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Residual Operator of Type-2 Triangular Norms on Convex Normal Upper Semi-Continuous Fuzzy Truth Values

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Abstract Based on the residual implication of continuous triangular norms, we obtain the expression of residual operator for extended operators (type-2 triangular norms) of continuous triangular norms on convex normal upper semi-continuous fuzzy truth values, answering an open problem in [D. LI, Inf. Sci., 2015, **317**: 259–277].

Keywords type-2 fuzzy set; triangular norm (t-norm); residual operator

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1. Introduction and preliminaries

In order to better describe fuzziness and uncertainty, Zadeh [1] introduced the concept of type-2 fuzzy sets as a generalization of ordinary fuzzy sets in [2], by taking the mappings from the unit interval to itself as membership functions, which were called "fuzzy truth values". In the study of type-2 fuzzy sets, type-2 fuzzy operations on fuzzy truth values are the key and core. In particular, type-2 t-(co)norms [3–9], type-2 aggregation operations [10–12], and type-2 fuzzy implications [5, 9, 13, 14] based on Zadeh's extension principle or convolution operations have been further studied. Let I be the unit interval $[0, 1]$.

Definition 1.1 ([2]) *A fuzzy set A on the space X is a mapping from X to I, i.e., $A \in \mathrm{Map}(X, I)$.*

Let A be a fuzzy set on X. For $\alpha \in (0, 1]$, the α-cut $[A]_\alpha$ of A is $[A]_\alpha = \{x \in X \mid A(x) \geq \alpha\}$.

A fuzzy truth value is a fuzzy set on the unit interval I. The set of all fuzzy truth values is denoted as $\mathbf{M}$.

Definition 1.2 ([6]) *A fuzzy truth value $f \in \mathbf{M}$ is*

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(1) normal if $\sup\{f(x) \mid x \in I\} = 1$*;*

(2) convex if, for any $0 \le x \le y \le z \le 1$, $f(y) \ge f(x) \wedge f(z)$*;*

(3) upper semi-continuous if $[f]_\alpha$ *is a closed subset of* I *for any* $\alpha \in (0, 1]$.

For any subset B of X, a special fuzzy set $\mathbf{1}_B$, called the characteristic function of B, is defined by

$$\mathbf{1}_B(x) = \begin{cases} 1, & x \in B, \\ 0, & x \in X \backslash B. \end{cases}$$

Let $\mathbf{I}^{[2]} = \{[a, b] \mid 0 \le a \le b \le 1\}$, $\mathbf{N} = \{f \in \mathbf{M} \mid f \text{ is normal}\}$, $\mathbf{C} = \{f \in \mathbf{M} \mid f \text{ is convex}\}$, $\mathbf{U} = \{f \in \mathbf{M} \mid f \text{ is upper semi-continuous}\}$, and $\mathbf{L} = \mathbf{N} \cap \mathbf{C}$, $\mathbf{L}_u = \mathbf{L} \cap \mathbf{U}$.

Based on extension principle, Mizumoto and Tanaka [15] introduced the following basic operations on $\mathbf{M}$, which are the basis of type-2 fuzzy sets and type-2 fuzzy logic systems.

Definition 1.3 ([3, Definition 1.3.7]) *The operations of* $\sqcup$ *(union),* $\sqcap$ *(intersection),* $\neg$ *(complementation) on* $\mathbf{M}$ *are defined as follows: for* $f, g \in \mathbf{M}$,

$$(f \sqcup g)(x) = \sup\{f(y) \wedge g(z) \mid y \vee z = x\},$$
$$(f \sqcap g)(x) = \sup\{f(y) \wedge g(z) \mid y \wedge z = x\}$$

and

$$(\neg f)(x) = \sup\{f(y) \mid 1 - y = x\} = f(1 - x).$$

From [6], it follows that $\mathfrak{M} = (\mathbf{M}, \sqcup, \sqcap, \neg, \mathbf{1}_{\{0\}}, \mathbf{1}_{\{1\}})$ does not have a lattice structure, although $\sqcup$ and $\sqcap$ satisfy the De Morgan's laws with respect to the complementation $\neg$.

Walker and Walker [6] introduced the following partial orders $\sqsubseteq$ and $\preccurlyeq$ on $\mathbf{M}$.

