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Unconditional and Optimal Pointwise Error Estimates of Finite Difference Methods for the Two-Dimensional Complex Ginzburg-Landau Equation

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Abstract In this paper, we give improved error estimates for linearized and nonlinear Crank-Nicolson type finite difference schemes of Ginzburg-Landau equation in two dimensions. For linearized Crank-Nicolson scheme, we use mathematical induction to get unconditional error estimates in discrete L^2 and H^1 norm. However, it is not applicable for the nonlinear scheme. Thus, based on a 'cut-off' function and energy analysis method, we get unconditional L^2 and H^1 error estimates for the nonlinear scheme, as well as boundedness of numerical solutions. In addition, if the assumption for exact solutions is improved compared to before, unconditional and optimal pointwise error estimates can be obtained by energy analysis method and several Sobolev inequalities. Finally, some numerical examples are given to verify our theoretical analysis.

Keywords complex Ginzburg-Landau equation; finite difference method; unconditional convergence; optimal estimates; pointwise error estimates

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1. Introduction

Complex valued Ginzburg Landau (GL) equation is an important nonlinear evolution equation in the fields of physics and mechanics [1]. It is widely used in fields such as superfluid, superconductivity, Bose Einstein condensation and so on. In this paper, we focus on investigating the following complex Ginzburg-Landau (GL) equation in two dimensions

$$\partial_t u - (\nu + i\alpha)\Delta u + (\kappa + i\beta)\left|u\right|^2 u - \gamma u = 0, \quad (x,y) \in \Omega,\ 0 < t \le T, \tag{1.1}$$

$$u(x,y,t) = 0, \quad (x,y) \in \partial\Omega,\ 0 < t \le T, \tag{1.2}$$

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in \bar{\Omega}, \tag{1.3}$$

where $i = \sqrt{-1}$ is the imaginary unit, $u = u(x,y,t)$ is a complex-valued scalar field, $\Delta = \partial_{xx} + \partial_{yy}$ is the Laplace operator, $\Omega = (a,b) \times (c,d)$, $\partial\Omega$ is the boundary of Ω, $\nu > 0$, $\kappa > 0$, α, β are four given real constants, the initial function u_0 is a given complex function, and γ is the coefficient of the linear evolution term.

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A large number of literatures have extensively studied the existence, uniqueness and regularity of solutions of complex valued GL equations. [2] studied the long-time behavior and global solution of two-dimensional complex valued GL equation, and makes an upper bound error estimation for the global solution. For the study of the existence of the solution of the equation, the literature [3] discussed the global existence and uniqueness of the solution of the complex valued GL equation. The literature [4] used the Galerkin method to analyze the existence and uniqueness of the global solution of the equation. Literatures [5, 6] gave the results of well posedness and regularity estimation of complex valued GL equation, respectively.

For the numerical aspects of the GL equation (1.1), many numerical methods have been proposed and analyzed in the literatures, such as the finite difference methods [7–14], finite element methods [15–22], spectral methods [23–29], and meshless methods [30–34]. Tsertsvadze [35] constructed a nonlinear implicit scheme for the one dimensional (1D) Kuratomo-Tsuzuki equation and proved that the scheme is convergent at the rate of order $O(h^{3/2})$ in the discrete L^2-norm and the rate of $O(h)$ in the uniform norm under the requirement $\tau = O(h^{2+\varepsilon})$ for any $\varepsilon > 0$. Then, in [9], for one-dimensional Kuramoto-Tsuzuki equation, Sun and Zhu proved the scheme in [35] is unconditionally and optimally convergent at $O(h^2 + \tau^2)$ in L^∞-norm. However, for two-dimensional GL equation, due to the difficulty in estimating the numerical solutions in L^∞-norm, error estimates for high-dimensional GL equation has lots of works to do. In [12], for two-dimensional GL equation, Wang and Guo gave proofs at second-order convergence in L^2-norm for a Crank-Nicolson scheme and a semiexplicit linearized Crank-Nicolson scheme by the mathematical induction method, respectively.

Since the convergent rate at $O(h^2+\tau^2)$ in L^∞-norm can be obtained in numerical studies [12], it is a natural question to ask how to improve the theoretical proof in two dimensions for GL equation's numerical studies. In other words, one has to give boundedness of numerical solutions for (1.1). The purpose of this paper is to give rigorous error estimates of two Crank-Nicolson type finite difference methods for the GL equation (1.1) in L^∞-norm.

Nowadays, methods of proof in error estimates have been greatly developed, such as the cut off technique [36] and the lifting technique [37]. In this paper, we use these techniques as: First, a linearized Crank-Nicolson finite-difference scheme of the GL equation is analyzed in depth, and the optimal error estimate of the scheme in L^∞ norm sense is established without any requirement on the grid ratio by using energy analysis methods combined with mathematical induction and lifting techniques. It is proved that the scheme is convergent in both time and space directions with 2 order accuracy in L^∞-norm sense. Second, a nonlinear Crank-Nicolson finite difference scheme is analyzed in depth. The existence and uniqueness of the nonlinear schemes solution are discussed in [12]. Then the optimal error estimate of the scheme in L^∞-norm sense is established by using the energy analysis method and the cut-off technique combined with the lifting technique, and the scheme is proved to be accurate to the 2 order in both space-time directions.

The rest of this paper is organized as follows. In Section 2, two different Crank-Nicolson finite difference schemes for GL equation are given. In Section 3, optimal error estimates in L^2

and H^1 norm of two Crank-Nicolson schemes are analyzed, and the improved error estimates in H^2-norm are given in appendix as long as the regularity assumption of numerical solution is improved. In Section 4, several numerical examples are presented to verify our theoretical analysis. Finally, some conclusions are given.

2. Two finite difference methods

In this section, we develop two Crank-Nicolson finite difference methods for the two dimensional GL equations (1.1)–(1.3). Throughout this paper, the constants C may be different as long as they are not decided by discrete parameters. When it is necessary to indicate the dependence on some parameters, we will use the notation $C(\cdot)$.

2.1. Notation and definitions

We denote the time step size $\tau = T/N$ with a positive integer N and the time grid points $t_n = n\tau$ for $0 \le n \le N$. Given a temporal grid function $\{w^n | 0 \le n \le N\}$, we denote $w^{n+\frac{1}{2}} = \frac{1}{2}(w^{n+1} + w^n)$, $\tilde{w}^{n+\frac{1}{2}} = \frac{1}{2}(3w^n - w^{n-1})$ and introduce the following difference quotient operators

$$\delta_t^+ w^n = \frac{1}{\tau}(w^{n+1} - w^n).$$

For the spatial discretization, we take the grid sizes $h_1 = (b-a)/J$, $h_2 = (d-c)/K$ with two positive integers J, K. Define two function spaces as follows

$$\begin{aligned} V_h^0 &= \{w = (w_{j,k})_{(j,k)\in\mathcal{T}_h^0} \mid w_{j,k} = 0 \text{ when } (j,k) \in \Gamma_h\}, \\ V_h &= \{w = (w_{j,k})_{(j,k)\in\mathcal{T}_h} \mid (w_{j,k})_{(j,k)\in\mathcal{T}_h^0} \in V_h^0\}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{T}_h^0 &= \{(j,k) \mid j = 0,1,2,\ldots,J,\ k = 0,1,2,\ldots,K\}, \\ \mathcal{T}_h &= \{(j,k) \mid j = 1,2,\ldots,J-1,\ k = 1,2,\ldots,K-1\}, \\ \Gamma_h &= \mathcal{T}_h^0/\mathcal{T}_h. \end{aligned}$$

Then for a given spatial grid function $v \in V_h^0$, we introduce the following difference quotient operators for $(j,k) \in \mathcal{T}_h$ as

$$\begin{aligned} &\delta_x^+ v_{j,k} = \frac{1}{h_1}(v_{j+1,k} - v_{j,k}),\ \ \delta_x^2 v_{j,k} = \frac{1}{h_1^2}(v_{j+1,k} - 2v_{j,k} + v_{j-1,k}), \\ &\delta_y^+ v_{j,k} = \frac{1}{h_2}(v_{j,k+1} - v_{j,k}),\ \ \delta_y^2 v_{j,k} = \frac{1}{h_2^2}(v_{j,k+1} - 2v_{j,k} + v_{j,k-1}), \\ &\nabla_h v_{j,k} = (\delta_x^+ v_{j,k}, \delta_y^+ v_{j,k})^\top,\ \ \Delta_h v_{j,k} = \delta_x^2 v_{j,k} + \delta_y^2 v_{j,k}. \end{aligned}$$

For two spatial grid functions $u, v \in V_h^0$, define discrete inner product as

$$(u,v)_h = h_1 h_2 \sum_{j=1}^{J-1} \sum_{k=1}^{K-1} u_{j,k}(v_{j,k})^*,$$

where $(s)^*$ represents taking the conjugate of s for $s \in \mathbb{C}$. For $p \ge 1$, the Sobolev norms (or seminorms) are defined as

$$\|u\|_h = \sqrt{(u,u)_h}, \quad \|u\|_p = \Big(h_1 h_2 \sum_{j=1}^{J-1}\sum_{k=1}^{K-1} |u_{j,k}|^p\Big)^{\frac{1}{p}},$$
$$|u|_{h,1} = \sqrt{\|\delta_x^+ u\|_h^2 + \|\delta_y^+ u\|_h^2}, \quad |u|_{h,2} = \|\Delta_h u\|_h,$$
$$\|u\|_{h,1} = \sqrt{\|u\|_h^2 + |u|_{h,1}^2}, \quad \|u\|_{h,2} = \sqrt{\|u\|_{h,1}^2 + |u|_{h,2}^2},$$
$$\|u\|_\infty = \max_{(j,k)\in\mathcal{T}_h} |u_{j,k}|,$$

for grid functions $u, v \in V_h$.

