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On the Normalized Laplacian Spectral Radius of Traceable Graphs

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Abstract In this paper, we give some sufficient conditions for a graph to be traceable in terms of its order and size. As applications, the normalized Laplacian spectral conditions for a graph to be traceable are established.

Keywords traceable graphs; normalized Laplacian spectral radius; degree sequence

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1. Introduction

All graphs considered in this paper are undirected simply connected graphs. Let $G=(V,E)$ denote a connected graph with vertex set V and edge set E. Let $|V(G)|=n$ and $|E(G)|=m=e(G)$ be the order and the size of G, respectively. For any vertex $v_i\in V(G)$, we denote by $d_i=d_{v_i}=d_G(v_i)$ the degree of v_i. Let $\pi=(d_1,d_2,\ldots,d_n)$ be the degree sequence of G, where $d_1\le d_2\le\cdots\le d_n$. Denote by $\delta(G)$ or simple δ the minimum degree of G.

Let G_1 and G_2 be two graphs. We use G_1+G_2 to denote the disjoint union of G_1 and G_2, and $G_1\vee G_2$ to denote the join of G_1 and G_2. As usual, let P_n and C_n denote the path and cycle on n vertices, respectively. The dumbbell graph, denoted by $D_{p,k,q}$, is the graph obtained from two cycles C_p, C_q and a path P_{k+2} by identifying each pendant vertex of P_{k+2} with a vertex of a cycle, respectively. The θ-graph, denoted by $\theta_{r,s,t}$, is the graph formed by joining two given vertices via three disjoint paths P_r, P_s and P_t, respectively. The p-rose graph is obtained by p cycles sharing a common vertex v, which is recorded as $R(C_1,\ldots,C_p)$, where $C_1,\ldots,C_p$ represent p cycles with a common vertex v, respectively.

Let $D(G)$ be the diagonal degree matrix, and $A(G)$ the adjacent matrix of G. The matrix $L(G)=D(G)-A(G)$ and $\mathfrak{L}(G)=D(G)^{-1/2}(D(G)-A(G))D(G)^{-1/2}$ (i.e., $\mathfrak{L}(G)=D(G)^{-1/2}L(G)D(G)^{-1/2}$) are defined the Laplacian matrix and the normalized Laplacian matrix of G, respectively. The largest eigenvalue of $\mathfrak{L}(G)$, denoted by $\rho(G)$, is called the normalized Laplacian spectral radius of G. In [1], Chung proved that $\frac{n-1}{n}\le\rho(G)\le 2$ for a connected graph

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G with $n \geq 2$ vertices, the left equality holds if and only if G is a complete graph, and the right equality holds if and only if G is a bipartite graph.

A path (cycle) that contains every vertex of G is called a Hamilton path (cycle) of G. A graph is traceable (Hamiltonian) if it contains a Hamilton path (cycle). And G is Hamilton-connected if every two vertices of G are connected by a Hamilton path.

It is an old problem to determine whether a given graph is traceable or not. Recently, there are many reasonable sufficient conditions that were given for a graph to be Hamiltonian, traceable or Hamilton-connected, see references [2–7] and therein. In 2010, Fiedler and Nikiforov [8] gave strict sufficient conditions for the existence of Hamilton paths and cycles in terms of the adjacency spectral radius of graphs or the complement of graphs, and Zhou [9] studied the signless Laplacian spectral radius of the complement of a graph, and presented some conditions for the existence of Hamilton cycles or paths. Later, Lu et al. [10] showed sufficient conditions for Hamilton paths in connected graphs and Hamilton cycles in bipartite graphs in terms of the adjacency spectral radius of a graph. Liu et al. [11] mentioned sufficient conditions on the adjacency spectral radius for a graph or a bipartite graph to be Hamiltonian and traceable. Recently, Wang et al. in [12–18] gave some sufficient conditions on adjacency spectral radius, distance signless Laplacian spectral radius and signless Laplacian spectral radius of G for the graph to be Hamiltonian, Hamilton-connected and k-connected, respectively. Indeed, more scholars have focused on sufficient conditions for the hamiltonicity of G in terms of lower bounds on the adjacency spectral radius and the signless Laplacian spectral radius of G, respectively, but for the normalized Laplacian spectral radius, it has been rarely mentioned. In fact, the problem of the sufficient conditions for a graph to be traceable in terms of the normalized Laplacian spectral radius of the graph has far from been resolved.