Definition 1.4 ([6, Definition 13]) $f \sqsubseteq g$ *if* $f \sqcap g = f$*;* $f \preccurlyeq g$ *if* $f \sqcup g = g$.

It follows from [6, Proposition 14] that both $\sqsubseteq$ and $\preccurlyeq$ are partial orders on $\mathbf{M}$. Generally, the partial orders $\sqsubseteq$ and $\preccurlyeq$ do not coincide. In [6, 16], it was proved that $\sqsubseteq$ and $\preccurlyeq$ coincide on $\mathbf{L}$, and the subalgebra $\mathfrak{L} = (\mathbf{L}, \sqcup, \sqcap, \neg, \mathbf{1}_{\{0\}}, \mathbf{1}_{\{1\}})$ is a bounded complete lattice. In particular, $\mathbf{1}_{\{0\}}$ and $\mathbf{1}_{\{1\}}$ are the minimum and the maximum of $\mathfrak{L}$, respectively.

Kulisch and Miranker [17] introduced the following partial order $\lesssim_{\mathrm{KM}}$ for $\mathbf{I}^{[2]}$: for $[a, b]$, $[c, d] \in \mathbf{I}^{[2]}$,

$$[a, b] \lesssim_{\mathrm{KM}} [c, d] \text{ if and only if } a \le c \text{ and } b \le d.$$

By Definition 1.3, it can be verified that $[a, b] \lesssim_{\mathrm{KM}} [c, d]$ if and only if $\mathbf{1}_{[a,b]} \sqsubseteq \mathbf{1}_{[c,d]}$.

Noting that every element of $\mathbf{L}_u$ is a fuzzy number on $[0, 1]$, by [18, Chapter 4], we have

Lemma 1.5 ([18, Chapter 4]) *Let* $f \in \mathbf{M}$*. Then,* $f \in \mathbf{L}_u$ *if and only if* $[f]_\alpha$ *is a nonempty closed subinterval of* I *for all* $0 < \alpha \le 1$.

Lemma 1.6 ([18, Chapter 4]) *Let* $f, g \in \mathbf{L}_u$*. Then, the following statements are equivalent:*

(i) $f \sqsubseteq g$*;*

(ii) For any $0 < \alpha \le 1$, $\mathbf{1}_{[f]_\alpha} \sqsubseteq \mathbf{1}_{[g]_\alpha}$*;*

(iii) For any $0 < \alpha \le 1$, $[f]_\alpha \lesssim_{\mathrm{KM}} [g]_\alpha$.

Definition 1.7 ([19]) *A binary operation $\Delta : I^2 \to I$ is called triangular norm (or briefly, t-norm) on I if, for any x, y, $z \in I$, it satisfies the following properties:*

(1) $x\Delta y = y\Delta x$ *(commutativity);*

(2) $(x\Delta y)\Delta z = x\Delta(y\Delta z)$ *(associativity);*

(3) $x\Delta z \le y\Delta z$ *for* $x \le y$ *(monotonicity);*

(4) $1\Delta x = x$ *(identity).*

Recently, Hernández et al. [4] systematically studied type-2 t-norms for the following general convolution on $\mathbf{M}$.

Definition 1.8 ([4, Definition 14]) *Let $*$ be a binary operation on I, Δ be a t-norm on I, and ∇ be a t-conorm on I. Define the convolution operations $\widetilde{\Delta}_*$ and $\widetilde{\nabla}_* : \mathbf{M}^2 \to \mathbf{M}$ as follows: for $f, g \in \mathbf{M}$,*

$$(f\widetilde{\Delta}_* g)(x) = \sup\{f(y) * g(z) \mid y\Delta z = x\} \tag{1.1}$$

and

$$(f\widetilde{\nabla}_* g)(x) = \sup\{f(y) * g(z) \mid y\nabla z = x\}. \tag{1.2}$$

Clearly, $\widetilde{\min}_{\min} = \sqcap$ and $\widetilde{\max}_{\min} = \sqcup$.