2.2. Two finite difference methods

The approximation of a linearized Crank-Nicolson finite difference (LCNFD) method to solve (1.1)–(1.3) is as follows:

(1) The LCNFD method

$$\delta_t^+ u_{j,k}^n - (\nu + i\alpha)\Delta_h u_{j,k}^{n+\frac{1}{2}} + (\kappa + i\beta)|\tilde{u}_{j,k}^{n+\frac{1}{2}}|^2 u_{j,k}^{n+\frac{1}{2}} - \gamma u_{j,k}^{n+\frac{1}{2}} = 0,$$
$$(j,k)\in\mathcal{T}_h, \quad 1 \le n \le N-1, \tag{2.1}$$
$$u_{j,k}^1 = u_{j,k}^0 + \tau u_t(x_j, y_k, 0), \quad (j,k)\in\mathcal{T}_h, \tag{2.2}$$

where the initial value

$$u_{j,k}^0 = u_0(x_j, y_k), \quad (j,k)\in\mathcal{T}_h^0, \tag{2.3}$$

the boundary value

$$u_{j,k}^n = 0, \quad (j,k)\in\Gamma_h, \ 1\le n\le N \tag{2.4}$$

and

$$u_t(x_j, y_k, 0) = (\nu + i\alpha)\Delta u(x_j, y_k, 0) - (\kappa + i\beta)\left|u_{j,k}^0\right|^2 u_{j,k}^0 + \gamma u_{j,k}^0.$$

The approximation of a nonlinear Crank-Nicolson finite difference (NLCNFD) method to solve (1.1)–(1.3) is as follows:

(2) The NLCNFD method

$$\delta_t^+ u_{j,k}^n - (\nu + i\alpha)\Delta_h u_{j,k}^{n+\frac{1}{2}} + (\kappa + i\beta)|u_{j,k}^{n+\frac{1}{2}}|^2 u_{j,k}^{n+\frac{1}{2}} - \gamma u_{j,k}^{n+\frac{1}{2}} = 0,$$
$$(j,k)\in\mathcal{T}_h, \quad 0 \le n \le N-1, \tag{2.5}$$

where the initial value

$$u_{j,k}^0 = u_0(x_j, y_k), \quad (j,k)\in\mathcal{T}_h^0, \tag{2.6}$$

the boundary value

$$u_{j,k}^n = 0, \ (j,k)\in\Gamma_h, \quad 1\le n\le N. \tag{2.7}$$

Since the LCNFD method is a linearized scheme, the solvability of it is apparent. For the solvability of NLCNFD method, refer to [12].

3. Unconditional optimal error estimates

In this section, we state and prove optimal error estimates for LCNFD and NLCNFD methods proposed in section 2. For the proof, we make some assumptions on the exact solution of (1.1) as

$$u \in W^{3,\infty}\big([0,T];L^{\infty}(\Omega)\big) \cap W^{2,\infty}\big([0,T];W^{2,\infty}(\Omega)\big) \cap L^{\infty}\big([0,T];W^{4,\infty}(\Omega)\big). \tag{A}$$

Define the 'error' function $e^n \in V_h^0$ as

$$e^n_{j,k} = U^n_{j,k} - u^n_{j,k}, \quad (j,k) \in \mathcal{T}_h^0,\ 0 \le n \le N,$$

where $U^n_{j,k}$ and $u^n_{j,k}$ represent exact solutions and numerical solutions for (1.1)–(1.3) at (x_j, y_k, t_n), respectively. Let $h = \max\{h_1, h_2\}$, then there is the following estimates:

Theorem 3.1 *Under the assumption* (A), *LCNFD method is unconditionally convergent and there exists* $\tau_0 > 0$, *when* $\tau \le \tau_0$,

$$\|e^n\|_h \le C(\tau^2 + h^2), \quad |e^n|_{h,1} \le C(\tau^2 + h^2),\ 0 \le n \le N.$$

Theorem 3.2 *Under the assumption* (A), *NLCNFD method is unconditionally convergent and there exist* $\tau_0' > 0$ *and* $h_0' > 0$, *when* $\tau \le \tau_0'$ *and* $h \le h_0'$,

$$\|e^n\|_h \le C(\tau^2 + h^2), \quad |e^n|_{h,1} \le C(\tau^2 + h^2), \quad 0 \le n \le N.$$

3.1. A priori error estimation

In this subsection, we present a priori error estimation of the numerical solution in the discrete l^2 norm.

Lemma 3.3 *Under assumption* (A), *for* $u^n \in V_h^0$ *of LCNFD and NLCNFD method, there exists* τ_*, *when* $\tau \le \tau_*$,

$$\|u^n\|_h \le C, \quad 0 \le n \le N.$$

Proof Making inner product of (2.1) with $u^{n+\frac{1}{2}}$ and by using Green formula, then taking the real part, we get

$$\frac{\|u^{n+1}\|_h^2 - \|u^n\|_h^2}{2\tau} + \nu\|\nabla_h u^{n+\frac{1}{2}}\|_h^2 + \kappa h_1 h_2 \sum_{j=1}^{M_1-1}\sum_{k=1}^{M_2-1} |\tilde{u}^{n+\frac{1}{2}}_{j,k}|^2 |u^{n+\frac{1}{2}}_{j,k}|^2 = \gamma \|u^{n+\frac{1}{2}}\|_h^2.$$

If $\gamma \le 0$, because of $\nu, \kappa > 0$, we can directly get

$$\|u^{n+1}\|_h^2 \le \|u^n\|_h^2, \quad 1 \le n \le N-1.$$

From (2.2), (2.3) and assumption (A), we get that

$$\|u^1\|_h \le C, \quad \|u^0\|_h \le C. \tag{3.1}$$

So when $\gamma \le 0$,

$$\|u^n\|_h \le C, \quad 0 \le n \le N.$$

If $\gamma > 0$, there is

$$\|u^{n+1}\|_h^2 - \|u^n\|_h^2 \le 2\tau\gamma\|u^{n+\frac{1}{2}}\|_h^2 \le \tau\gamma(\|u^{n+1}\|_h^2 + \|u^n\|_h^2).$$

Using Gronwall's inequality and defining $\tau_* = \frac{1}{2\gamma}$, when $\tau \le \tau_*$,

$$\|u^n\|_h^2 \le \exp(4\gamma T)\|u^1\|_h^2, \quad 2 \le n \le N.$$

Then combining (3.1), when $\gamma > 0$,

$$\|u^n\|_h \le C, \quad 0 \le n \le N.$$

Thus, we get the proof of the LCNFD method in Lemma 3.3. By similar proof methods, numerical solutions of the NLCNFD method are bounded in discrete L2-norm, so we omit it here. □

3.2. Convergence analysis

In this subsection, the following two lemmas will play an important role in theoretical analysis,

Lemma 3.4 ([38]) *For any grid function $u \in V_h^0$, there is*

$$\|u\|_p \le \|u\|_h^{\frac{2}{p}}(C_p|u|_{h,1} + \frac{1}{l}\|u\|_h)^{1-\frac{2}{p}}, \quad 2 \le p < \infty,$$

where $C_p = \max\{2\sqrt{2}, \frac{p}{\sqrt{2}}\}, l = \min\{l_1, l_2\}, l_1 = b - a, l_2 = d - c.$

Lemma 3.5 ([39]) *For any grid function $u \in V_h^0$, there is*

$$\|u\|_\infty \le C\|u\|_h^{\frac{1}{2}}(\|u\|_h + |u|_{h,2})^{\frac{1}{2}}.$$

First, we consider the local truncation of LCNFD defined as

$$\begin{aligned} R_{j,k}^{n+\frac{1}{2}} &= \delta_t^+ U_{j,k}^n - (\nu + i\alpha)\Delta_h U_{j,k}^{n+\frac{1}{2}} + (\kappa + i\beta)|\tilde{U}_{j,k}^{n+\frac{1}{2}}|^2 U_{j,k}^{n+\frac{1}{2}} - \gamma U_{j,k}^{n+\frac{1}{2}}, \\ &\quad (j,k) \in \mathcal{T}_h, \quad 1 \le n \le N-1, \\ R_{j,k}^{n+\frac{1}{2}} &= 0, \quad (j,k) \in \Gamma_h, \ 1 \le n \le N-1. \end{aligned}$$

Then there are estimates for the local truncation as follows

Lemma 3.6 *Under the assumption (A), we can obtain*

$$\|R^{n+\frac{1}{2}}\|_h \le C(\tau^2 + h^2), \quad 1 \le n \le N-1. \tag{3.2}$$

Proof By Taylor expansion, the proof can be easily got, so we omit it here. □

Then we begin our main proof for Theorem 3.1.

Proof Subtracting (2.1) from (3.2), we get the 'error' system

$$\begin{aligned} &\delta_t^+ e_{j,k}^n - (\nu + \mathrm{i}\alpha)\Delta_h e_{j,k}^{n+\frac{1}{2}} + (\kappa + \mathrm{i}\beta)P_{j,k}^{n+\frac{1}{2}} - \gamma e_{j,k}^{n+\frac{1}{2}} = R_{j,k}^{n+\frac{1}{2}}, \\ &\qquad (j,k) \in \mathcal{T}_h, \quad 1 \le n \le N-1, &(3.3) \\ &e_{j,k}^1 = R_{j,k}^1, \quad (j,k) \in \mathcal{T}_h, &(3.4) \\ &e_{j,k}^n = 0, \quad (j,k) \in \Gamma_h, 1 \le n \le N, &(3.5) \end{aligned}$$

$$e^0_{j,k} = 0, (j,k) \in \mathcal{T}^0_h, \tag{3.6}$$

where $P^{n+\frac{1}{2}}_{j,k} = |\tilde{U}^{n+\frac{1}{2}}_{j,k}|^2 U^{n+\frac{1}{2}}_{j,k} - |\tilde{u}^{n+\frac{1}{2}}_{j,k}|^2 u^{n+\frac{1}{2}}_{j,k}$. Taking inner product of the equality above with $e^{n+\frac{1}{2}}$ and using the Green formula, we obtain the real part of the result as follows

$$\begin{aligned}&\frac{\|e^{n+1}\|_h^2 - \|e^n\|_h^2}{2\tau} + \nu |e^{n+\frac{1}{2}}|^2_{h,1} + \mathrm{Re}\big((\kappa + i\beta)(P^{n+\frac{1}{2}}, e^{n+\frac{1}{2}})_h\big) - \gamma \|e^{n+\frac{1}{2}}\|_h^2 \\ &= \mathrm{Re}(R^{n+\frac{1}{2}}, e^{n+\frac{1}{2}})_h. \end{aligned} \tag{3.7}$$