In this paper, we first give some sufficient conditions for a graph to be traceable in terms of its size and order, then establish the sufficient conditions for a graph to be traceable in terms of the normalized Laplacian spectral radius of the graph.

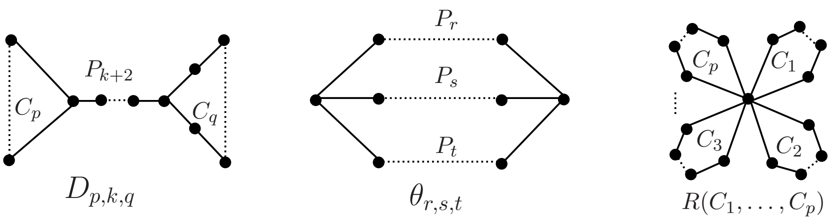

Figure 1 Graphs $D_{p,k,q}$, $\theta_{r,s,t}$ and $R(C_1, \ldots, C_p)$

2. Lemmas and main results

Firstly, we give some lemmas which are used to prove the main results.

Lemma 2.1 ([19]) *Let G be a nontrivial simple graph with degree sequence $\pi = (d_1, d_2, \ldots, d_n)$, where $d_1 \leq d_2 \leq \cdots \leq d_n$ and $n \geq 4$. Suppose that there is no integer $k < \frac{n+1}{2}$ such that*

$d_k \le k-1$ and $d_{n-k+1} \le n-k-1$. *Then G is traceable.*

Lemma 2.2 ([20]) *Let G be a connected graph on $n \ge 4$ vertices of m edges with $\delta \ge 1$. If*

$$m \ge \binom{n-2}{2} + 2,$$

then G contains a Hamilton path unless $G \in \{K_1 \vee (K_{n-3} + 2K_1), K_1 \vee (K_{1,3} + K_1), K_{2,4}, K_2 \vee 4K_1, K_2 \vee (3K_1 + K_2), K_1 \vee K_{2,5}, K_3 \vee 5K_1, K_2 \vee (K_{1,4} + K_1), K_4 \vee 6K_1\}$.

A sequence π is called a permissible graphic sequence if there is a simple graph with degree sequence π. According to Lemma 2.2, we can obtain the following corollary. In order to enhance its readability, we also give a new proof subsequently.

Corollary 2.3 *Let G be a connected graph on $n \ge 4$ vertices of m edges with $\delta \ge 2$. If*

$$m > \frac{n^2 - 4n + 1}{2}, \tag{2.1}$$

then G is traceable unless $G \in \{K_3 \vee 5K_1, K_2 \vee (K_1 + K_{1,4}), K_1 \vee K_{2,5}, K_2 \vee (K_2 + 3K_1), K_2 \vee 4K_1, K_{2,4}\}$.

Proof Let π be a permissible graphic sequence satisfying the condition of Lemma 2.1. Assume that G is not a traceable graph with degree sequence $d_1 \le d_2 \le \cdots \le d_n$. Then by Lemma 2.1, there is an integer $k < \frac{n+1}{2}$ such that $d_k \le k-1$ and $d_{n-k+1} \le n-k-1$. Obviously, $d_n \le n-1$. Thus,

$$\begin{aligned} 2m &= \sum_{i=1}^{n} d_i = \sum_{i=1}^{k} d_i + \sum_{i=k+1}^{n-k+1} d_i + \sum_{i=n-k+2}^{n} d_i \\ &\le k(k-1) + (n-2k+1)(n-k-1) + (n-1)(k-1) \\ &= n^2 - 4n + 1 + f(k), \end{aligned} \tag{2.2}$$

where $f(k) = 3k^2 - (2n+1)k + 3n - 1$. Combining (2.1) and (2.2) we have $f(k) > 0$. Moreover, we notice that $5 \le 2\delta + 1 \le 2(d_k + 1) - 1 \le 2k - 1 < n$ and $3 \le \delta + 1 \le d_k + 1 \le k < \frac{n+1}{2}$, which implies that $n \ge 6$ and $3 \le k < \frac{n+1}{2}$.