Definition 1.9 ([19]) *Let Δ be a left continuous t-norm. The residual implication $\to_\Delta : I^2 \to I$ associated with Δ is defined as*

$$x \to_\Delta y = \sup\{z \in I \mid x\Delta z \le y\}.$$

Let Δ be a continuous t-norm on I. The residual operator $\to_{\widetilde{\Delta}_{\min}}$ of $\sqcap$ and $\widetilde{\Delta}_{\min}$ on $\mathbf{L}_u$ is defined in [9, 14] as follows: for f, $g \in \mathbf{L}_u$,

$$f \to_{\widetilde{\Delta}_{\min}} g = \sqcup\{h \in \mathbf{L}_u \mid f\widetilde{\Delta}_{\min} h \sqsubseteq g\}, \tag{1.3}$$

where $\sqcup A$ is the supremum of a subset A of $\mathbf{L}_u$ in $(\mathbf{L}_u, \sqsubseteq)$.

From [3, Theorem 6.8.12], it follows that $\widetilde{\Delta}_{\min}$ is left-continuous and $f\widetilde{\Delta}_{\min}(f \to_{\widetilde{\Delta}_{\min}} g) = \sqcup\{f\widetilde{\Delta}_{\min}h \mid f\widetilde{\Delta}_{\min}h \sqsubseteq g\} \sqsubseteq g$. Li [5] studied the residual operators of some special type-2 operations on $(\mathbf{L}_u, \sqsubseteq)$ and proposed the following problem for the residual operator:

Problem 1 ([5, Problem (2)]). What is the residual operator of a left-continuous type-2 t-norm $\widetilde{\Delta}_T$?

Knowing from [20, Theorem 5.10.58], each α-cut of $f\widetilde{\Delta}_{\min}h$ is equal to $[f]_\alpha \Delta [h]_\alpha$, this prompts us to stratify and obtain each α-cut of $f \to_{\widetilde{\Delta}_{\min}} g$, thereby deriving the specific expression of $f \to_{\widetilde{\Delta}_{\min}} g$ using the representation theorem of fuzzy sets. Therefore, this paper is devoted to obtaining the specific expression of the residual operator "$\to_{\widetilde{\Delta}_{\min}}$" to solve Problem 1. For a general t-norm T, firstly, the existing literature has not yet provided a characterization of the left-continuity of $\widetilde{\Delta}_T$. Secondly, there is not a theorem as convenient as [20, Theorem 5.10.58] to achieve an α-cuts representation of $f \to_{\widetilde{\Delta}_T} g$. Therefore, we are currently unable to solve Problem 1 in the general case.

2. Answer to Problem 1

Given any fixed f, $g \in \mathbf{L}_u$, to obtain the expression of $f \rightarrow_{\widetilde{\Delta}_{\min}} g$, we follow the steps below:
Step 1. For any $\alpha \in (0,1]$, take the interval $[\mathcal{T}_\alpha^{(-)}, \mathcal{T}_\alpha^{(+)}]$ as follows:

- If $\min[f]_\alpha \rightarrow_\Delta \min[g]_\alpha \le \max[f]_\alpha \rightarrow_\Delta \max[g]_\alpha$, then $[\mathcal{T}_\alpha^{(-)}, \mathcal{T}_\alpha^{(+)}] = [\min[f]_\alpha \rightarrow_\Delta \min[g]_\alpha, \max[f]_\alpha \rightarrow_\Delta \max[g]_\alpha]$;
- If $\min[f]_\alpha \rightarrow_\Delta \min[g]_\alpha > \max[f]_\alpha \rightarrow_\Delta \max[g]_\alpha$, then $[\mathcal{T}_\alpha^{(-)}, \mathcal{T}_\alpha^{(+)}] = [\max[f]_\alpha \rightarrow_\Delta \max[g]_\alpha, \max[f]_\alpha \rightarrow_\Delta \max[g]_\alpha]$.

