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Cohyponormal Weighted Composition Operators on the Fock Space over $\mathbb{C}^N$

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Abstract In this paper, cohyponormal weighted composition operators on the Fock space over $\mathbb{C}^N$ are characterized completely. We also consider a class of weighted composition operators on the Fock space over $\mathbb{C}^N$ which are both posinormal and coposinormal. As an application, we obtain the characterization of hyponormal, cohyponormal, posinormal and coposinormal composition operators on the Fock space over $\mathbb{C}^N$.

Keywords Fock space; weighted composition operator; hyponormal; cohyponormal; posinormal; coposinormal

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1. Introduction

Let $\mathbb{C}^N$ $(N \geq 1)$ be the N-dimensional complex Euclidean space and $\mathrm{d}m_{2N}$ be the usual area measure on $\mathbb{C}^N$. For $z, w \in \mathbb{C}^N$, $z \cdot w$ denotes the dot product in $\mathbb{C}^N$ and $|z|^2 = z \cdot z$. The Fock space $\mathcal{F}^2$ over $\mathbb{C}^N$ is the Hilbert space of analytic functions f on $\mathbb{C}^N$ with inner product

$$\langle f, g\rangle = \frac{1}{(2\pi)^N} \int_{\mathbb{C}^N} f(z)\overline{g(z)} e^{-\frac{|z|^2}{2}} \mathrm{d}m_{2N}(z), \quad f, g \in \mathcal{F}^2.$$

$\mathcal{F}^2$ is a reproducing kernel Hilbert space with reproducing kernel functions

$$K_w(z) = e^{\frac{z\cdot w}{2}}, \quad w, z \in \mathbb{C}^N.$$

Let k_w be the normalization of K_w, that is,

$$k_w(z) = \frac{K_w(z)}{\|K_w\|} = e^{\frac{z\cdot w}{2} - \frac{|w|^2}{4}}.$$

Fock space is not only an important space of analytic functions, but also plays an important role in quantum physics, harmonic analysis, and partial differential equations. We refer readers to the book by Zhu [1] for more information on the Fock spaces over $\mathbb{C}$ and operator theory on them.

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Let $\psi \in \mathcal{F}^2$ and φ be an analytic self-map of $\mathbb{C}^N$. The weighted composition operator $C_{\psi,\varphi}$ on $\mathcal{F}^2$ induced by ψ, φ is defined as

$$C_{\psi,\varphi} f = \psi(f \circ \varphi), \quad f \in \mathcal{F}^2.$$

If $\psi(z) = 1$, then $C_{\psi,\varphi}$ is the composition operator C_φ. If $\varphi(z) = z$, then $C_{\psi,\varphi}$ is the multiplication operator M_ψ. Due to the Liouville Theorem, it is well-known that M_ψ is bounded on $\mathcal{F}^2$ if and only if ψ is constant. The boundedness and compactness of composition operators on $\mathcal{F}^2$ were characterized by Carswell, MacCluer and Schuster [2], in which it was first established that a composition operator C_φ is bounded on the Fock space over $\mathbb{C}^N$, then $\varphi(z) = Az + b$ with A an operator on $\mathbb{C}^N$ and $b \in \mathbb{C}^N$. In 2007, Ueki characterized bounded and compact weighted composition operators on $\mathcal{F}^2$ by properties of the corresponding Berezin transforms of the operators [3]. In 2014, Le gave new and simple characterizations for bounded and compact weighted composition operators on $\mathcal{F}^2$ in the case of $N = 1$ (see [4]), in which it was first proved that if $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$, then $\varphi(z) = az + b$ for some $a, b \in \mathbb{C}$. Le's results were extended completely to the case of several variables in [5] and the norms and essential norms of weighted composition operators on Fock spaces are also estimated [5]. Normal and isometric weighted composition operators on $\mathcal{F}^2$ in the case of $N = 1$ were also completely characterized in [4]. We should note that many ideas in [4] play an important role in the characterization of a great variety of weighted composition operators on $\mathcal{F}^2$ later. Unitary, normal, isometric, invertible and Fredholm weighted composition operators on $\mathcal{F}^2$ over $\mathbb{C}^N$ and their spectrum are characterized completely [6–10]. In [11], complex symmetric weighted composition operators on $\mathcal{F}^2$ over $\mathbb{C}^N$ are considered. The (weighted) composition operators on Fock-type spaces are also studied [12, 13].