Rewrite the nonlinear term $P^{n+\frac{1}{2}}$ as follows

$$P^{n+\frac{1}{2}}_{j,k} = (|\tilde{U}^{n+\frac{1}{2}}_{j,k}|^2 - |\tilde{u}^{n+\frac{1}{2}}_{j,k}|^2) U^{n+\frac{1}{2}}_{j,k} + |\tilde{u}^{n+\frac{1}{2}}_{j,k}|^2 e^{n+\frac{1}{2}}_{j,k}. \tag{3.8}$$

Then the third term of the left hand of (3.3) will be

$$\begin{aligned}\mathrm{Re}\big((\kappa + i\beta)(P^{n+\frac{1}{2}}, e^{n+\frac{1}{2}})_h\big) =&\mathrm{Re}\Big((\kappa + i\beta) h_1 h_2 \sum_{j=1}^{M_1-1} \sum_{k=1}^{M_2-1} (|\tilde{U}^{n+\frac{1}{2}}_{j,k}|^2 - |\tilde{u}^{n+\frac{1}{2}}_{j,k}|^2) U^{n+\frac{1}{2}}_{j,k} \overline{e}^{n+\frac{1}{2}}_{j,k}\Big) + \\ &\kappa h_1 h_2 \sum_{j=1}^{M_1-1} \sum_{k=1}^{M_2-1} |\tilde{u}^{n+\frac{1}{2}}_{j,k}|^2 |e^{n+\frac{1}{2}}_{j,k}|^2. \end{aligned} \tag{3.9}$$

From assumption (A), there is $|U^n_{j,k}| \le C$ for $(j,k) \in \mathcal{T}_h,\ 1 \le n \le N$, then we can make some detailed estimates on the equality above

$$\begin{aligned}&|(|\tilde{U}^{n+\frac{1}{2}}_{j,k}|^2 - |\tilde{u}^{n+\frac{1}{2}}_{j,k}|^2) U^{n+\frac{1}{2}}_{j,k}| \le (|\tilde{U}^{n+\frac{1}{2}}_{j,k}| + |\tilde{u}^{n+\frac{1}{2}}_{j,k}|) |\tilde{e}^{n+\frac{1}{2}}_{j,k}| |U^{n+\frac{1}{2}}_{j,k}| \\ &\le (2|\tilde{U}^{n+\frac{1}{2}}_{j,k}| + |\tilde{e}^{n+\frac{1}{2}}_{j,k}|) |\tilde{e}^{n+\frac{1}{2}}_{j,k}| |U^{n+\frac{1}{2}}_{j,k}| \le C\big(|\tilde{e}^{n+\frac{1}{2}}_{j,k}| + |\tilde{e}^{n+\frac{1}{2}}_{j,k}|^2\big) \\ &\le C\big(|e^n_{j,k}| + |e^{n-1}_{j,k}| + |e^n_{j,k}|^2 + |e^{n-1}_{j,k}|^2\big). \end{aligned} \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.7), then using Cauchy-Schwarz inequality and Young inequality, we get

$$\frac{\|e^{n+1}\|_h^2 - \|e^n\|_h^2}{2\tau} + \nu |e^{n+\frac{1}{2}}|^2_{h,1}$$

$$\le C(\|e^{n+1}\|_h^2 + \|e^n\|_h^2 + \|e^{n-1}\|_h^2) + \tag{3.11}$$

$$C(\|e^n\|_4^2 \|e^{n+\frac{1}{2}}\|_h + \|e^{n-1}\|_4^2 \|e^{n+\frac{1}{2}}\|_h) + \|R^{n+\frac{1}{2}}\|_h^2. \tag{3.12}$$

Now we estimate $\|e^n\|_4^2 \|e^{n+\frac{1}{2}}\|_h$ and $\|e^{n-1}\|_4^2 \|e^{n+\frac{1}{2}}\|_h$. From Lemma 3.4 and using Young inequality, there is

$$\begin{aligned}\|e^n\|_4^2 \|e^{n+\frac{1}{2}}\|_h \le& \|e^n\|_h \Big(C_p |e^n|_{h,1} + \frac{\|e^n\|_h}{l}\Big) \|e^{n+\frac{1}{2}}\|_h \\ \le& C\big(|e^n|_{h,1} \|e^n\|_h \|e^{n+\frac{1}{2}}\|_h + \|e^n\|_h^2 \|e^{n+\frac{1}{2}}\|_h\big) \\ \le& C\big(|e^n|^2_{h,1} + \|e^n\|_h^2 \|e^{n+\frac{1}{2}}\|_h^2 + \|e^n\|_h^2 \|e^{n+\frac{1}{2}}\|_h\big). \end{aligned} \tag{3.13}$$

Besides, according to assumption (A) and Lemma 3.3, we get

$$\|e^{n+\frac{1}{2}}\|_h \le \|U^{n+\frac{1}{2}}\|_h + \|u^{n+\frac{1}{2}}\|_h \le C. \tag{3.14}$$

Then we can get

$$\|e^n\|_4^2 \|e^{n+\frac{1}{2}}\|_h \le C(|e^n|^2_{h,1} + \|e^n\|_h^2). \tag{3.15}$$

Similarly, there is

$$\|e^{n-1}\|_4^2\|e^{n+\frac{1}{2}}\|_h \le C(|e^{n-1}|_{h,1}^2 + \|e^{n-1}\|_h^2). \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.11), we obtain

$$\|e^{n+1}\|_h^2 - \|e^n\|_h^2 \le C\tau(|e^n|_{h,1}^2 + |e^{n-1}|_{h,1}^2 + \|e^{n+1}\|_h^2 + \|e^n\|_h^2 + \|e^{n-1}\|_h^2 + \|R^{n+\frac{1}{2}}\|_h^2), \tag{3.17}$$

for $1 \le n \le N-1$. Then we use Mathematical induction in the following. When $n = 0, 1$, under assumption (A), there is $\|e^n\|_h + |e^n|_{h,1} \le C(\tau^2 + h^2)$. Assume that when $2 \le m \le n < N$, there is $\|e^m\|_h + |e^m|_{h,1} \le C(\tau^2 + h^2)$. Then from (3.17), we get

$$(1 - C\tau)\|e^{n+1}\|_h^2 \le C(\tau^2 + h^2)^2. \tag{3.18}$$

So take $\tau_1 = \frac{1}{2C}$, for $\tau \le \tau_1$, there is

$$\|e^{n+1}\|_h \le C(\tau^2 + h^2). \tag{3.19}$$

Here we get the convergent estimates of $\|e^n\|_h$ for $0 \le n \le N$. Next, making inner product of (3.3) with $\frac{1}{\nu - i\alpha}\delta_t^+ e^n$ and taking the real part, we get

$$\begin{aligned} &\frac{\nu}{\nu^2+\alpha^2}\|\delta_t^+ e^n\|_h^2 + \frac{1}{2\tau}(|e^{n+1}|_{h,1}^2 - |e^n|_{h,1}^2) \\ &\quad = \frac{\nu}{\nu^2+\alpha^2}\mathrm{Re}\big(-(\kappa + i\beta)P^{n+\frac{1}{2}} + \gamma e^{n+\frac{1}{2}} + R^{n+\frac{1}{2}}, \delta_t^+ e^n\big)_h. \end{aligned} \tag{3.20}$$

Using Cauchy-Schwarz inequality and Young inequality, we get

$$\begin{aligned} &\frac{\nu}{\nu^2+\alpha^2}\|\delta_t^+ e^n\|_h^2 + \frac{1}{2\tau}(|e^{n+1}|_{h,1}^2 - |e^n|_{h,1}^2) \\ &\quad \le C\| - (\kappa + i\beta)P^{n+\frac{1}{2}} + \gamma e^{n+\frac{1}{2}} + R^{n+\frac{1}{2}}\|_h \|\delta_t^+ e^n\|_h \\ &\quad \le C(\varepsilon_1)\| - (\kappa + i\beta)P^{n+\frac{1}{2}} + \gamma e^{n+\frac{1}{2}} + R^{n+\frac{1}{2}}\|_h^2 + \varepsilon_1\|\delta_t^+ e^n\|_h^2, \end{aligned} \tag{3.21}$$

for any $\varepsilon_1 > 0$. Take

$$\varepsilon_1 = \frac{\nu}{\nu^2+\alpha^2},$$

then using Minkowski inequality, there is

$$\begin{aligned} \frac{1}{2\tau}(|e^{n+1}|_{h,1}^2 - |e^n|_{h,1}^2) &\le C\big(\|P^{n+\frac{1}{2}}\|_h^2 + \|e^{n+\frac{1}{2}}\|_h^2 + \|R^{n+\frac{1}{2}}\|_h^2\big) \\ &\le C\|P^{n+\frac{1}{2}}\|_h^2 + C(\tau^2 + h^2)^2. \end{aligned} \tag{3.22}$$

About $P_{j,k}^{n+\frac{1}{2}}$, we have

$$\begin{aligned} P_{j,k}^{n+\frac{1}{2}} =& |\tilde{U}_{j,k}^{n+\frac{1}{2}}|^2 U_{j,k}^{n+\frac{1}{2}} - |\tilde{u}_{j,k}^{n+\frac{1}{2}}|^2 u_{j,k}^{n+\frac{1}{2}} \\ =& |\tilde{U}_{j,k}^{n+\frac{1}{2}}|^2 e_{j,k}^{n+\frac{1}{2}} + \big(|\tilde{U}_{j,k}^{n+\frac{1}{2}}|^2 - |\tilde{u}_{j,k}^{n+\frac{1}{2}}|^2\big) u_{j,k}^{n+\frac{1}{2}} \\ \le& |\tilde{U}_{j,k}^{n+\frac{1}{2}}|^2 |e_{j,k}^{n+\frac{1}{2}}| + |\tilde{e}_{j,k}^{n+\frac{1}{2}}|\big(2|\tilde{U}_{j,k}^{n+\frac{1}{2}}| + |\tilde{e}_{j,k}^{n+\frac{1}{2}}|\big)\big(|U_{j,k}^{n+\frac{1}{2}}| + |e_{j,k}^{n+\frac{1}{2}}|\big) \\ \le& C\big(|e_{j,k}^{n+\frac{1}{2}}| + |e_{j,k}^{n+\frac{1}{2}}|^2 + |\tilde{e}_{j,k}^{n+\frac{1}{2}}| + |\tilde{e}_{j,k}^{n+\frac{1}{2}}|^2 + |\tilde{e}_{j,k}^{n+\frac{1}{2}}|^4\big). \end{aligned} \tag{3.23}$$

