(\nu)+M\psi_3^{l}(\mathsf{T})+\frac{\|w\|_c\mathsf{T}^{\varrho}}{\varrho}E_{\varrho,\varrho}(\|\Omega\|\mathsf{T}^{\varrho})<\eta.$$

The proof is completed. □

Theorem 4.4 *Assume that* $[E_3]$ *and* $[E_4]$ *are met. System* (1.2) *is FTS in relation to* $\{0,J,\varsigma,\delta,\eta\}$

in the event of

$$\big(\delta\psi_1(\mathsf{T})+\sup_{\nu\in J}\delta\psi_2(\nu)+M\psi_3^l(\mathsf{T})\big)E_\varrho(L\Gamma(\varrho)E_{\varrho,\varrho}(\|\Omega\|\mathsf{T}^\varrho)\mathsf{T}^\varrho)<\eta.$$

Proof In accordance with the Lemmas 2.8, 2.9, 2.11 and 2.13, for $\nu\in((\mathsf{k}-1)\varsigma,\mathsf{k}\varsigma]$ and $\mathsf{k}\in\{1,2,\ldots,l\}$, we have

$$\begin{aligned}\|\Upsilon(\nu)\|\le&\|\mathcal{P}_\varrho^\varsigma(\nu)\mathbf{b}\|+\|\mathcal{H}_\varrho^\varsigma(\nu)\mathbf{a}\|+\int_{-\varsigma}^0\|\mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)({}^R\mathsf{D}^\varrho_{-\varsigma^+}\varphi)(\zeta)\|\mathrm{d}\zeta+\\&\int_0^\nu\|\mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)f(\zeta,\Upsilon(\zeta))\|\mathrm{d}\zeta\\\le&(\delta\psi_1(\nu)+\delta\psi_2(\nu)+M\psi_3^{\mathsf{k}}(\nu))+L\int_0^\nu\|\mathcal{P}_\varrho^\varsigma(\nu-\varsigma-\zeta)\|\|\Upsilon(\zeta)\|\mathrm{d}\zeta\\\le&(\delta\psi_1(\nu)+\delta\psi_2(\nu)+M\psi_3^{\mathsf{k}}(\nu))+LE_{\varrho,\varrho}(\|\Omega\|\nu^\varrho)\int_0^\nu(\nu-\zeta)^{\varrho-1}\|\Upsilon(\zeta)\|\mathrm{d}\zeta\\\le&(\delta\psi_1(\nu)+\delta\psi_2(\nu)+M\psi_3^{\mathsf{k}}(\nu))E_\varrho(L\Gamma(\varrho)E_{\varrho,\varrho}(\|\Omega\|\nu^\varrho)\nu^\varrho)\\\le&(\delta\psi_1(\mathsf{T})+\sup_{\nu\in J}\delta\psi_2(\nu)+M\psi_3^l(\mathsf{T}))E_\varrho(L\Gamma(\varrho)E_{\varrho,\varrho}(\|\Omega\|\mathsf{T}^\varrho)\mathsf{T}^\varrho),\end{aligned}$$

which implies

$$\|\Upsilon\|=\sup_{\nu\in J}\|\Upsilon(\nu)\|\le(\delta\psi_1(\mathsf{T})+\sup_{\nu\in J}\delta\psi_2(\nu)+M\psi_3^l(\mathsf{T}))E_\varrho(L\Gamma(\varrho)E_{\varrho,\varrho}(\|\Omega\|\mathsf{T}^\varrho)\mathsf{T}^\varrho)<\eta.$$

The proof is completed. □

5. Example

The rationality of the results is proved by an instance. In order to get more accurate results, we calculate the data in this article to four decimal places.

Example 5.1 Set $\varrho=1.6$, $\varsigma=0.3$, $l=3$ and $\mathsf{T}=0.9$. Consider

$$\begin{cases}{}^R\mathsf{D}^{1.6}_{-0.3^+}\Upsilon(\nu)=\Omega\Upsilon(\nu-0.3)+f(\nu,\Upsilon(\zeta)), & 0<\nu\le0.9,\\ \varphi(\nu)=\begin{pmatrix}(\nu+0.3)^2\\ \frac{(\nu+0.3)^3}{2}\end{pmatrix}, & -0.3<\nu\le0,\\ \mathsf{J}^{0.4}_{-0.3^+}\Upsilon(-0.3^+)=\mathbf{a}=\mathbf{0},\quad {}^R\mathsf{D}^{0.6}_{-0.3^+}\Upsilon(-0.3^+)=\mathbf{b}=\mathbf{0}, \end{cases}\tag{5.1}$$

where $\Upsilon(\nu)=\begin{pmatrix}\Upsilon_1(\nu)\\\Upsilon_2(\nu)\end{pmatrix}$, $\Omega=\begin{pmatrix}0.86&0\\0&0.9\end{pmatrix}$ and $f(\nu,\Upsilon(\nu))=\begin{pmatrix}\nu^2\frac{|\Upsilon_1(\nu)|}{1+|\Upsilon_1(\nu)|}\\\nu^3\frac{|\Upsilon_2(\nu)|}{1+|\Upsilon_2(\nu)|}\end{pmatrix}$.