In this paper, we study cohyponormal weighted composition operators on $\mathcal{F}^2$ and extend the corresponding result [14, Theorem 2.6] in the case of $\mathbb{C}$ to the case of $\mathbb{C}^N$. We also consider a class of weighted composition operator on $\mathcal{F}^2$ which are both posinormal and coposinormal.

Recall that a bounded linear operator T on a Hilbert space H is said to be hyponormal if

$$TT^* \leq T^*T.$$

If T^* is hyponormal, then T is called cohyponormal, that is $T^*T \leq TT^*$. Moreover, T is called posinormal if there exists a positive operator P on H such that

$$TT^* = T^*PT.$$

If T^* is posinormal, then we say that T is coposinormal. Posinormal operators and coposinormal operators are introduced by Rhaly firstly in [15]. In [15], Rhaly gives some useful equivalent characterizations for posinormality. In particular, T is posinormal if and only if there exists a positive constant λ such that

$$TT^* \leq \lambda T^*T.$$

It follows that hyponormal operators are posinormal operators. Obviously, hyponormality and posinormality are unitary invariants.

Since it follows from [4, Proposition 2.1] and [10, Proposition 7] that if $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$, then there exists an operator A on $\mathbb{C}^N$ with the norm of A, $|A| \le 1$ and $b \in \mathbb{C}^N$ such that

$$\varphi(z) = Az + b, \quad z \in \mathbb{C}^N,$$

in the following, we always assume that $\varphi(z) = Az + b$ with A an operator on $\mathbb{C}^N$, $|A| \le 1$ and $b \in \mathbb{C}^N$.

Now we state the main results in this paper.

Theorem 1.1 *Let $\psi \in \mathcal{F}^2$, $\psi \neq 0$ and $\varphi(z) = Az + b$, where A is an operator on $\mathbb{C}^N$ with $|A| \le 1$ and $b \in \mathbb{C}^N$. Suppose that $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$. Then the following statements are equivalent:*

(i) $C_{\psi,\varphi}$ is cohyponormal;

(ii) $C_{\psi,\varphi}$ is normal;

(iii) A is normal on $\mathbb{C}^N$ and there exists a vector $c \in \mathbb{C}^N$ and a nonzero constant s such that $\psi(z) = sK_c(z)$ and $Ac + b = A^*b + c$, $|b| = |c|$.

Theorem 1.2 *Let $\psi(z) = K_c(z)$ and $\varphi(z) = Az + b$, where A is an operator on $\mathbb{C}^N$ with $|A| \le 1$ and $b, c \in \mathbb{C}^N$. Suppose that $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$. Then the following statements are equivalent:*

(i) $C_{\psi,\varphi}$ is posinormal;

(ii) $C_{\psi,\varphi}$ is coposinormal;

(iii) A is normal on $\mathbb{C}^N$ and $Ac + b = A^*b + c$. In this case,

$$C_{\psi,\varphi}C^*_{\psi,\varphi} = \exp(\frac{|b|^2 - |c|^2}{2})C^*_{\psi,\varphi}C_{\psi,\varphi}.$$

Corollary 1.3 *Let $\varphi(z) = Az + b$, where A is an operator on $\mathbb{C}^N$ with $|A| \le 1$ and $b \in \mathbb{C}^N$. Suppose that C_φ is bounded on $\mathcal{F}^2$. Then the following statements are equivalent:*

(i) C_φ is hyponormal;

(ii) C_φ is cohyponormal;

(iii) C_φ is posinormal;

(iv) C_φ is coposinormal;

(v) C_φ is normal;

(vi) A is normal on $\mathbb{C}^N$ and $b = 0$.

Although Theorem 1.1 and Corollary 1.3 show that except the normal (weighted) composition operators, there are no nontrivial cohyponormal weighted composition operators and posinormal (coposinormal) composition operators on the Fock space, Theorem 1.2 implies that there exist nontrivial posinormal and coposinormal weighted composition operators on the Fock space. For more information on hyponormal, cohyponormal, posinormal and coposinormal (weighted) composition operators on other analytic function spaces such as Hardy spaces and Bergman spaces, see [16–21] and the references therein.

2. Proof of main results

In this section, we begin gathering some elementary properties of weighted composition operators on $\mathcal{F}^2$. Then we give proofs of main results.