Then we get estimate on upper bound of $\|P^{n+\frac{1}{2}}\|_h$ as follows

$$\|P^{n+\frac{1}{2}}\|_h \le C\big(\|e^{n+\frac{1}{2}}\|_h + \|e^{n+\frac{1}{2}}\|_4^2 + \|\tilde{e}^{n+\frac{1}{2}}\|_h + \|\tilde{e}^{n+\frac{1}{2}}\|_4^2 + \|\tilde{e}^{n+\frac{1}{2}}\|_8^4\big). \tag{3.24}$$

From Lemma 3.4 and (3.19), we get

$$\|e^{n+\frac{1}{2}}\|_4^2 \le \|e^{n+\frac{1}{2}}\|_h\big(C_p|e^{n+\frac{1}{2}}|_1 + \frac{1}{l}\|e^{n+\frac{1}{2}}\|_h\big) \le C\big(|e^{n+\frac{1}{2}}|_1 + (\tau^2+h^2)\big). \tag{3.25}$$

Combining inductive hypothesis and Lemma 3.4 gives

$$\|\tilde{e}^{n+\frac{1}{2}}\|_4^2 \le \|\tilde{e}^{n+\frac{1}{2}}\|_h\big(C_p|\tilde{e}^{n+\frac{1}{2}}|_{h,1} + \frac{1}{l}\|\tilde{e}^{n+\frac{1}{2}}\|_h\big) \le C(\tau^2+h^2) \tag{3.26}$$

and

$$\|\tilde{e}^{n+\frac{1}{2}}\|_8^4 \le \|\tilde{e}^{n+\frac{1}{2}}\|_h\big(C_p|\tilde{e}^{n+\frac{1}{2}}|_{h,1} + \frac{1}{l}\|\tilde{e}^{n+\frac{1}{2}}\|_h\big)^3 \le C(\tau^2+h^2). \tag{3.27}$$

Substituting (3.24)–(3.27) into (3.22), we obtain

$$\begin{aligned}\frac{1}{2\tau}(|e^{n+1}|_{h,1}^2 - |e^n|_{h,1}^2) &\le C|e^{n+\frac{1}{2}}|_{h,1}^2 + C(\tau^2+h^2)^2\\ &\le C|e^{n+1}|_{h,1}^2 + C(\tau^2+h^2)^2.\end{aligned} \tag{3.28}$$

Then

$$(1-2C\tau)|e^{n+1}|_{h,1}^2 \le C(\tau^2+h^2)^2. \tag{3.29}$$

Similar to (3.19), taking $\tau_2 = \frac{1}{4C}$ when $\tau \le \tau_2$, we have

$$|e^{n+1}|_{h,1} \le C(\tau^2+h^2). \tag{3.30}$$

Thus, taking $\tau_0 = \min\{\tau_1, \tau_2\}$, we get the proof of Theorem 3.1. $\Box$

However, for the nonlinear numerical method NLCNFD, mathematical induction is no longer applicable. Thus, we search a smooth function to establish a replacing scheme for giving the proof of Theorem 3.2.

Proof Concretely, choose a smooth function $\rho \in C^\infty(\mathbb{R})$ as

$$\rho(s) = \begin{cases} 1, & |s| \le 1,\\ \in [0,1], & 1 \le |s| \le 2,\\ 0, & |s| \ge 2.\end{cases}$$

It is easy to see that ρ and ρ' have compact support. Following assumption (A), we define $M_0 = \|u\|_{L^\infty([0,T];L^\infty(\Omega))}$. Take $M = (M_0+1)^2$, for $s \ge 0$, and define

$$\rho_M(s) = s\rho(s/M).$$

Then ρ_M is global Lipschitz as

$$|\rho_M(s_1) - \rho_M(s_2)| \le C\left|\sqrt{s_1} - \sqrt{s_2}\right|, \quad \forall s_1,\ s_2 \ge 0. \tag{3.31}$$

Let $\hat{u}^0 = u^0$ and define $\hat{u}^n \in V_h^0$ $(n = 1, 2, \ldots, N)$ as solutions of the following replacing scheme

$$\begin{aligned}&\delta_t^+ \hat{u}_{j,k}^n - (\nu + i\alpha)\Delta_h \hat{u}_{j,k}^{n+\frac{1}{2}} + (\kappa + i\beta)\rho_M\big(\big|\hat{u}_{j,k}^{n+\frac{1}{2}}\big|^2\big)\hat{u}_{j,k}^{n+\frac{1}{2}} - \gamma\hat{u}_{j,k}^{n+\frac{1}{2}} = 0,\\ &\qquad (j,k) \in \mathcal{T}_h,\ 0 \le n \le N-1,\end{aligned} \tag{3.32}$$

$$\hat{u}_{j,k}^n = 0,\ (j,k) \in \Gamma_h,\ 1 \le n \le N. \tag{3.33}$$

Define the local truncation $\hat{R}^{n+\frac{1}{2}} \in V_h^0$ of the replacing scheme as

$$\hat{R}_{j,k}^{n+\frac{1}{2}} = \delta_t^+ U_{j,k}^n - (\nu + i\alpha)\Delta_h U_{j,k}^{n+\frac{1}{2}} + (\kappa + i\beta)\rho_M\big(|U_{j,k}^{n+\frac{1}{2}}|^2\big)U_{j,k}^{n+\frac{1}{2}} - \gamma U_{j,k}^{n+\frac{1}{2}},$$
$$(j,k) \in \mathcal{T}_h, \quad 0 \le n \le N-1. \tag{3.34}$$

$$\hat{R}_{j,k}^{n+\frac{1}{2}} = 0, \quad (j,k) \in \Gamma_h, \quad 0 \le n \le N-1. \tag{3.35}$$

Following the definition of ρ_M, we get

$$\rho_M\big(|U_{j,k}^{n+\frac{1}{2}}|^2\big) = |U_{j,k}^{n+\frac{1}{2}}|^2, \quad (j,k) \in \mathcal{T}_h.$$

Thus, under assumption (A), the replacing scheme (3.32)-(3.33) shares the same local truncation results as the NLCNFD method, i.e.,

$$\|\hat{R}^{n+\frac{1}{2}}\|_h \le C(\tau^2 + h^2), \quad 0 \le n \le N-1. \tag{3.36}$$

Remark 3.7 Here, we can see replacing scheme (3.32)–(3.33) as another approximate of GLE (1.1)–(1.3). Once the boundedness of solutions of the replacing scheme is obtained, error estimates of NLCNFD can be obtained by its analysis results due to the global Lipschitz property of function ρ_M.

Next, we use energy method to give the error estimate of the replacing scheme (3.32)–(3.33). Define 'error' function $\hat{e}^n \in V_h^0$ $(n = 0, 1, \dots, N)$ as follows

$$\hat{e}_{j,k}^n = U_{j,k}^n - \hat{u}_{j,k}^n.$$

Subtracting (3.32) from (3.34), we get the following 'error' equation

$$\delta_t^+ \hat{e}_{j,k}^n - (\nu + i\alpha)\Delta_h \hat{e}_{j,k}^{n+\frac{1}{2}} + (k + i\beta)\hat{P}_{j,k}^{n+\frac{1}{2}} - \gamma \hat{e}_{j,k}^{n+\frac{1}{2}} = \hat{R}_{j,k}^{n+\frac{1}{2}},$$
$$(j,k) \in \mathcal{T}_h, \quad 0 \le n \le N-1, \tag{3.37}$$

where $\hat{P}^{n+\frac{1}{2}} \in V_h^0$ is defined by

$$\begin{aligned}\hat{P}_{j,k}^{n+\frac{1}{2}} =& \rho_M\big(|U_{j,k}^{n+\frac{1}{2}}|^2\big)U_{j,k}^{n+\frac{1}{2}} - \rho_M\big(|\hat{u}_{j,k}^{n+\frac{1}{2}}|^2\big)\hat{u}_{j,k}^{n+\frac{1}{2}}\\ =& \big(\rho_M\big(|U_{j,k}^{n+\frac{1}{2}}|^2\big) - \rho_M\big(|\hat{u}_{j,k}^{n+\frac{1}{2}}|^2\big)\big)U_{j,k}^{n+\frac{1}{2}} + \rho_M\big(|\hat{u}_{j,k}^{n+\frac{1}{2}}|^2\big)\hat{e}_{j,k}^{n+\frac{1}{2}},\\ & (j,k) \in \mathcal{T}_h\end{aligned}$$

and

$$\hat{P}_{j,k}^{n+\frac{1}{2}} = 0, \quad (j,k) \in \Gamma_h.$$

Collecting the boundedness of the exact solution and the global Lipschitz property of ρ_M, we get

$$\big|\hat{P}_{j,k}^{n+\frac{1}{2}}\big| \le C\big|\hat{e}_{j,k}^{n+\frac{1}{2}}\big|. \tag{3.38}$$

Make the inner product of (3.37) with $\hat{e}^{n+\frac{1}{2}}$, then take the real part of the result as follows

$$\frac{\|\hat{e}^{n+1}\|_h^2 - \|\hat{e}^n\|_h^2}{2\tau} + \nu|\hat{e}^{n+\frac{1}{2}}|_{h,1}^2 + \mathrm{Re}\big((k + i\beta)\hat{P}^{n+\frac{1}{2}}, \hat{e}^{n+\frac{1}{2}}\big) - \gamma\|\hat{e}^{n+\frac{1}{2}}\|_h^2 = \mathrm{Re}(\hat{R}^{n+\frac{1}{2}}, \hat{e}^{n+\frac{1}{2}})_h. \tag{3.39}$$

Combining Cauchy-Schwarz inequality, Young inequality and (3.38), we obtain

$$\|\hat{e}^{n+1}\|_h^2 - \|\hat{e}^n\|_h^2 + 2\tau\nu|\hat{e}^{n+\frac{1}{2}}|_{h,1}^2$$

$$
\begin{aligned}
&\le 2C\tau\|\hat{P}^{n+\frac{1}{2}}\|_h\|\hat{e}^{n+\frac{1}{2}}\|_h + 2\tau\gamma\|\hat{e}^{n+\frac{1}{2}}\|_h^2 + 2\tau\|\hat{R}^{n+\frac{1}{2}}\|_h\|\hat{e}^{n+\frac{1}{2}}\|_h \\
&\le C\tau\big((\tau^2+h^2)^2 + \|\hat{P}^{n+\frac{1}{2}}\|_h^2 + \|\hat{e}^{n+\frac{1}{2}}\|_h^2\big) \\
&\le C\tau\big((\tau^2+h^2)^2 + \|\hat{e}^{n+\frac{1}{2}}\|_h^2\big).
\end{aligned} \tag{3.40}
$$