On the basis of Definition 3.1 and $\nu\in[0,0.9]$, the solution to (5.1) could be displayed as follows:

$$\begin{aligned}\Upsilon(\nu)=\mathcal{P}^{0.3}_{1.6}(\nu)\mathbf{b}+\mathcal{H}^{0.3}_{1.6}(\nu)\mathbf{a}+\int_{-0.3}^0\mathcal{P}^{0.3}_{1.6}(\nu-0.3-\zeta)({}^R\mathsf{D}^{1.6}_{-0.3^+}\varphi)(\zeta)\mathrm{d}\zeta+\\\int_0^\nu\mathcal{P}^{0.3}_{1.6}(\nu-0.3-\zeta)\begin{pmatrix}\zeta^2\frac{|\Upsilon_1(\zeta)|}{1+|\Upsilon_1(\zeta)|}\\\zeta^3\frac{|\Upsilon_2(\zeta)|}{1+|\Upsilon_2(\zeta)|}\end{pmatrix}\mathrm{d}\zeta,\end{aligned}$$

where

$$\int_{-0.3}^{0} \mathcal{P}_{1.6}^{0.3}(\nu-0.3-\zeta)({}^{R}\mathsf{D}_{-0.3^+}^{1.6}\varphi)(\zeta)\mathrm{d}\zeta = \int_{-0.3}^{0} \mathcal{P}_{1.6}^{0.3}(\nu-0.3-\zeta)\begin{pmatrix}\frac{3.36}{\Gamma(0.4)}(\zeta+0.3)^{0.4}B[3,0.4]\\ \frac{4.08}{\Gamma(0.4)}(\zeta+0.3)^{1.4}B[4,0.4]\end{pmatrix}\mathrm{d}\zeta.$$

When we estimate the norm of $\Upsilon(\nu)$, we can consider it as a linear problem because of functions $\frac{|y_1(\nu)|}{1+|y_1(\nu)|} < 1$ and $\frac{|y_2(\nu)|}{1+|y_2(\nu)|} < 1$ (see Figure 1).

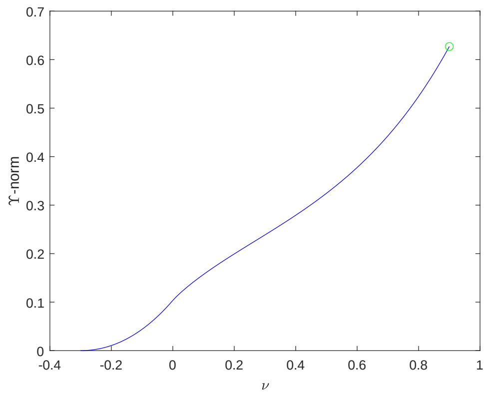

Figure 1 The norm of Υ

Obviously, $\|f(\nu,\Upsilon)-f(\nu,z)\| \le (\nu^2+\nu^3)\|\Upsilon-z\|$ for $\Upsilon, z \in \mathbb{R}^n$ and $\|f(\nu,\Upsilon)\| < \nu^2+\nu^3$, $\forall\,\nu \in [0,0.9]$. Assume that $p=2$, $q=2$, $K=L=1.5390$, $\rho=0.9444$ and $w(\nu)=\psi(\nu)=\nu^2+\nu^3$.

After a simple calculation, one obtains $\|\varphi\|=0.1035$, $M=1.8402$, $\|w\|=1.5390$, $E_{\varrho,\varrho}(\|\Omega\|\mathsf{T}^{\varrho}) = 1.4672$ and $E_{\varrho}(L\Gamma(\varrho)E_{\varrho,\varrho}(\|\Omega\|\mathsf{T}^{\varrho})\mathsf{T}^{\varrho}) = 2.6306$. Furthermore, $\psi_1(\nu) \le \psi_1(0.9) = 1.9024$, $\psi_2(\nu) \le \sup_{\nu\in[0,0.9]}\psi_2(\nu) = 1.9649$, $\psi_3^3(\nu) \le \psi_3^3(0.9) = 0.4064$ and $\psi_4^3(\nu) \le \psi_4^3(0.9) = 0.6136$.

Let $\delta = 0.12$. We obtain Table 1.

Theorem	$\|\varphi\|$	ϱ	T	ς	δ	$\|\Upsilon\|$	η	**FTS**
4.1	0.1035	1.6	0.9	0.3	0.12	2.1563	2.16	Yes
4.2	0.1035	1.6	0.9	0.3	0.12	1.6414 (Optimal)	1.65	Yes
4.3	0.1035	1.6	0.9	0.3	0.12	2.4043	2.41	Yes
4.4	0.1035	1.6	0.9	0.3	0.12	3.9147	3.92	Yes

Table 1 FTS results of system (5.1)

Analyses. In the light of Definition 2.7, we ought to determine a relative threshold η to ensure the system is FTS. Table 1 tells us the FTS results of system (5.1) when the fixed time $\mathsf{T} = 0.9$. From the data in Table 1, we can get a relatively optimum threshold $\eta = 1.65$.

Acknowledgements We thank the referees for their time and comments.

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