Lemma 2.1 *Let $\psi_1, \psi_2 \in \mathcal{F}^2$ and φ_1, φ_2 be analytic self-maps of $\mathbb{C}^N$. If $C_{\psi_1,\varphi_1}, C_{\psi_2,\varphi_2}$ are bounded on $\mathcal{F}^2$, then*

$$C_{\psi_1,\varphi_1} C_{\psi_2,\varphi_2} = C_{\psi_1\cdot(\psi_2\circ\varphi_1),\ \varphi_2\circ\varphi_1}.$$

Lemma 2.2 *Let $\psi \in \mathcal{F}^2$ and φ be an analytic self-map of $\mathbb{C}^N$. If $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$, then for any $z \in \mathbb{C}^N$,*

$$C^*_{\psi,\varphi} K_z = \overline{\psi(z)} K_{\varphi(z)}.$$

Lemmas 2.1 and 2.2 are well-known and the conclusions hold in any Hilbert space with reproducing kernels. The following result follows in [10, Lemma 6, Proposition 7]. For completeness, we give a brief proof here.

Lemma 2.3 *Let $\psi \in \mathcal{F}^2, \psi \neq 0$ and $\varphi(z) = Az + b$, where A is an operator on $\mathbb{C}^N$ with $|A| \leq 1$ and $b \in \mathbb{C}^N$. Suppose that $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$. If $|A\xi| = |\xi|$ for $\xi \in \mathbb{C}^N$, then there exists a constant $s(\xi) \in \mathbb{C}$ such that*

$$\psi_\xi(u) := \psi(u\xi) = s(\xi) \exp(-\frac{A\xi \cdot b}{2} u - \frac{|b|^2}{4}), \quad u \in \mathbb{C}.$$

As a consequence,

$$\psi(\xi) = \psi(0) \exp(-\frac{A\xi \cdot b}{2}). \tag{2.1}$$

Proof Since $C_{\psi,\varphi}$ is bounded, there exists a positive constant M such that for any $w \in \mathbb{C}^N$,

$$|\psi(w)|^2 \|K_{\varphi(w)}\|^2 = \|C^*_{\psi,\varphi} K_w\|^2 \leq M \|K_w\|^2.$$

It follows from the facts $\|K_w\|^2 = e^{\frac{|w|^2}{2}}$ and $\varphi(w) = Aw + b$ that

$$|\psi(w)|^2 \exp(\frac{|Aw+b|^2}{2} - \frac{|w|^2}{2}) \leq M, \quad w \in \mathbb{C}^N,$$

that is

$$|\psi(w) \exp(\frac{Aw \cdot b}{2} + \frac{|b|^2}{4})|^2 \leq M \exp(\frac{|w|^2 - |Aw|^2}{2}), \quad w \in \mathbb{C}^N. \tag{2.2}$$

Assume $|A\xi| = |\xi|$ for $\xi \in \mathbb{C}^N$. If $\xi = 0$, then $\psi_\xi(u) = \psi(0)$ for any $u \in \mathbb{C}$. In this case, take $s(\xi) = \psi(0) e^{\frac{|b|^2}{4}}$. If $|\xi| \neq 0$, then by (2.2), $\psi_\xi(u) \exp(\frac{A\xi \cdot b}{2} u + \frac{|b|^2}{4})$ is a bounded analytic function on $\mathbb{C}$. Hence there exists a constant $s(\xi)$ such that

$$\psi_\xi(u) \exp(\frac{A\xi \cdot b}{2} u + \frac{|b|^2}{4}) = s(\xi).$$

That is

$$\psi_\xi(u) = s(\xi) \exp(-\frac{A\xi \cdot b}{2} u - \frac{|b|^2}{4}), \quad u \in \mathbb{C}.$$

Let $u = 0$ and $u = 1$, respectively. Then we have $\psi(0) = s(\xi) \exp(-\frac{|b|^2}{4})$ and

$$\psi(\xi) = \psi(0) \exp(-\frac{A\xi \cdot b}{2}). \quad \square$$

Lemma 2.4 *Let* $\varphi(z) = Az + b$ *and* $\psi(z) = K_c(z)$ *with* A *an operator on* $\mathbb{C}^N$, $|A| \leq 1$ *and* $b, c \in \mathbb{C}^N$. *Suppose that* $C_{\psi,\varphi}$ *is bounded on* $\mathcal{F}^2$. *Then* $C^*_{\psi,\varphi} = C_{\psi_1,\varphi_1}$ *with*