From inequality above, it follows

$$
\|\hat{e}^{n+1}\|_h^2 - \|\hat{e}^n\|_h^2 \le C\tau\big((\tau^2+h^2)^2 + \|\hat{e}^{n+\frac{1}{2}}\|_h^2\big). \tag{3.41}
$$

Summing (3.41) for n from 0 to m, and replacing m by n, we get

$$
\begin{aligned}
\|\hat{e}^{n+1}\|_h^2 \le& \|\hat{e}^0\|_h^2 + C(n+1)\tau(\tau^2+h^2)^2 + C\tau\sum_{m=0}^{n}\|\hat{e}^{m+\frac{1}{2}}\|_h^2 \\
\le& \|\hat{e}^0\|_h^2 + CT(\tau^2+h^2)^2 + C\tau\sum_{m=0}^{n}\|\hat{e}^{m+\frac{1}{2}}\|_h^2 \\
\le& C(\tau^2+h^2)^2 + C\tau\sum_{m=0}^{n+1}\|\hat{e}^{m}\|_h^2.
\end{aligned} \tag{3.42}
$$

Following Gronwall's inequality, we have $\tau_1' = \frac{1}{C}$, when $\tau \le \tau_1'$,

$$
\|\hat{e}^n\|_h \le C(\tau^2+h^2), \quad n = 1, 2, \dots, N. \tag{3.43}
$$

Here we get the convergence of $\|\hat{e}^n\|_h$, then we will use the 'lifting' technique and Sobolev inverse inequality to get the equivalence of the replacing scheme (3.32)–(3.33) with NLCNFD method. Specifically, rewrite the 'error' equation (3.37) as follows

$$
(\nu + i\alpha)\Delta_h\hat{e}_{j,k}^{n+\frac{1}{2}} = \delta_t^+\hat{e}_{j,k}^{n+\frac{1}{2}} + \hat{P}_{j,k}^{n+\frac{1}{2}} - \gamma\hat{e}_{j,k}^{n+\frac{1}{2}} - \hat{R}_{j,k}^{n+\frac{1}{2}}. \tag{3.44}
$$

Making inner product of (3.44) with itself on both sides of the equation, we get

$$
\begin{aligned}
\sqrt{\nu^2+\alpha^2}|\hat{e}^{n+\frac{1}{2}}|_{h,2} =& \|\delta_t^+\hat{e}^n + \hat{P}^n - \gamma\hat{e}^{n+\frac{1}{2}} - \hat{R}^{n+\frac{1}{2}}\|_h \\
&\le C(\|\delta_t^+\hat{e}^n\|_h + \|\hat{P}^n\|_h + \|\hat{e}^n + \hat{e}^{n+1}\|_h + \|\hat{R}^{n+\frac{1}{2}}\|_h).
\end{aligned} \tag{3.45}
$$

Using Minkowski inequality and (3.43) yields

$$
\big\|\delta_t^+\hat{e}^n\big\| \le \tau^{-1}(\big\|\hat{e}^{n+1}\big\| + \|\hat{e}^n\|) \le C\tau^{-1}(h^2+\tau^2), \quad n = 0, 1, \dots, N-1. \tag{3.46}
$$

Combining (3.38) and (3.43), we get

$$
\|\hat{P}^{n+\frac{1}{2}}\|_h \le C(\tau^2+h^2), \quad n = 0, 1, \dots, N-1. \tag{3.47}
$$

Substituting (3.36), (3.43), (3.46) and (3.47) into (3.45), one obtains

$$
|\hat{e}^n + \hat{e}^{n+1}|_{h,2} \le C\tau^{-1}(\tau^2+h^2), \quad n = 0, 1, \dots, N-1.
$$

From Lemma 3.5 and (3.43),

$$
\begin{aligned}
\|\hat{e}^n + \hat{e}^{n+1}\|_\infty \le& \|\hat{e}^n + \hat{e}^{n+1}\|^{\frac{1}{2}}(\|\hat{e}^n + \hat{e}^{n+1}\| + |\hat{e}^n + \hat{e}^{n+1}|_{h,2})^{\frac{1}{2}} \\
\le& C\tau^{-\frac{1}{2}}(\tau^2+h^2), \quad n = 0, 1, \dots, N-1.
\end{aligned} \tag{3.48}
$$

Using Minkowski inequality and (3.48), we get

$$
\big\|\hat{e}^{n+1}\big\|_\infty - \|\hat{e}^n\|_\infty \le \big\|\hat{e}^n + \hat{e}^{n+1}\big\|_\infty \le C\tau^{-\frac{1}{2}}(h^2+\tau^2). \tag{3.49}
$$

Summing (3.49) for n from 0 to m, and replacing m by n, we get

$$\begin{aligned}\|\hat{e}^{n+1}\|_\infty \leq & C(n+1)\tau^{-\frac{1}{2}}(h^2+\tau^2) \leq C\tau^{-\frac{3}{2}}(h^2+\tau^2) \\ \leq & C\tau^{-\frac{3}{2}}h^2 + \tau^{\frac{1}{2}}, \quad n=0,1,\dots,N-1.\end{aligned}$$

Therefore, define $\tau_2' = \frac{1}{(C+1)^2}$, when $\tau \leq \tau_2'$, for any $h \leq \tau$,

$$\|\hat{e}^n\|_\infty \leq 1, \quad n=1,2,\dots,N. \tag{3.50}$$

On the other hand, using Lemma 3.5 and inverse inequality, we get

$$\begin{aligned}\|\hat{e}^n\|_\infty &\leq \|\hat{e}^n\|_h^{\frac{1}{2}}(\|\hat{e}^n\|_h + |\hat{e}^n|_{h,2})^{\frac{1}{2}} \\ &\leq \|\hat{e}^n\|_h^{\frac{1}{2}}(\|\hat{e}^n\|_h + C(h^{-2})\|\hat{e}^n\|_h)^{\frac{1}{2}} \\ &\leq Ch^{-1}(h^2+\tau^2) \\ &= C(h^{-1}\tau^2 + h), \quad n=0,1,2,\dots,N-1.\end{aligned}$$

Hence, define $h_1' = \frac{1}{2C}$, when $h \leq h_1'$, for any $\tau \leq h$,

$$\|\hat{e}^n\|_\infty \leq 1, \quad n=0,1,\dots,N. \tag{3.51}$$

Collecting (3.50) and (3.51), for any $\tau, h \leq \min\{\tau_2', h_1'\}$, we have

$$\|\hat{e}^n\|_\infty \leq 1, \quad n=0,1,\dots,N$$

and

$$\|\hat{u}\|_\infty \leq \|u^n\|_\infty + \|\hat{e}^n\|_\infty \leq M_0 + 1, \quad n=1,2,\dots,N.$$

Thus, we obtain the equivalence of replacing scheme (3.32) and (3.33) with NLCNFD, then we begin to give a further estimate of $\hat{e}^n$, for $n=0,1,\dots,N$. Making inner product of (3.37) with $\delta_t^+\hat{e}^n$ and taking the real part, we get

$$\begin{aligned}&\|\delta_t^+\hat{e}^n\|_h^2 + \frac{\nu}{2\tau}\big(|\hat{e}^{n+1}|_{h,1}^2 - |\hat{e}^n|_{h,1}^2\big) + \mathrm{Re}\big(\hat{P}^{n+\frac{1}{2}}, \delta_t^+\hat{e}^n\big)_h - \gamma\mathrm{Re}\big(\hat{e}^{n+\frac{1}{2}}, \delta_t^+\hat{e}^n\big)_h \\ &\quad = \mathrm{Re}\big(\hat{R}^{n+\frac{1}{2}}, \delta_t^+\hat{e}^n\big)_h.\end{aligned} \tag{3.52}$$

Combining Cauchy-Schwarz inequality and Young inequality gives

$$\begin{aligned}&\|\delta_t^+\hat{e}^n\|_h^2 + \frac{\nu}{2\tau}\big(|\hat{e}^{n+1}|_{h,1}^2 - |\hat{e}^n|_{h,1}^2\big) \\ &\leq \|\hat{P}^{n+\frac{1}{2}}\|_h\|\delta_t^+\hat{e}^n\|_h + \gamma\|\hat{e}^{n+\frac{1}{2}}\|_h\|\delta_t^+\hat{e}^n\|_h + \|\hat{R}^{n+\frac{1}{2}}\|_h\|\delta_t^+\hat{e}^n\|_h \\ &\leq 2\|\hat{P}^{n+\frac{1}{2}}\|_h^2 + \frac{1}{8}\|\delta_t^+\hat{e}^n\|_h^2 + 2\gamma\|\hat{e}^{n+\frac{1}{2}}\|_h^2 + \frac{1}{8}\|\delta_t^+\hat{e}^n\|_h^2 + 2\|\hat{R}^{n+\frac{1}{2}}\|_h^2 + \frac{1}{8}\|\delta_t^+\hat{e}^n\|_h^2 \\ &\leq C\Big(\|\hat{P}^{n+\frac{1}{2}}\|_h^2 + \|\hat{e}^{n+\frac{1}{2}}\|_h^2 + \|\hat{R}^{n+\frac{1}{2}}\|_h^2\Big) + \frac{3}{8}\|\delta_t^+\hat{e}^n\|_h^2.\end{aligned} \tag{3.53}$$

From (3.38) and (3.34),

$$|\hat{e}^{n+1}|_{h,1}^2 - |\hat{e}^n|_{h,1}^2 \leq C\tau\big((h^2+\tau^2)^2 + \|\hat{e}^{n+\frac{1}{2}}\|_h^2\big). \tag{3.54}$$