Clearly,

$$\mathcal{T}_\alpha^{(-)} \le \min[f]_\alpha \rightarrow_\Delta \min[g]_\alpha \text{ and } \mathcal{T}_\alpha^{(+)} \le \max[f]_\alpha \rightarrow_\Delta \max[g]_\alpha. \tag{2.1}$$

Step 2. For any $\alpha \in (0,1]$, calculate $L(\alpha)$ and $R(\alpha)$ as follows:

$$L(\alpha) = \inf\{\mathcal{T}_\lambda^{(-)} \mid \lambda \ge \alpha\} \tag{2.2}$$

and

$$R(\alpha) = \inf\{\mathcal{T}_\lambda^{(+)} \mid \lambda \le \alpha\}. \tag{2.3}$$

By (2.1), we have

$$L(\alpha) \le \mathcal{T}_\alpha^{(-)} \le \min[f]_\alpha \rightarrow_\Delta \min[g]_\alpha \text{ and } R(\alpha) \le \mathcal{T}_\alpha^{(+)} \le \max[f]_\alpha \rightarrow_\Delta \max[g]_\alpha, \tag{2.4}$$

implying that

$$L(\alpha)\Delta \min[f]_\alpha \le \min[g]_\alpha \text{ and } R(\alpha)\Delta \max[f]_\alpha \le \max[g]_\alpha. \tag{2.5}$$

Proposition 2.1 *For $0 < \alpha_1 \le \alpha_2 \le 1$, $L(\alpha_1) \le L(\alpha_2)$ and $R(\alpha_1) \ge R(\alpha_2)$.*

Proof It follows directly from $\{\mathcal{T}_\lambda^{(-)} \mid \lambda \ge \alpha_1\} \supseteq \{\mathcal{T}_\lambda^{(-)} \mid \lambda \ge \alpha_2\}$ and $\{\mathcal{T}_\lambda^{(+)} \mid \lambda \ge \alpha_1\} \subseteq \{\mathcal{T}_\lambda^{(+)} \mid \lambda \ge \alpha_2\}$. □

Step 3. For any $\alpha \in (0,1]$, take the interval $[\widetilde{\mathcal{T}}_\alpha^{(-)}, \widetilde{\mathcal{T}}_\alpha^{(+)}]$ as follows:
(1) If $L(1) \le R(1)$, then $[\widetilde{\mathcal{T}}_\alpha^{(-)}, \widetilde{\mathcal{T}}_\alpha^{(+)}] = [L(\alpha), R(\alpha)]$ for all $\alpha \in (0,1]$;
(2) If $L(1) > R(1)$, then

$$[\widetilde{\mathcal{T}}_\alpha^{(-)}, \widetilde{\mathcal{T}}_\alpha^{(+)}] = \begin{cases} [R(1), R(\alpha)], & L(\alpha) \ge R(1), \\ [L(\alpha), R(\alpha)], & L(\alpha) < R(1). \end{cases} \tag{2.6}$$

Proposition 2.2 *(1) $\widetilde{\mathcal{T}}_\alpha^{(-)} \le L(\alpha)$ and $\widetilde{\mathcal{T}}_\alpha^{(+)} = R(\alpha)$.*
(2) For $0 < \alpha_1 \le \alpha_2 \le 1$, $[\widetilde{\mathcal{T}}_{\alpha_1}^{(-)}, \widetilde{\mathcal{T}}_{\alpha_1}^{(+)}] \supseteq [\widetilde{\mathcal{T}}_{\alpha_2}^{(-)}, \widetilde{\mathcal{T}}_{\alpha_2}^{(+)}] \ne \varnothing$.

Proof It follows directly from Proposition 2.1 and the choice of the interval $[\widetilde{\mathcal{T}}_\alpha^{(-)}, \widetilde{\mathcal{T}}_\alpha^{(+)}]$. □

Step 4. By the representation theorem of fuzzy sets, take the fuzzy set $\phi : I \rightarrow I$ by

$$\phi(x) = \sup\{\alpha \in (0,1] \mid x \in [\widetilde{\mathcal{T}}_\alpha^{(-)}, \widetilde{\mathcal{T}}_\alpha^{(+)}]\}, \tag{2.7}$$

where $\sup \varnothing = 0$.

It follows directly from the representation theorem of fuzzy sets and Proposition 2.2 (2) that, for any $\alpha \in (0,1]$,

$$[\phi]_\alpha = \bigcap_{\lambda<\alpha} [\widetilde{\mathcal{T}}_\lambda^{(-)}, \widetilde{\mathcal{T}}_\lambda^{(+)}] = [\lim_{\lambda\to\alpha^-} \widetilde{\mathcal{T}}_\lambda^{(-)}, \lim_{\lambda\to\alpha^-} \widetilde{\mathcal{T}}_\lambda^{(+)}] \supseteq [\widetilde{\mathcal{T}}_\alpha^{(-)}, \widetilde{\mathcal{T}}_\alpha^{(+)}]. \tag{2.8}$$

This, together with Lemma 1.5, implies that $\phi \in \mathbf{L}_u$.