It follows that the two roots of $f(k) = 0$ are

$$k_1 = \frac{2n+1-\sqrt{4n^2-32n+13}}{6}, \quad k_2 = \frac{2n+1+\sqrt{4n^2-32n+13}}{6},$$

respectively. Since $f(k) > 0$ and $3 \le k < \frac{n+1}{2}$, we have either $3 \le k < k_1$ or $k_2 < k < \frac{n+1}{2}$.

For $3 \le k < k_1$, together with $k_1 = \frac{2n+1-\sqrt{4n^2-32n+13}}{6}$ we have $n \le 7$.

For $k_2 < k < \frac{n+1}{2}$, if n is even, then $k_2 \le \frac{n}{2}$, together with $k_2 = \frac{2n+1+\sqrt{4n^2-32n+13}}{6}$ we have $2 \le n \le 8$. Otherwise, $k_2 \le \frac{n-1}{2}$, similarly, $1 \le n \le 7$. Thus, $1 \le n \le 8$.

On the other hand, we notice that $n \ge 6$. Hence, from above we have $n = 6, 7, 8$. Now we discuss two cases below.

Case 1. $n = 7$ or $n = 8$.

Note that if $n = 7$ or $n = 8$, then $\lfloor \frac{n^2-4n+1}{2} \rfloor + 1 = \binom{n-2}{2} + 2$. Since m is an integer and

$m > \frac{n^2-4n+1}{2}$, we have

$$m \geq \lfloor \frac{n^2-4n+1}{2} \rfloor + 1 = \binom{n-2}{2} + 2.$$

Thus, from Lemma 2.2 and $\delta \geq 2$ one can obtain that

$$G \in \{K_3 \vee 5K_1, K_2 \vee (K_1 + K_{1,4}), K_1 \vee K_{2,5}, K_2 \vee (K_2 + 3K_1)\}.$$

Case 2. $n = 6$.

If $n = 6$, then $7 = \lfloor \frac{n^2-4n+1}{2} \rfloor + 1 < \binom{n-2}{2} + 2 = 8$. Since $m > \frac{n^2-4n+1}{2}$, we have

$$\lfloor \frac{n^2-4n+1}{2} \rfloor + 1 \leq m < \binom{n-2}{2} + 2 \text{ or } m \geq \binom{n-2}{2} + 2,$$

i.e., $7 \leq m < 8$ or $m \geq 8$. On the other hand, since $3 \leq k < \frac{n+1}{2} = 3.5$, we get $k = 3$, and so $f(3) = 5$. Furthermore, combining (2.1) and (2.2), one can obtain $13 < \sum_{i=1}^{6} d_i = 2m \leq 18$, and so $\sum_{i=1}^{6} d_i = 2m = 14, 16, 18$, i.e., $m = 7, 8, 9$.

Subcase 2.1. $m = 8$ or $m = 9$.

Clearly, $m = 8 = \binom{n-2}{2} + 2$ and $m = 9 > \binom{n-2}{2} + 2$ if $n = 6$. Thus, it follows from Lemma 2.2 and $\delta \geq 2$ that $G \in \{K_2 \vee 4K_1, K_{2,4}\}$.

Subcase 2.2. $m = 7$.

It is easy to see that $k = 3$. Then by Lemma 2.1 we have that $d_1 \leq d_2 \leq d_3 \leq d_4 \leq 2$. Obviously, $d_5 \leq d_6 \leq 5$. Hence, the permissible graphic sequences π and G are as follows.

$$\begin{cases} \pi = (2,2,2,2,2,4), & G \cong R(C_3, C_4), \\ \pi = (2,2,2,2,3,3), & G \cong \theta_{4,3,3},\ D_{3,0,3},\ \theta_{4,4,2},\ \theta_{5,3,2}. \end{cases}$$

Now we prove that those permissible graphic sequences do exist. Since $d_1 \leq d_2 \leq d_3 \leq d_4 \leq 2$ and $\delta(G) \geq 2$, we have $d_1 = d_2 = d_3 = d_4 = 2$, and thus, it follows from $\sum_{i=1}^{6} d_i = 2m = 14$ that $d_5 + d_6 = 6$. Since $d_5 \leq d_6 \leq 5$ and $d_5 \geq d_4 = 2$, we have either $d_5 = 2, d_6 = 4$ or $d_5 = d_6 = 3$. Hence the permissible graphic sequences are $\pi = (2,2,2,2,2,4)$ and $\pi = (2,2,2,2,3,3)$.