Summing the above inequality for n from 0 to m, and replacing m by n, we get

$$|\hat{e}^{n+1}|_{h,1}^2 \leq |\hat{e}^0|_{h,1}^2 + C(n+1)\tau(h^2+\tau^2)^2 + C\tau\sum_{m=0}^{n}\|\hat{e}^{m+\frac{1}{2}}\|_h^2$$

$$\leq C(h^2+\tau^2)^2 + CT(h^2+\tau^2)^2 + C\tau\sum_{m=0}^{n}\|\hat{e}^{m+\frac{1}{2}}\|_h^2$$
$$\leq C(h^2+\tau^2)^2 + C\tau\sum_{m=0}^{n}\|\hat{e}^{m+\frac{1}{2}}\|_h^2. \tag{3.55}$$

Substituting (3.43) into the above inequality, we have

$$\begin{aligned}|\hat{e}^{n+1}|_{h,1}^2 &\leq C(h^2+\tau^2)^2 + C(n+1)\tau(h^2+\tau^2)^2\\ &\leq C(h^2+\tau^2)^2 + CT(h^2+\tau^2)^2\\ &\leq C(h^2+\tau^2)^2.\end{aligned} \tag{3.56}$$

Then we have

$$|\hat{e}^n|_{h,1} \leq C(h^2+\tau^2), \quad n = 1, 2, \ldots, N. \tag{3.57}$$

Thus, from (3.43), (3.57) and the equivalence of replacing scheme (3.32) and (3.33) with NLCNFD, taking $\tau_0' = \min\{\tau_1', \tau_2', h_1'\}$ and $h_0' = \min\{\tau_2', h_1'\}$, we finish the proof of Theorem 3.2. □

Remark 3.8 An improved regularity assumption on the exact solution u and energy analysis method can make $|e^n|_{h,2}$ be bounded by $C(\tau^2 + h^2)$. Then using Sobolev inequalities, the pointwise error estimate of LCNFD and NLCNFD can be obtained immediately. The proof will be presented in Appendix A.

Remark 3.9 The above results for GL equations with Dirichlet boundary can be extended to periodic boundary problems.

4. Numerical experiments

In this section, we will use some numerical experiments to validate our theoretical analysis of two proposed finite difference methods LCNFD and NLCNFD for the two dimensional GL equation.

Example 4.1 First, we take $\nu = \alpha = \kappa = 1, \beta = 2, \gamma = 3$, the initial function as

$$u(x, y, 0) = e^{-3(x^2+y^2)},$$

the computational area as $\Omega = (-10, 10) \times (-10, 10)$, and the computational time as $T = 1$, such that when $0 \leq t \leq T$, the area is big enough to see the initial boundary problem as a Dirichlet boundary problem. Since this problem has no exact solution, we use TSFP method with $h_1 = h_2 = 1/128$, $\tau = 1e-4$ to get a 'numerical' exact solution.

It can be seen from Tables 1–4 that both LCNFD and NLCNFD method converge well in spatial and temporal direction consisting with theoretical analysis. Comparing Table 2 with Table 4, we find that NLCNFD method has better computational efficiency than LCNFD method.

h	$h_0 = 1/4$	$h_0/2$	$h_0/2^2$	$h_0/2^3$	$h_0/2^4$
$\|e^N\|_\infty$	5.69E-03	1.41E-03	3.52E-04	8.79E-05	2.20E-05
Order	-	2.01	2.00	2.00	2.00
$\|e^N\|_{h,1}$	3.23E-02	8.05E-03	2.01E-03	5.02E-04	1.26E-04
Order	-	2.01	2.00	2.00	2.00
$\|e^N\|_h$	2.06E-02	5.12E-03	1.28E-03	3.19E-04	7.98E-05
Order	-	2.01	2.00	2.00	2.00

Table 1 Spatial convergence order of LCNFD with τ =1e-4 at $T = 1$ in Dirichlet problem

τ	$\tau_0 = 1/16$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$
$\|e^N\|_\infty$	1.59E-02	3.70E-03	9.18E-04	2.30E-04	6.12E-05
Order	-	2.11	2.01	2.00	1.91
$\|e^N\|_{h,1}$	6.21E-02	1.09E-02	2.41E-03	5.76E-04	1.44E-04
Order	-	2.51	2.17	2.07	2.00
$\|e^N\|_h$	8.15E-02	1.68E-02	3.88E-03	9.43E-04	2.38E-04
Order	-	2.27	2.12	2.04	1.99

Table 2 Temporal convergence order of LCNFD with $h = 1/128$ at $T = 1$ in Dirichlet problem

h	$h_0 = 1/4$	$h_0/2$	$h_0/2^2$	$h_0/2^3$	$h_0/2^4$
$\|e^N\|_\infty$	5.69E-03	1.41E-03	3.52E-04	8.79E-05	2.20E-05
Order	-	2.01	2.00	2.00	2.00
$\|e^N\|_{h,1}$	3.23E-02	8.05E-03	2.01E-03	5.02E-04	1.26E-04
Order	-	2.01	2.00	2.00	2.00
$\|e^N\|_h$	2.06E-02	5.12E-03	1.28E-03	3.19E-04	7.98E-05
Order	-	2.01	2.00	2.00	2.00

Table 3 Spatial convergence order of NLCNFD with τ =1e-4 at $T = 1$ in Dirichlet problem

τ	$\tau_0 = 1/8$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$
$\|e^N\|_\infty$	1.02E-02	2.38E-03	5.89E-04	1.47E-04	3.72E-05
Order	-	2.09	2.02	2.00	1.98
$\|e^N\|_{h,1}$	6.32E-02	1.45E-02	3.59E-03	8.97E-04	2.26E-04
Order	-	2.12	2.01	2.00	1.99
$\|e^N\|_h$	8.00E-02	1.94E-02	4.80E-03	1.19E-03	2.95E-04
Order	-	2.04	2.01	2.01	2.01

Table 4 Temporal convergence order of NLCNFD with $h = 1/128$ at $T = 1$ in Dirichlet problem

Example 4.2 ([38]) Secondly, we consider a $(2\pi, 2\pi)$-periodic initial-value problem as

$$u_t - (1+i)\Delta u + (1+2i)|u|^2 u - 3u = 0, \quad (x,y) \in \mathbb{R} \times \mathbb{R},\ 0 < t \le T, \tag{4.1}$$

$$u(x,y,t) = u(x+2\pi, y, t), \quad u(x,y,t) = u(x, y+2\pi, t),\ 0 < t \le T. \tag{4.2}$$

The initial function is defined by the following exact solution

$$u(x,y,t) = e^{i(x+y-4t)}.$$

It can be seen from Tables 5–8, GL equation with periodic boundary also has second order convergent rate.

h	$h_0 = \pi/8$	$h_0/2$	$h_0/2^2$	$h_0/2^3$	$h_0/2^4$
$\|e^N\|_\infty$	1.16E-02	2.91E-03	7.28E-04	1.82E-04	4.55E-05
Order	-	2.00	2.00	2.00	2.00
$\|e^N\|_{h,1}$	9.94E-02	2.54E-02	6.41E-03	1.61E-03	4.03E-04
Order	-	1.97	1.99	1.99	2.00
$\|e^N\|_h$	7.30E-02	1.83E-02	4.57E-03	1.14E-03	2.86E-04
Order	-	2.00	2.00	2.00	2.00

Table 5 Spatial convergence order of LCNFD with τ =1e-4 at $T = 1$ in periodic problem

τ	$\tau_0 = 1/32$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$
$\|e^N\|_\infty$	1.81E-02	4.63E-03	1.17E-03	2.94E-04	7.38E-05
Order	-	1.97	1.99	1.99	1.99
$\|e^N\|_{h,1}$	1.61E-01	4.11E-02	1.04E-03	2.61E-03	6.55E-04
Order	-	1.97	1.99	1.99	1.99
$\|e^N\|_h$	1.14E-01	2.91E-02	7.34E-03	1.85E-03	4.64E-04
Order	-	1.97	1.99	1.99	1.99

Table 6 Temporal convergence order of LCNFD with $h = \pi/512$ at $T = 1$ in periodic problem

h	$h_0 = \pi/8$	$h_0/2$	$h_0/2^2$	$h_0/2^3$	$h_0/2^4$
$\|e^N\|_\infty$	1.16E-02	2.91E-03	7.28E-04	1.82E-04	4.54E-05
Order	-	2.00	2.00	2.00	2.00
$\|e^N\|_{h,1}$	9.94E-02	2.54E-02	6.41E-03	1.61E-03	4.03E-04
Order	-	1.97	1.99	1.99	2.00
$\|e^N\|_h$	7.30E-02	1.83E-02	4.57E-03	1.14E-03	2.86E-04
Order	-	2.00	2.00	2.00	2.00

Table 7 Spatial convergence order of NLCNFD with τ =1e-4 at $T = 1$ in periodic problem

τ	$\tau_0 = 1/16$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$
$\|e^N\|_\infty$	3.47E-02	8.73E-03	2.19E-03	5.46E-04	1.36E-04
Order	-	1.99	2.00	2.00	2.01
$\|e^N\|_{h,1}$	3.08E-01	7.75E-02	1.94E-02	4.84E-03	1.21E-03
Order	-	1.99	2.00	2.00	2.01
$\|e^N\|_h$	2.18E-01	5.48E-02	1.37E-02	3.43E-03	8.54E-04
Order	-	1.99	2.00	2.00	2.01

Table 8 Temporal convergence order of NLCNFD with $h = \pi/256$ at $T = 1$ in periodic problem

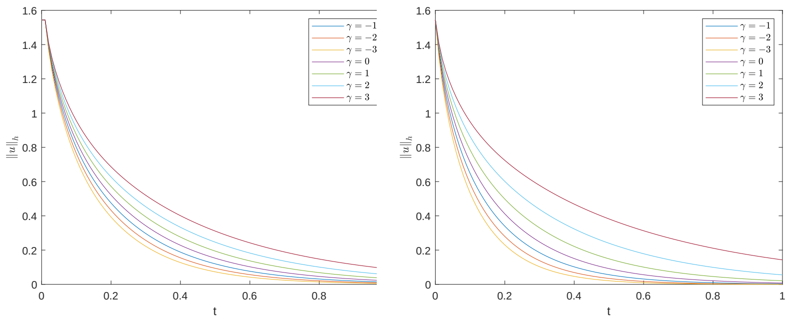
Example 4.3 We consider the following Ginzburg-Landau equation with homogeneous boundary conditions

$$\partial_t u - (\nu + i\alpha)\Delta u + (\kappa + i\beta)\left|u\right|^2 u - \gamma u = 0,$$
$$(x, y) \in \Omega = [-1, 1] \times [-1, 1],\ 0 < t \leq 1, \tag{4.3}$$
$$u(x, y, t) = 0, \quad (x, y) \in \partial\Omega,\ 0 < t \leq T, \tag{4.4}$$
$$u(x, y, 0) = \mathrm{sech}(x)\mathrm{sech}(y)\exp(i(x + y)), \quad (x, y) \in \bar{\Omega} = \Omega \cup \partial\Omega, \tag{4.5}$$

where $\nu = \alpha = \kappa = \beta = 1, T = 1$. We take $\tau = 0.01, h_1 = h_2 = 0.05$ to verify the dissipative nature of the schemes in Figure 1. Figure 2 shows the change of the numerical solution with $\gamma = 3$ at different times.