$$\psi_1(z) = K_b(z), \quad \varphi_1(z) = A^*z + c, \; z \in \mathbb{C}^N.$$

Proof Since

$$\begin{aligned}(C_{\psi,\varphi}K_z)(w) = \psi(w)K_z(\varphi(w)) &= \exp\Big(\frac{w \cdot c}{2} + \frac{(Aw + b) \cdot z}{2}\Big) \\ &= \exp\Big(\frac{b \cdot z}{2} + \frac{w \cdot (c + A^*z)}{2}\Big) \\ &= \overline{K_b(z)} K_{A^*z+c}(w),\end{aligned}$$

we have

$$C_{\psi,\varphi}K_z = \overline{K_b(z)} K_{A^*z+c}.$$

Therefore, for any $f \in \mathcal{F}^2$,

$$(C^*_{\psi,\varphi}f)(z) = \langle C^*_{\psi,\varphi}f, K_z\rangle = \langle f, C_{\psi,\varphi}K_z\rangle = K_b(z)f(A^*z + c) = (C_{\psi_1,\varphi_1}f)(z). \quad \square$$

For $p \in \mathbb{C}^N$, let k_p be the normalization of K_p, $\varphi_p(z) = z - p$, $z \in \mathbb{C}^N$ and $U_p = C_{k_p,\varphi_p}$.

Lemma 2.5 ([7, Proposition 2.3]) U_p *is a unitary operator on* $\mathcal{F}^2$ *and* $U_p^{-1} = U_{-p}$.

Lemma 2.6 *Let* $\psi \in \mathcal{F}^2$, $\psi \neq 0$ *and* $\varphi(z) = Az + b$, *where* A *is an operator on* $\mathbb{C}^N$ *with* $|A| \leq 1$ *and* $b \in \mathbb{C}^N$. *Suppose that* $C_{\psi,\varphi}$ *is bounded on* $\mathcal{F}^2$. *There there exist* $\Psi \in \mathcal{F}^2$, $d \in \mathbb{C}^N$ *and* $\Phi(z) = Az + d$ *such that* $Ad = d$ *and* $C_{\psi,\varphi}$ *is unitarily equivalent to* $C_{\Psi,\Phi}$.

Proof Let

$$b = c + d, \quad c \in \operatorname{ran}(I - A), d \in (\operatorname{ran}(I - A))^{\perp}.$$

Then there exists a vector $p \in \mathbb{C}^N$ such that $(I - A)p = c$.

Denote $\Phi = \varphi_p \circ \varphi \circ \varphi_{-p}$ and $\Psi = k_{-p} \cdot (\psi \circ \varphi_{-p}) \cdot (k_p \circ \varphi \circ \varphi_{-p})$. Then

$$\Phi(z) = \varphi(z + p) - p = A(z + p) + b - p = Az + d$$

and $Ad = d$ since $d \in (ran(I - A))^{\perp} = \ker(I - A^*)$ and $|A| \leq 1$ (see [8, Lemma 2.8]).

By Lemma 2.1,

$$U_{-p}C_{\psi,\varphi}U_p = C_{k_{-p}(\psi\circ\varphi_{-p})(k_p\circ\varphi\circ\varphi_{-p}),\varphi_p\circ\varphi\circ\varphi_{-p}} = C_{\Psi,\Phi}.$$

It follows from Lemma 2.5 that $C_{\psi,\varphi}$ is unitarily equivalent to $C_{\Psi,\Phi}$. $\square$

Now we present proofs of our main results.

Proof of Theorem 1.1 Let $\psi \in \mathcal{F}^2$, $\psi \neq 0$ and $\varphi(z) = Az + b$, where A is an operator on $\mathbb{C}^N$ with $|A| \leq 1$ and $b \in \mathbb{C}^N$. Suppose that $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$. By Lemma 2.6, without loss of generality, we assume that $Ab = b$.

If $C_{\psi,\varphi}$ is cohyponormal on $\mathcal{F}^2$, then

$$\|C_{\psi,\varphi}K_b\|^2 \leq \|C^*_{\psi,\varphi}K_b\|^2 = |\psi(b)|^2\|K_{\varphi(b)}\|^2. \tag{2.3}$$

Since $Ab = b$, applying Eq. (2.1) in Lemma 2.3, we have

$$\psi(b) = \psi(0)\exp(-\frac{Ab\cdot b}{2}) = \psi(0)\exp(-\frac{|b|^2}{2}).$$

Since $\varphi(b) = 2b$, the right hand side of formula (2.3) becomes

$$|\psi(b)|^2 \exp(2|b|^2) = |\psi(0)|^2 \exp(-|b|^2)\exp(2|b|^2) = |\psi(0)|^2 \exp(|b|^2).$$