Next we show that $R(C_3, C_4)$ is determined by the degree sequence $\pi = (2,2,2,2,2,4)$. Clearly, the vertex of degree 4 must be adjacent to four vertices of degree 2, it implies that there is one vertex of degree 2 adjacent to two vertices of degree 2, and so, together with its adjacent vertices and the vertex of degree 4 these form a C_4. At this time, the other two vertices of degree 2 must be adjacent to meet the requirement of degree 2, so they form a C_3 with the vertex of degree 4. Hence G is 2-rose graph with C_3 and C_4, i.e., $G \cong R(C_3, C_4)$.

Finally, we verify that $\theta_{4,3,3}$, $D_{3,0,3}$, $\theta_{4,4,2}$ and $\theta_{5,3,2}$ are also determined by $\pi = (2,2,2,2,3,3)$.

(i) If the two vertices of degree 3 are not adjacent, then they should be adjacent to three vertices of degree 2, but their neighbors in these vertices of degree 2 are not same exactly, since if not, a loop must appear, it is a contradiction. At this time, in these four vertices of degree 2, there must be two vertices receiving degree 2 and two vertices receiving degree 1 from others, so the temporary two vertices of degree 1 must be adjacent. Hence G is θ-graph, obtained by joining two vertices of degree 3 via three disjoint paths P_4, P_3, P_3, respectively, i.e., $G \cong \theta_{4,3,3}$.

(ii) If two vertices of degree 3 are adjacent in $\pi = (2,2,2,2,3,3)$, then they should be adjacent to two vertices of degree 2. Next we discuss whether their neighbors of the 3-degree vertices are the same or not. When these neighbours are different exactly and two parts are not adjacent each other, it implies that two vertices of degree 2 are adjacent in each part. Thus, the two vertices of degree 3 and its neighbours form two distinct C_3. So, G is dumbbell graph and $G \cong D_{3,0,3}$. When the neighbours of two vertices of degree 3 are different exactly and they are adjacent each other between two parts. Hence G is obtained by joining two vertices of degree 3 via three disjoint paths P_4, P_4, P_2, respectively. That is, $G \cong \theta_{4,4,2}$. When the neighbours of two vertices of degree 3 are not same exactly, there must be one vertex receiving degree 2, two vertices receiving degree 1 and an isolated vertex in these four vertices of degree 2, so the temporary two vertices of degree 1 and an isolated vertex are adjacent to satisfy the requirement of degree 2. Hence G can be obtained by joining two vertices of degree 3 via three disjoint paths P_5, P_3, P_2, respectively, i.e., $G \cong \theta_{5,3,2}$. When the neighbours of two vertices of degree 3 are same exactly, the two vertices of degree 2 form two loops, hence it is impossible.

In fact, $R(C_3, C_4), \theta_{4,3,3}, D_{3,0,3}, \theta_{4,4,2}$ and $\theta_{5,3,2}$ in above contain Hamilton path, which contradict our assumption.

Summing up above, $G \in \{K_3 \vee 5K_1, K_2 \vee (K_1 + K_{1,4}), K_1 \vee K_{2,5}, K_2 \vee (K_2 + 3K_1), K_2 \vee 4K_1, K_{2,4}\}$, and thus, the proof is completed. □

Lemma 2.4 ([21]) *Let G be a graph of order n and size m. Let $a = \max_{uv \in E(G)}\{d(u) + d(v)\}$ and $b = \max_{uv \notin E(G)}\{d(u) + d(v)\}$. Then*

(1) $\rho(G) \geq \frac{2m(a-2)}{a(2m-a)}$ and the equality holds if $G \cong K_n$;

(2) $\rho(G) \geq \frac{2m}{2m-b}$ and the equality holds if $G \cong K_{2,n-2}$, where $G \cong K_{2,n-2}$ is the complete bipartite graph with parts of cardinalities 2 and $n-2$.