	τ	$\tau_0 = 1/32$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$
LCNFD	$\|e^N\|_\infty$	1.81E-02	4.63E-03	1.17E-03	2.94E-04
	Order	-	1.97	1.99	1.99
NLCNFD	$\|e^N\|_\infty$	8.73E-03	2.19E-03	5.46E-04	1.36E-04
	Order	-	2.00	2.00	2.01
LCNFD	$\|e^N\|_{h,1}$	1.61E-01	4.11E-02	1.04E-03	2.61E-03
	Order	-	1.97	1.99	1.99
NLCNFD	$\|e^N\|_{h,1}$	7.75E-02	1.94E-02	4.84E-03	1.21E-03
	Order	-	2.00	2.00	2.01
LCNFD	$\|e^N\|_h$	1.14E-01	2.91E-02	7.34E-03	1.85E-03
	Order	-	1.97	1.99	1.99
NLCNFD	$\|e^N\|_h$	5.48E-02	1.37E-02	3.43E-03	8.54E-04
	Order	-	2.00	2.00	2.01

Table 9 Comparison of computational efficiency between LCNFD and NLCNFD

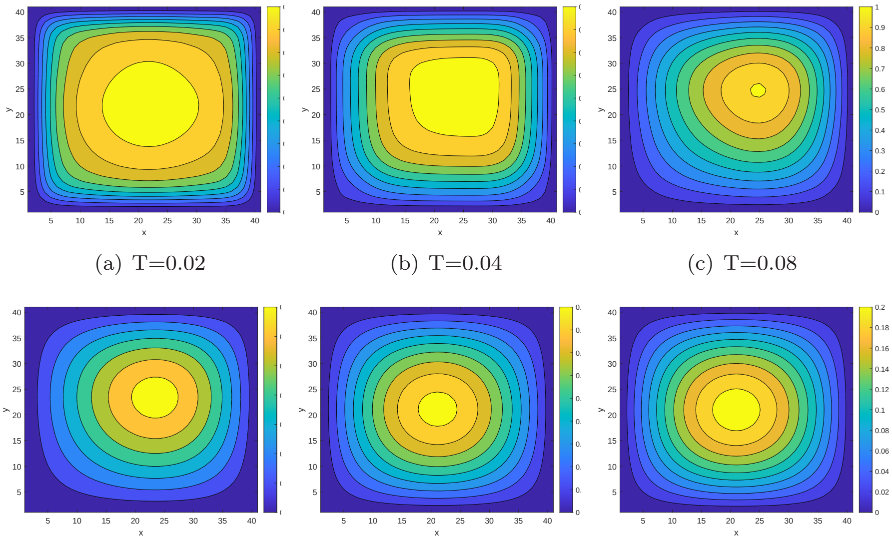


(a) LCNFD (b) NLCNFD

Figure 1 Dissipation law of two schemes with different γ

5. Conclusion

In this paper, we give detailed analysis for two Crank-Nicolson type finite difference schemes of Ginzburg-Landau equation to get their unconditional and optimal pointwise error estimates. With different assumptions of the exact solution, we get error estimates in different norm. unfortunately, the key Lemma 3.4 in our proof cannot be generalized to three dimensions. In future, we will try to search new ways to obtain optimal error estimates in three dimensions. Finally, when carrying our experiments, we find that iterations in calculating cannot be ignored for both Crank-Nicolson schemes and the nonlinear one has better computational efficiency although it performs worse in the analysis.



(a) T=0.02 (b) T=0.04 (c) T=0.08

(d) T=0.2 (e) T=0.4 (f) T=0.8

Figure 2 The change of the numerical solution at different times with $\tau = 0.01, h_1 = h_2 = 0.05$

Appendix: the proof of H2-norm error estimate

For getting the proof of Remark 3.8, we give another assumption of the exact solution as follows

$$u \in W^{3,\infty}\big([0,T]; W^{1,\infty}(\Omega)\big) \cap W^{2,\infty}\big([0,T]; W^{3,\infty}(\Omega)\big) \cap L^{\infty}\big([0,T]; W^{5,\infty}(\Omega)\big). \tag{B}$$

Part I: for LCNFD

Theorem A.1 *Under assumption* (B), *LCNFD method* (2.1)–(2.4) *is unconditionally convergent and satisfies*

$$|e^n|_{h,2} \le C(\tau^2 + h^2), \quad 0 \le n \le N. \tag{A.1}$$

For giving the proof of Theorem A.1, we give the discrete H^1-seminorm estimate on the truncation error as

Lemma A.2 *Under the assumption* (B), *we can obtain*

$$|R^{n+\frac{1}{2}}|_{h,1} \le C(\tau^2 + h^2), \quad 1 \le n \le N-1. \tag{A.2}$$

Proof By Taylor expansion, the proof can be easily got, so we omit here. □

Then we give proof of Theorem A.1 in the following.

Proof Here we will use mathematical induction again. When $n = 0, 1$, under assumption (B), there is $|e^n|_{h,2} \le C(\tau^2 + h^2)$. Assume that when $1 \le m \le n < N$, there is $|e^m|_{h,2} \le C(\tau^2 + h^2)$. With Lemma 3.5 and Theorem 3.1, and assumption (B), there are

$$\|e^m\|_\infty \le C\|e^m\|_h^{\frac{1}{2}}(\|e^m\|_h + |e^m|_{h,2})^{\frac{1}{2}} \le C(\tau^2 + h^2) \tag{A.3}$$

and

$$\|u^m\|_\infty \le \|e^m\|_\infty + \|U^m\|_\infty \le C, \quad 1 \le m \le n \le N. \tag{A.4}$$

Making inner product of (3.3) with $2\tau\delta_t^+\Delta_h e^n$, then taking the real part, we get

$$\begin{aligned}&\nu(|e^{n+1}|_{h,2}^2 - |e^n|_{h,2}^2) + 2\tau|\delta_t^+ e^n|_{h,1}^2 \\ &\quad = 2\tau\mathrm{Re}\big((\kappa+i\beta)P^{n+\frac{1}{2}} - \gamma e^{n+\frac{1}{2}} - R^{n+\frac{1}{2}}, \delta_t^+\Delta_h e^n\big)_h.\end{aligned} \tag{A.5}$$

Using partial summation formula, Cauchy-Schwarz inequality, Young inequality and Minkowski inequality, we get

$$\begin{aligned}&\nu(|e^{n+1}|_{h,2}^2 - |e^n|_{h,2}^2) + 2\tau|\delta_t^+ e^n|_{h,1}^2 \\ &\quad \le C\tau|(\kappa+i\beta)P^{n+\frac{1}{2}} - \gamma e^{n+\frac{1}{2}} - R^{n+\frac{1}{2}}|_{h,1}|\delta_t^+ e^n|_{h,1} \\ &\quad \le C(\varepsilon_2)\tau|(\kappa+i\beta)P^{n+\frac{1}{2}} - \gamma e^{n+\frac{1}{2}} - R^{n+\frac{1}{2}}|_{h,1}^2 + \varepsilon_2\tau|\delta_t^+ e^n|_{h,1}^2 \\ &\quad \le C(\varepsilon_2)\tau\big(|P^{n+\frac{1}{2}}|_{h,1}^2 + |e^{n+\frac{1}{2}}|_{h,1}^2 + |R^{n+\frac{1}{2}}|_{h,1}^2\big) + \varepsilon_2\tau|\delta_t^+ e^n|_{h,1}^2,\end{aligned} \tag{A.6}$$

for any $\varepsilon_2 > 0$. About estimates of $|P^{n+\frac{1}{2}}|_{h,1}$, we have

$$\begin{aligned}\delta_x^+ P_{j,k}^{n+\frac{1}{2}} =& |\tilde{U}_{j,k}^{n+\frac{1}{2}}|^2\delta_x^+ U_{j,k}^{n+\frac{1}{2}} - |\tilde{u}_{j,k}^{n+\frac{1}{2}}|^2\delta_x^+ u_{j,k}^{n+\frac{1}{2}} + \\ & U_{j+1,k}^{n+\frac{1}{2}}(\delta_x^+\tilde{U}_{j,k}^{n+\frac{1}{2}}(\tilde{U}_{j+1,k}^{n+\frac{1}{2}})^*) - u_{j+1,k}^{n+\frac{1}{2}}(\delta_x^+\tilde{u}_{j,k}^{n+\frac{1}{2}}(\tilde{u}_{j+1,k}^{n+\frac{1}{2}})^*) + \\ & U_{j+1,k}^{n+\frac{1}{2}}(\delta_x^+(\tilde{U}_{j,k}^{n+\frac{1}{2}})^*\tilde{U}_{j,k}^{n+\frac{1}{2}}) - u_{j+1,k}^{n+\frac{1}{2}}(\delta_x^+(\tilde{u}_{j,k}^{n+\frac{1}{2}})^*\tilde{u}_{j,k}^{n+\frac{1}{2}}) \\ =& \big(|\tilde{U}_{j,k}^{n+\frac{1}{2}}|^2 - |\tilde{u}_{j,k}^{n+\frac{1}{2}}|^2\big)\delta_x^+ U_{j,k}^{n+\frac{1}{2}} + |\tilde{u}_{j,k}^{n+\frac{1}{2}}|^2\delta_x^+ e_{j,k}^{n+\frac{1}{2}} + \\ & e_{j+1,k}^{n+\frac{1}{2}}(\delta_x^+\tilde{U}_{j,k}^{n+\frac{1}{2}}(\tilde{U}_{j+1,k}^{n+\frac{1}{2}})^*) + u_{j+1,k}^{n+\frac{1}{2}}\big(\delta_x^+\tilde{U}_{j,k}^{n+\frac{1}{2}}(\tilde{e}_{j+1,k}^{n+\frac{1}{2}})^* + \delta_x^+\tilde{e}_{j,k}^{n+\frac{1}{2}}(\tilde{u}_{j+1,k}^{n+\frac{1}{2}})^*\big) + \\ & e_{j,k}^{n+\frac{1}{2}}(\delta_x^+(\tilde{U}_{j,k}^{n+\frac{1}{2}})^*\tilde{U}_{j,k}^{n+\frac{1}{2}}) + u_{j,k}^{n+\frac{1}{2}}\big(\delta_x^+(\tilde{U}_{j,k}^{n+\frac{1}{2}})^*\tilde{e}_{j,k}^{n+\frac{1}{2}} + \delta_x^+(\tilde{e}_{j,k}^{n+\frac{1}{2}})^*\tilde{u}_{j,k}^{n+\frac{1}{2}}\big).\end{aligned} \tag{A.7}$$