Define

$$F(z) = (C_{\psi,\varphi}K_b)(z) = \psi(z)K_b(\varphi(z)).$$

Then

$$F(0) = \psi(0)K_b(\varphi(0)) = \psi(0)K_b(b) = \psi(0)\exp(\frac{|b|^2}{2}).$$

Formula (2.3) then says

$$\|F\|^2 \le |F(0)|^2.$$

This implies that F must be a constant function and hence

$$\begin{aligned}\psi(z) &= F(0)K_{-b}(\varphi(z)) = \psi(0)\exp(\frac{|b|^2}{2})K_{-b}(Az+b)\\ &= \psi(0)\exp(\frac{|b|^2 + (Az+b)\cdot(-b)}{2})\\ &= \psi(0)\exp(\frac{z\cdot(-A^*b)}{2})\\ &= \psi(0)\exp(\frac{z\cdot(-b)}{2})\\ &= \psi(0)K_{-b}(z),\quad z\in\mathbb{C}^N.\end{aligned}$$

Since

$$(I-A^*)b = (I-A)(-b) = 0,\quad |-b| = |b|,$$

it follows from [8, Theorem 1.1] that $C_{\psi,\varphi}$ is normal. (i) $\Rightarrow$ (ii) is proved. (ii) $\Rightarrow$ (i) is obvious. Hence (i) is equivalent to (ii).

The equivalence of (ii) and (iii) follows from [8, Theorem 1.1]. We complete the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2 Let $\psi(z) = K_c(z)$ and $\varphi(z) = Az + b$, where A is an operator on $\mathbb{C}^N$ with $|A| \le 1$ and $b, c \in \mathbb{C}^N$. Suppose that $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$. Then by Lemma 2.4, $C^*_{\psi,\varphi} = C_{\psi_1,\varphi_1}$ with

$$\psi_1(z) = K_b(z),\ \varphi_1(z) = A^*z + c,\quad z\in\mathbb{C}^N.$$

Hence by Lemma 2.1,

$$C^*_{\psi,\varphi}C_{\psi,\varphi} = C_{\psi_1,\varphi_1}C_{\psi,\varphi} = C_{\psi_1\cdot\psi\circ\varphi_1,\varphi\circ\varphi_1},$$

$$C_{\psi,\varphi}C^*_{\psi,\varphi} = C_{\psi,\varphi}C_{\psi_1,\varphi_1} = C_{\psi\cdot\psi_1\circ\varphi,\varphi_1\circ\varphi}.$$

A direct computation shows that

$$\Psi(z) := (\psi_1\cdot\psi\circ\varphi_1)(z) = \psi_1(z)\psi(\varphi_1(z)) = K_b(z)K_c(A^*z+c)$$

$$= \exp(\frac{z\cdot b}{2}+\frac{(A^*z+c)\cdot c}{2}) = \exp(\frac{z\cdot(Ac+b)}{2}+\frac{|c|^2}{2})$$
$$= \exp\frac{|c|^2}{2}K_{Ac+b}(z), \tag{2.4}$$
$$\Phi(z) := (\varphi\circ\varphi_1)(z) = \varphi(\varphi_1(z)) = A(A^*z+c)+b = AA^*z+Ac+b, \tag{2.5}$$

$$\Psi_1(z) := (\psi\cdot\psi_1\circ\varphi)(z) = \psi(z)\psi_1(\varphi(z)) = K_c(z)K_b(Az+b)$$
$$= \exp(\frac{z\cdot c}{2}+\frac{(Az+b)\cdot b}{2}) = \exp(\frac{z\cdot(A^*b+c)}{2}+\frac{|b|^2}{2})$$
$$= \exp\frac{|b|^2}{2}K_{A^*b+c}(z), \tag{2.6}$$
$$\Phi_1(z) := (\varphi_1\circ\varphi)(z) = \varphi_1(\varphi(z)) = A^*(Az+b)+c = A^*Az+A^*b+c \tag{2.7}$$