Lemma 2.5 ([22]) *Let X be an eigenvector of $\mathfrak{L}(G)$ corresponding to $\rho(G)$. Then, for any $v \in V(G)$, we have*

$$\sum_{uv \in E(G)} \frac{X_u}{\sqrt{d_u}} = \sqrt{d_v}(1 - \rho(G))X_v. \tag{2.3}$$

For $n \geq 4$ and $2 \leq k < \frac{n+1}{2}$, we define

$$H_n^k = K_{k-1} \vee (K_{n-(2k-1)} + kK_1).$$

Furthermore, for $\ell \geq 1$, let $H_n^k(\ell)$ denote the set of all possible graphs obtained from H_n^k by deleting exactly ℓ edges such that $\delta \geq 3$. Obviously, $H_n^k(0) = H_n^k$ if $\ell = 0$.

Theorem 2.6 *Let G be a connected graph on $n \geq 10$ vertices of m edges with $\delta \geq 3$. If*

$$m \geq \binom{n-3}{2} + 7, \tag{2.4}$$

then G is traceable unless $G \in \{H_{10}^5(\ell), H_{11}^5, H_{12}^6(\ell), H_{14}^7, K_5 \vee (K_1 + K_{1,7}), K_4 \vee K_{2,8} | \ell = 0,1,2\}$.

Proof Let π be a permissible graphic sequence satisfying the condition of Lemma 2.1. Assume that G is not traceable graph with $\pi = (d_1, d_2, \ldots, d_n)$, where $d_1 \leq d_2 \leq \cdots \leq d_n$ and $n \geq 10$.

By Lemma 2.1 there exists an integer $k < \frac{n+1}{2}$ such that $d_k \le k-1$ and $d_{n-k+1} \le n-k-1$. Clearly, $d_n \le n-1$. Thus,

$$\begin{aligned} m &= \frac{1}{2}\sum_{i=1}^{n} d_i = \frac{1}{2}\Big(\sum_{i=1}^{k} d_i + \sum_{i=k+1}^{n-k+1} d_i + \sum_{i=n-k+2}^{n} d_i\Big) \\ &\le \frac{1}{2}(k(k-1) + (n-2k+1)(n-k-1) + (n-1)(k-1)) \\ &= \binom{n-3}{2} + 7 + \frac{f(k)}{2}, \end{aligned} \tag{2.5}$$

where $f(k) = 3k^2 - (2n+1)k + 6n - 26$. Combining (2.4) and (2.5) we have $f(k) \ge 0$. Moreover, we notice that $4 \le \delta + 1 \le d_k + 1 \le k < \frac{n+1}{2}$, which implies that $4 \le k < \frac{n+1}{2}$.

It is easy to obtain that the two roots of $f(k) = 0$ are

$$k_1 = \frac{2n+1-\sqrt{4n^2-68n+313}}{6}, \quad k_2 = \frac{2n+1+\sqrt{4n^2-68n+313}}{6}.$$

Since $f(k) \ge 0$ and $4 \le k < \frac{n+1}{2}$, we have either $4 \le k \le k_1$ or $k_2 \le k < \frac{n+1}{2}$.

For $4 \le k \le k_1$, it follows that $n < 9$. In fact, $n \ge 10$. Thus, there is no required graph.

For $k_2 \le k < \frac{n+1}{2}$, if n is even, then $k_2 \le \frac{n}{2}$, we notice that $k_2 = \frac{2n+1+\sqrt{4n^2-68n+313}}{6}$, it follows that $6 \le n \le 14$. Otherwise, $k_2 \le \frac{n-1}{2}$. Similarly we can get $9 \le n \le 11$. Thus, combining with $n \ge 10$ we have $n = 10, 11, 12, 14$.

Now we consider k and $f(k)$ again in the following.

If $n = 10$, then $k_2 \approx 4.46 < k < \frac{10+1}{2}$, i.e., $k = 5$. By a simple calculation, we have $f(5) = 4$;

If $n = 11$, then $k_2 = 5 \le k < \frac{11+1}{2}$, i.e., $k = 5$, and so $f(5) = 0$;

If $n = 12$, then $k_2 \approx 5.59 < k < \frac{12+1}{2}$, i.e., $k = 6$, and so $f(6) = 4$;

If $n = 14$, then $k_2 \approx 6.84 < k < \frac{14+1}{2}$, i.e., $k = 7$, and so $f(7) = 2$.