Lemma 2.3 ([20, Theorem 5.10.58]) *Let Δ be a continuous t-norm on I. Then, for any f, $g \in \mathbf{L}_u$ and any $\alpha \in (0,1]$, we have $[f\widetilde{\Delta}_{\min} g]_\alpha = [f]_\alpha \Delta [g]_\alpha$.*

Proposition 2.4 *Let ϕ be defined by Eq. (2.7). Then, $f\widetilde{\Delta}_{\min}\phi \sqsubseteq g$.*

Proof Given any fixed $\alpha \in (0,1]$, it is clear that $\min[f]_{\alpha-\frac{1}{n}} \nearrow \min[f]_\alpha$ and $\max[f]_{\alpha-\frac{1}{n}} \searrow \max[f]_\alpha$. Together with Lemma 2.3, Proposition 2.2 (1), and Eq. (2.8), since Δ is continuous and increasing, we have

(i) $\min[f\widetilde{\Delta}_{\min}\phi]_\alpha = \min[f]_\alpha \Delta \min[\phi]_\alpha = \min[f]_\alpha \Delta (\lim_{\lambda\to\alpha^-} \widetilde{\mathcal{T}}_\lambda^{(-)})$

$$= \lim_{n\to+\infty} (\min[f]_{\alpha-\frac{1}{n}} \Delta \widetilde{\mathcal{T}}^{(-)}_{\alpha-\frac{1}{n}}) \le \lim_{n\to+\infty} (\min[f]_{\alpha-\frac{1}{n}} \Delta L(\alpha - \frac{1}{n}))$$
$$\le \lim_{n\to+\infty} \min[g]_{\alpha-\frac{1}{n}} = \min[g]_\alpha \quad \text{by Eq. (2.5)};$$

(ii) $\max[f\widetilde{\Delta}_{\min}\phi]_\alpha = \max[f]_\alpha \Delta \max[\phi]_\alpha = \max[f]_\alpha \Delta (\lim_{\lambda\to\alpha^-} \widetilde{\mathcal{T}}_\lambda^{(+)})$

$$= \lim_{n\to+\infty} (\max[f]_{\alpha-\frac{1}{n}} \Delta \widetilde{\mathcal{T}}^{(+)}_{\alpha-\frac{1}{n}}) = \lim_{n\to+\infty} (\max[f]_{\alpha-\frac{1}{n}} \Delta R(\alpha - \frac{1}{n}))$$
$$\le \lim_{n\to+\infty} \max[g]_{\alpha-\frac{1}{n}} = \max[g]_\alpha \quad \text{by Eq. (2.5)},$$

implying that $[f\widetilde{\Delta}_{\min}\phi]_\alpha \lesssim_{\mathrm{KM}} [g]_\alpha$. Thus, $f\widetilde{\Delta}_{\min}\phi \sqsubseteq g$ by Lemma 1.6. □

Theorem 2.5 *The fuzzy set ϕ obtained by Eq. (2.7) is equal to $f \to_{\widetilde{\Delta}_{\min}} g$.*

Proof By Proposition 2.4 and Eq. (1.3), it suffices to show that, for any $h \in \mathbf{L}_u$ with $f\widetilde{\Delta}_{\min}h \sqsubseteq g$, we have $h \sqsubseteq \phi$.

Claim 1. For any $\alpha \in (0,1]$, $[h]_\alpha \lesssim_{\mathrm{KM}} [\mathcal{T}_\alpha^{(-)}, \mathcal{T}_\alpha^{(+)}]$.

By Lemmas 1.6 and 2.3, noting that $f\widetilde{\Delta}_{\min}h \sqsubseteq g$, we have $\min[f]_\alpha \Delta \min[h]_\alpha \le \min[g]_\alpha$ and $\max[f]_\alpha \Delta \max[h]_\alpha \le \max[g]_\alpha$, implying that $\min[h]_\alpha \le \min[f]_\alpha \to_\Delta \min[g]_\alpha$ and $\max[h]_\alpha \le \max[f]_\alpha \to_\Delta \max[g]_\alpha$. Then $[h]_\alpha \lesssim_{\mathrm{KM}} [\mathcal{T}_\alpha^{(-)}, \mathcal{T}_\alpha^{(+)}]$ by the construction of $[\mathcal{T}_\alpha^{(-)}, \mathcal{T}_\alpha^{(+)}]$.