and

$$\begin{aligned}
\langle C^*_{\psi,\varphi}C_{\psi,\varphi}K_w,K_w\rangle &= \langle C_{\Psi,\Phi}K_w,K_w\rangle = \langle K_w, C^*_{\Psi,\Phi}K_w\rangle \\
&= \langle K_w, \overline{\Psi(w)}K_{\Phi(w)}\rangle = \Psi(w)K_w(\Phi(w)) \\
&= \exp(\frac{|c|^2}{2}+\frac{w\cdot(Ac+b)}{2}+\frac{(AA^*w+Ac+b)\cdot w}{2}) \\
&= \exp(\frac{|c|^2}{2}+\frac{w\cdot(Ac+b)+(Ac+b)\cdot w}{2}+\frac{|A^*w|^2}{2}), \\
\langle C_{\psi,\varphi}C^*_{\psi,\varphi}K_w,K_w\rangle &= \langle C_{\Psi_1,\Phi_1}K_w,K_w\rangle = \langle K_w, C^*_{\Psi_1,\Phi_1}K_w\rangle \\
&= \langle K_w, \overline{\Psi_1(w)}K_{\Phi_1(w)}\rangle = \Psi_1(w)K_w(\Phi_1(w)) \\
&= \exp(\frac{|b|^2}{2}+\frac{w\cdot(A^*b+c)}{2}+\frac{(A^*Aw+A^*b+c)\cdot w}{2}) \\
&= \exp(\frac{|b|^2}{2}+\frac{w\cdot(A^*b+c)+(A^*b+c)\cdot w}{2}+\frac{|Aw|^2}{2}).
\end{aligned}$$

If $C_{\psi,\varphi}$ is posinormal, then there exists a positive constant λ such that

$$C_{\psi,\varphi}C^*_{\psi,\varphi} \le \lambda C^*_{\psi,\varphi}C_{\psi,\varphi}.$$

So for all $w\in\mathbb{C}^N$,

$$\begin{aligned}
&\exp(\frac{|b|^2}{2}+\frac{w\cdot(A^*b+c)+(A^*b+c)\cdot w}{2}+\frac{|Aw|^2}{2}) \\
&\le \lambda\exp(\frac{|c|^2}{2}+\frac{w\cdot(Ac+b)+(Ac+b)\cdot w}{2}+\frac{|A^*w|^2}{2}).
\end{aligned}$$

That is

$$\exp(\frac{|b|^2-|c|^2+w\cdot((A^*b+c)-(Ac+b))+((A^*b+c)-(Ac+b))\cdot w+|Aw|^2-|A^*w|^2}{2}) \le \lambda. \tag{2.8}$$

By the arbitrariness of $w\in\mathbb{C}^N$, it follows that

$$|Aw|^2-|A^*w|^2 \le 0,$$

which implies that A is a cohyponormal operator on $\mathbb{C}^N$. It is well-known that an operator on $\mathbb{C}^N$ is hyponormal (cohyponormal) if and only if it is normal. Hence A is normal. Then by the normality of A and (2.8), we have

$$w\cdot((A^*b+c)-(Ac+b))+((A^*b+c)-(Ac+b))\cdot w=0$$

for all $w\in\mathbb{C}^N$. It follows that

$$Ac+b=A^*b+c.$$

Therefore, (iii) follows from (i).

Similarly, we can obtain that (ii)⇒(iii).

If (iii) holds, then by (2.4)–(2.7), we have

$$\exp(-\frac{|b|^2}{2})\Psi_1(z)=\exp(-\frac{|c|^2}{2})\Psi(z),\quad \Phi_1(z)=\Phi(z).$$

So

$$\exp(-\frac{|b|^2}{2})C_{\psi,\varphi}C^*_{\psi,\varphi}=\exp(-\frac{|c|^2}{2})C^*_{\psi,\varphi}C_{\psi,\varphi}.$$

Therefore, $C_{\psi,\varphi}$ is both posinormal and coposinormal. (iii) $\Rightarrow$ (i) and (ii) follows.

The proof is completed. □

In the last, we give a sketch for the proof of Corollary 1.3.

Let $\varphi(z)=Az+b$, where A is an operator on $\mathbb{C}^N$ with $|A|\le 1$ and $b\in\mathbb{C}^N$. Suppose that C_φ is bounded on $\mathcal{F}^2$. Under these conditions, the statement (iii) in Theorem 1.2 is equivalent to that A is a normal operator on $\mathbb{C}^N$ and $A^*b=b$. On the other hand, it follows from [22, Lemma 5.2] that $b\in\ker(I-A^*)^\perp$ if C_φ is bounded on $\mathcal{F}^2$ with $\varphi(z)=Az+b$. Therefore, the statements (iii) in Theorem 1.2 is reduced to the statement (vi) in Corollary 1.3. The equivalence of (v) and (vi) in Corollary 1.3 is well-known [23, Lemma 2.4]. Based on the reasoning above, Corollary 1.3 easily follows from Theorem 1.2.

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