From the calculating results above, we distinguish the following four cases.

Case 1. $n = 10$ and $k = 5$.

By Lemma 2.1 we have

$$d_1 \le d_2 \le d_3 \le d_4 \le d_5 \le d_6 \le 4. \tag{2.6}$$

Clearly,

$$d_7 \le d_8 \le d_9 \le d_{10} \le 9. \tag{2.7}$$

Furthermore, we notice that $f(5) = 4$ when $n = 10$. Combining (2.4) and (2.5), one can obtain $28 \le m \le 30$.

If $m = 30$, then all inequalities in (2.5) should hold. Thus, G is a graph with $d_1 = d_2 = d_3 = d_4 = d_5 = d_6 = 4$, $d_7 = d_8 = d_9 = d_{10} = 9$, which implies $G \cong H_{10}^5$. Next we prove the uniqueness of the graph G determined by the degree sequence $\pi = (4,4,4,4,4,4,9,9,9,9)$. Clearly, the four vertices of degree 9 must be adjacent to every vertex, so these induce a subgraph K_4. Furthermore, the remaining six vertices now have degree 4, so they induce a $6K_1$. Hence $G \cong H_{10}^5$. If $28 \le m \le 29$, we may assume that

$$e(G) = 30 - \ell = e(H_{10}^5) - \ell, \ell \in \{1, 2\}. \tag{2.8}$$

From $\sum_{i=1}^{10} d_i = 60 - 2\ell \geq 56$ we conclude that G has at least one 4-degree vertex since if not, we have $d_1 = d_2 = d_3 = d_4 = d_5 = d_6 = 3$ due to $\delta \geq 3$, $d_7 \leq d_8 \leq d_9 \leq d_{10} \leq 9$, i.e., $\sum_{i=1}^{10} d_i = 60 - 2\ell \leq 54 < 56$, a contradiction.

Suppose that $d_G(x_0) = 4$ and $H_1 = G[V(G) \setminus \{x_0\}]$. It is easy to see that $|V(H_1)| = 9$ and $\delta(H_1) \geq 2$ since $\delta(G) \geq 3$. Moreover, we notice that $e(H_1) = e(G) - 4 \geq 28 - 4 = 24 > \binom{9-2}{2} + 2 = 23$, and thus, H_1 is traceable by Lemma 2.2, which implies that there exist two vertices w and w' such that they are connected by a path passing through all vertices in H_1. Assume that wPw' is a Hamilton path from w to w' (say) in H_1. Let u be a vertex of the path. We denote by u^+ and u^- the successor and predecessor of u, respectively. We suppose that y_1, y_2, y_3, y_4 are the neighbors of x_0 on wPw', successively. To promote the proof, we need to show the following claim.

Claim 2.1. $\{x_0, w, y_1^+, y_2^+, y_3^+, w'\}$ is an independent set.

Proof If at least one of $w = y_1$ and $w' = y_4$ is true, clearly, there exists a Hamilton path in G, which leads to a contradiction. Otherwise, we state that $y_{i+1} \neq y_i^+$ $(1 \leq i \leq 3)$ in wPw' since if not, there exists a Hamilton path $w \cdots y_i x_0 y_{i+1} \cdots w'$ in G, also a contradiction. Hence, at least one vertex exists between y_i and y_{i+1} $(1 \leq i \leq 3)$ in wPw'. Since $w \neq y_1$ and $w' \neq y_4$, there are at least six distinct vertices $w, y_1, y_2, y_3, y_4, w'$ on wPw'. On the other hand, since $|V(H_1)| = 9$ and at least one vertex exists between y_i and y_{i+1} $(1 \leq i \leq 3)$, there is just one vertex between y_i and y_{i+1} $(1 \leq i \leq 3)$, i.e., y_i^+ for $1 \leq i \leq 3$. Thus, wPw' can be written as $wy_1y_1^+y_2y_2^+y_3y_3^+y_4w'$. Clearly, we have $x_0w, x_0w', x_0y_i^+ \notin E(G)$ because $w \neq y_1$, $w' \neq y_4$ and $y_i^+ \neq y_{i+1}$ for $1 \leq i \leq 3$. Here one can conclude that $y_i^+y_j^+ \notin E(G)$ for $1 \leq i, j \leq 3$ since if not, there exists a Hamilton path $w \cdots y_i^+ y_j^+ \cdots y_{i+1} x_0 y_{j+1} \cdots w'$ in G, it leads to a contradiction. Similarly, one can obtain $wy_i^+, w'y_i^+, ww' \notin E(G)$ for $1 \leq i \leq 3$. Therefore, the claim holds. □