From assumption (B) and inductive hypothesis (A.4),

$$\begin{aligned}|\delta_x^+ P_{j,k}^{n+\frac{1}{2}}| \le& C(|\tilde{e}_{j,k}^{n+\frac{1}{2}}|^2 + |\tilde{e}_{j,k}^{n+\frac{1}{2}}| + |\delta_x^+ e_{j,k}^{n+\frac{1}{2}}| + \\ & |e_{j,k}^{n+\frac{1}{2}}| + |e_{j+1,k}^{n+\frac{1}{2}}| + |\tilde{e}_{j+1,k}^{n+\frac{1}{2}}| + |\delta_x^+\tilde{e}_{j,k}^{n+\frac{1}{2}}|).\end{aligned} \tag{A.8}$$

Using Cauchy-Schwarz inequality and Young inequality gives

$$\|\delta_x^+ P^{n+\frac{1}{2}}\|_h \le C(\|\tilde{e}^{n+\frac{1}{2}}\|_4^2 + \|\tilde{e}^{n+\frac{1}{2}}\|_h + \|\delta_x^+ e^{n+\frac{1}{2}}\|_h + \|e^{n+\frac{1}{2}}\|_h + \|\delta_x^+\tilde{e}^{n+\frac{1}{2}}\|_h). \tag{A.9}$$

From Lemma 3.4,

$$\|\tilde{e}^{n+\frac{1}{2}}\|_4^2 \le \|\tilde{e}^{n+\frac{1}{2}}\|_h\big(C_p|\tilde{e}^{n+\frac{1}{2}}|_{h,1} + \frac{1}{l}\|\tilde{e}^{n+\frac{1}{2}}\|_h\big). \tag{A.10}$$

Substituting (A.10) and Theorem 3.1 into (A.8), we get

$$\|\delta_x^+ P^{n+\frac{1}{2}}\|_h \le C(\tau^2 + h^2). \tag{A.11}$$

Similarly, we get

$$\|\delta_y^+ P^{n+\frac{1}{2}}\|_h \le C(\tau^2 + h^2). \tag{A.12}$$

Taking $\varepsilon_2 = 2$ and substituting Lemma A.2, Theorem 3.1, (A.11), and (A.12) into (A.6), one obtains

$$|e^{n+1}|_{h,2}^2 - |e^n|_{h,2}^2 \le C\tau\big(\tau^2+h^2\big)^2. \tag{A.13}$$

From inductive hypothesis,

$$|e^{n+1}|_{h,2}^2 \le C\big(\tau^2+h^2\big)^2. \tag{A.14}$$

Thus, we get the proof of Theorem A.1. □

Part II: for NLCNFD

Theorem A.3 *Under assumption (B), NLCNFD method (2.5)–(2.7) is unconditionally convergent and satisfies*

$$|e^n|_{h,2} \le C(\tau^2+h^2), \quad 0\le n\le N. \tag{A.15}$$

Lemma A.4 *Under the assumption (B), we can obtain*

$$|\hat{R}^{n+\frac12}|_{h,1} \le C(\tau^2+h^2), \quad 0\le n\le N-1. \tag{A.16}$$

Proof By Taylor expansion, the proof can be easily got, so we omit it here. □

Then we give proof of Theorem A.3 in the following.

Proof Making inner product of (3.37) with $\delta_t^+\Delta_h\hat{e}^n$, and then taking the real part, we get

$$\begin{aligned}
&|\delta_t^+\hat{e}^n|_{h,1}^2 + \frac{\nu}{2\tau}\big(|\hat{e}^{n+1}|_{h,2}^2 - |\hat{e}^n|_{h,2}^2\big)\\
&= \mathrm{Re}\big(\hat{P}^{n+\frac12} - \hat{e}^{n+\frac12} - \hat{R}^{n+\frac12}, \delta_t^+\Delta_h\hat{e}^n\big)_h\\
&\le C|\hat{P}^{n+\frac12} - \hat{e}^{n+\frac12} - \hat{R}^{n+\frac12}|_{h,1}|\delta_t^+\hat{e}^n|_{h,1}\\
&\le C(\varepsilon_3)|\hat{P}^{n+\frac12} - \hat{e}^{n+\frac12} - \hat{R}^{n+\frac12}|_{h,1}^2 + \varepsilon_3|\delta_t^+\hat{e}^n|_{h,1}^2\\
&\le C(\varepsilon_3)\big(|\hat{P}^{n+\frac12}|_{h,1}^2 + |\hat{e}^{n+\frac12}|_{h,1}^2 + |\hat{R}^{n+\frac12}|_{h,1}^2\big) + \varepsilon_3|\delta_t^+\hat{e}^n|_{h,1}^2.
\end{aligned} \tag{A.17}$$

About estimates of $|\hat{P}^{n+\frac12}|_{h,1}$, we have

$$\begin{aligned}
\delta_x^+\hat{P}_{j,k}^{n+\frac12} =& |U_{j,k}^{n+\frac12}|^2\delta_x^+U_{j,k}^{n+\frac12} - |u_{j,k}^{n+\frac12}|^2\delta_x^+u_{j,k}^{n+\frac12} +\\
& |U_{j+1,k}^{n+\frac12}|^2\delta_x^+U_{j,k}^{n+\frac12} - |u_{j+1,k}^{n+\frac12}|^2\delta_x^+u_{j,k}^{n+\frac12} +\\
& U_{j+1,k}^{n+\frac12}(\delta_x^+(U_{j,k}^{n+\frac12})^*U_{j,k}^{n+\frac12}) - u_{j+1,k}^{n+\frac12}(\delta_x^+(u_{j,k}^{n+\frac12})^*u_{j,k}^{n+\frac12})\\
=& \big(|U_{j,k}^{n+\frac12}|^2 - |u_{j,k}^{n+\frac12}|^2\big)\delta_x^+U_{j,k}^{n+\frac12} + |u_{j,k}^{n+\frac12}|^2\delta_x^+e_{j,k}^{n+\frac12} +\\
& \big(|U_{j+1,k}^{n+\frac12}|^2 - |u_{j+1,k}^{n+\frac12}|^2\big)\delta_x^+U_{j,k}^{n+\frac12} + |u_{j+1,k}^{n+\frac12}|^2\delta_x^+e_{j,k}^{n+\frac12} +\\
& e_{j,k}^{n+\frac12}(\delta_x^+(U_{j,k}^{n+\frac12})^*U_{j,k}^{n+\frac12}) + u_{j,k}^{n+\frac12}\big(\delta_x^+(U_{j,k}^{n+\frac12})^*e_{j,k}^{n+\frac12} + \delta_x^+(e_{j,k}^{n+\frac12})^*u_{j,k}^{n+\frac12}\big).
\end{aligned} \tag{A.18}$$

From assumption (B) and Theorem 3.2,

$$|\delta_x^+\hat{P}_{j,k}^{n+\frac12}| \le C(|e_{j,k}^{n+\frac12}|^2 + |e_{j,k}^{n+\frac12}| + |\delta_x^+e_{j,k}^{n+\frac12}| + |e_{j+1,k}^{n+\frac12}|^2). \tag{A.19}$$

Using Cauchy-Schwarz inequality and Young inequality gets

$$\|\delta_x^+\hat{P}^{n+\frac12}\|_h \le C(\|e^{n+\frac12}\|_4^2 + \|e^{n+\frac12}\|_h + \|\delta_x^+e^{n+\frac12}\|_h). \tag{A.20}$$

From Lemma 3.4,

$$\|e^{n+\frac{1}{2}}\|_4^2 \le \|e^{n+\frac{1}{2}}\|_h \big(C_p|e^{n+\frac{1}{2}}|_{h,1} + \frac{1}{l}\|e^{n+\frac{1}{2}}\|_h\big). \tag{A.21}$$

Substituting (A.21) and Theorem 3.2 into (A.20), we get

$$\|\delta_x^+ \hat{P}^{n+\frac{1}{2}}\|_h \le C(\tau^2 + h^2). \tag{A.22}$$

Similarly,

$$\|\delta_y^+ \hat{P}^{n+\frac{1}{2}}\|_h \le C(\tau^2 + h^2). \tag{A.23}$$

Taking $\varepsilon_3 = 1$ and substituting Lemma A.2, Theorem 3.2, (A.22), and (A.23) into (A.17), one obtains

$$|\hat{e}^{n+1}|_{h,2}^2 - |\hat{e}^n|_{h,2}^2 \le C\tau(\tau^2 + h^2)^2. \tag{A.24}$$

Summing the above inequality up for n from 0 to m, then replacing m by n, we get

$$\begin{aligned} |\hat{e}^{n+1}|_{h,2}^2 \le& C(n+1)\tau(\tau^2 + h^2)^2 + |\hat{e}^0|_{h,2}^2 \\ \le& CT(\tau^2 + h^2)^2. \end{aligned} \tag{A.25}$$

Then we complete the proof of Theorem A.3.

$$|\hat{e}^n|_{h,2} \le C(\tau^2 + h^2), \quad 0 \le n \le N. \ \square \tag{A.26}$$

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