Claim 2. For any $\alpha \in (0,1]$, $\min[h]_\alpha \le L(\alpha)$ and $\max[h]_\alpha \le R(\alpha)$.

It is clear that $\min[h]_\alpha = \inf\{\min[h]_\lambda \mid \lambda \ge \alpha\}$ and $\max[h]_\alpha = \inf\{\max[h]_\lambda \mid \lambda \le \alpha\}$. This, together with Claim 1, implies that $\min[h]_\alpha \le \inf\{\mathcal{T}_\lambda^{(-)} \mid \lambda \ge \alpha\} = L(\alpha)$ and $\max[h]_\alpha \le \inf\{\mathcal{T}_\lambda^{(+)} \mid \lambda \le \alpha\} = R(\alpha)$.

Claim 3. For any $\alpha \in (0,1]$, $[h]_\alpha \lesssim_{\mathrm{KM}} [\widetilde{\mathcal{T}}_\alpha^{(-)}, \widetilde{\mathcal{T}}_\alpha^{(+)}]$.

By the choice of the interval $[\widetilde{\mathcal{T}}_\alpha^{(-)}, \widetilde{\mathcal{T}}_\alpha^{(+)}]$, consider the following two cases:

Case 1. If $L(1) \le R(1)$, by Claim 2, it is clear that $[h]_\alpha \lesssim_{\mathrm{KM}} [L(\alpha), R(\alpha)] = [\widetilde{\mathcal{T}}_\alpha^{(-)}, \widetilde{\mathcal{T}}_\alpha^{(+)}]$;

Case 2. If $L(1) > R(1)$, by Claim 2, we have $\min[h]_\alpha \le \min[h]_1 \le \min\{L(1), R(1)\} = R(1)$

and $\max[h]_\alpha \le R(\alpha)$, and thus $[h]_\alpha \lesssim_{\mathrm{KM}} [\widetilde{\mathcal{T}}_\alpha^{(-)}, \widetilde{\mathcal{T}}_\alpha^{(+)}]$ by Eq. (2.6).

Claim 4. For any $\alpha \in (0,1]$, $[h]_\alpha \lesssim_{\mathrm{KM}} [\phi]_\alpha$, i.e., $h \sqsubseteq \phi$.

By $[h]_\alpha = \bigcap_{\lambda<\alpha}[h]_\lambda$, noting that each α-cut of h is a nonempty closed subinterval of I (by Lemma 1.5), we have $\min[h]_\alpha = \lim_{\lambda\to\alpha^-}\min[h]_\lambda$ and $\max[h]_\alpha = \lim_{\lambda\to\alpha^-}\max[h]_\lambda$. This, together with Eq. (2.8) and Claim 3, implies that $\min[h]_\alpha = \lim_{\lambda\to\alpha^-}\min[h]_\lambda \le \lim_{\lambda\to\alpha^-}\widetilde{\mathcal{T}}_\lambda^{(-)} = \min[\phi]_\alpha$ and $\max[h]_\alpha = \lim_{\lambda\to\alpha^-}\max[h]_\lambda \le \lim_{\lambda\to\alpha^-}\widetilde{\mathcal{T}}_\lambda^{(+)} = \max[\phi]_\alpha$, i.e., $[h]_\alpha \lesssim_{\mathrm{KM}} [\phi]_\alpha$. □

Processing Steps 1–4, by Theorem 2.5, we have

- $\mathbf{1}_{[a,b]} \to_{\widetilde{\Delta}_{\min}} \mathbf{1}_{[c,d]} = \begin{cases} \mathbf{1}_{[a\to_\Delta c,\, b\to_\Delta d]}, & a\to_\Delta c \le b\to_\Delta d, \\ \mathbf{1}_{\{b\to_\Delta d\}}, & a\to_\Delta c > b\to_\Delta d. \end{cases}$

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