Combining (2.6)–(2.8) and Claim 2.1, one can see that the graph G is obtained from H_{10}^5 by deleting any ℓ $(\ell \in \{1, 2\})$ edges such that $\delta(G) \geq 3$, i.e., $G \in H_{10}^5(\ell)$ where $\ell \in \{1, 2\}$.

Case 2. $n = 11$ and $k = 5$.

Similarly, by Lemma 2.1 we have

$$d_1 \leq d_2 \leq d_3 \leq d_4 \leq d_5 \leq 4, d_6 \leq d_7 \leq 5.$$

Obviously,

$$d_8 \leq d_9 \leq d_{10} \leq d_{11} \leq 10.$$

Furthermore, we notice that $f(5) = 0$ when $n = 11$. Combining (2.4) and (2.5) one can obtain $m = \binom{n-3}{2} + 7 = 35$, and thus, all inequalities in (2.5) should be held. Thus, G is a graph with $d_1 = d_2 = d_3 = d_4 = d_5 = 4$, $d_6 = d_7 = 5$, $d_8 = d_9 = d_{10} = d_{11} = 10$, which implies $G \cong H_{11}^5$. Next we prove the uniqueness of the graph G determined by degree sequence $\pi = (4, 4, 4, 4, 4, 5, 5, 10, 10, 10, 10)$. Clearly, the four vertices of degree 10 must be adjacent to every vertex, so these induce a subgraph K_4. Meanwhile, the five vertices of degree 4 are not adjacent to other vertices, thus, they induce a $5K_1$. Furthermore, the remaining two vertices of degree 5 must be adjacent to each other to make sure the requirement of the degree 5, so they induce a subgraph K_2. Hence $G \cong H_{11}^5$.

Case 3. $n = 12$ and $k = 6$.

As the same argument as above, we have $d_1 \le \cdots \le d_7 \le 5$, $d_8 \le \cdots \le d_{12} \le 11$ and $43 \le m \le 45$. From $\sum_{i=1}^{12} d_i \ge 86$ it follows that G has at least one 5-degree vertex since if not, we have $3 \le d_1 \le \cdots \le d_7 \le 4$, $d_8 \le \cdots \le d_{12} \le 11$, i.e., $\sum_{i=1}^{12} d_i \le 82 < 86$, a contradiction.

If $m = 45$, then all inequalities in (2.5) should hold. Thus, G is a graph with $d_1 = d_2 = d_3 = d_4 = d_5 = d_6 = d_7 = 5$, $d_8 = d_9 = d_{10} = d_{11} = d_{12} = 11$, which implies $G \cong H_{12}^6$. The proof of the uniqueness of the graph G determined by degree sequence $\pi = (5,5,5,5,5,5,5,11,11,11,11,11)$ is similar to that of Case 1.

Suppose that $d_G(x_0) = 5$ and $H_2 = G[V(G) \setminus \{x_0\}]$. Clearly, $|V(H_2)| = 11$ and $\delta(H_2) \ge 2$ since $\delta(G) \ge 3$.

If $m = 44$, then $e(H_2) = e(G) - 5 = 39 > \binom{11-2}{2} + 2 = 38$, and thus, H_2 is traceable by Lemma 2.2.

As the same argument as Case 1, one can obtain $G \cong H_{12}^6(1)$.

If $m = 43$, then $e(H_2) = e(G) - 5 = 38 = \binom{11-2}{2} + 2$, and thus, H_2 is traceable or $H_2 = K_1 \vee (K_8 + 2K_1)$ by Lemma 2.2. If H_2 is traceable, as the same argument as Case 1, one can obtain $G \cong H_{12}^6(2)$. Otherwise, $H_2 = K_1 \vee (K_8 + 2K_1)$. Since $\delta(H_2) \ge 2$, we omit it.

Case 4. $n = 14$ and $k = 7$.

Similarly, we have $d_1 \le \cdots \le d_8 \le 6$, $d_9 \le \cdots \le d_{14} \le 13$ and $62 \le m \le 63$. From the inequality $\sum_{i=9}^{14} d_i = 2m - \sum_{i=1}^{8} d_i \ge 124 - 48 = 76$, we conclude that there are at least four vertices of degree 13 since if not, without loss of generality, assume that $d_{11} = 12$, $d_{12} = d_{13} = d_{14} = 13$, then $\sum_{i=9}^{14} d_i \le 74 < 76$, which leads to a contradiction. Hence $d_{11} = d_{12} = d_{13} = d_{14} = 13$. On the other hand, since $\sum_{i=9}^{14} d_i \ge 76$, we have $d_9 + d_{10} = \sum_{i=9}^{14} d_i - \sum_{i=11}^{14} d_i \ge 76 - 52 = 24$. Also note that $\sum d_i$ is even and the total degree is between 124 and 126. Now we consider d_9 and d_{10} again.

If $d_9 = d_{10} = 13$, then the permissible graphic sequence is $\pi = (6,6,6,6,6,6,6,6,13,13,13,13,13,13)$, which implies $G \cong H_{14}^7$. The proof of the uniqueness of the graph G determined by degree sequence π is similar to that of Case 1.

If $d_9 = 12$ and $d_{10} = 13$, then the permissible graphic sequence is $\pi = (5,6,6,6,6,6,6,6,12,13,13,13,13,13)$, which implies $G \cong K_5 \vee (K_1 + K_{1,7})$. Next we prove the uniqueness of the graph G determined by degree sequence π. Clearly, the five vertices of degree 13 must be adjacent to every vertex, so these induce a subgraph K_5. Meanwhile, the vertex of degree 5 must be not adjacent to other vertices, so it induces a K_1. Furthermore, the vertex of degree 12 must be adjacent to the remaining vertices, so they induce a subgraph $K_{1,7}$. Hence $G \cong K_5 \vee (K_1 + K_{1,7})$.

If $d_9 = 11$ and $d_{10} = 13$, then the permissible graphic sequence is $\pi = (6,6,6,6,6,6,6,6,11,13,13,13,13,13)$, which implies $G = K_5 \vee (K_2 + K_{1,6})$. Next we prove the uniqueness of the graph G determined by degree sequence π. Clearly, the five vertices of degree 13 must be adjacent to every vertex of G, so they induce a subgraph K_5. Meanwhile, the vertex of degree 11 must be adjacent to six of the remaining vertices, thus, they induce a subgraph $K_{1,6}$. Furthermore, the remaining two vertices need to be adjacent to each other to meet the requirement of the degree

6, so they induce a subgraph K_2. Hence $G \cong K_5 \vee (K_2 + K_{1,6})$.

If $d_9 = d_{10} = 12$, then the permissible graphic sequence is $\pi = (6, 6, 6, 6, 6, 6, 6, 6, 12, 12, 13, 13, 13, 13)$. Next we prove the graph G determined by degree sequence π. Clearly, the four vertices of degree 13 induce a subgraph K_4, and they are adjacent to every vertex of G. Next we consider the two vertices of degree 12. When they are adjacent, they need to be adjacent to seven vertices of degree 6, but their neighbours in these vertices of degree 6 are not same exactly, since if not, there is one vertex whose degree is 4 at most, which can not meet the requirement of degree 6, hence it is impossible. At this time, in these eight vertices of degree 6 there must be six vertices receiving degree 6 and two vertices receiving degree 5 from others, so the temporary two vertices of degree 5 must be adjacent. Thus, G is obtained from $K_6 \vee 8K_1$ by deleting x_1y_1, x_2y_2 and adding a new edge y_1y_2, where $x_1, x_2 \in K_6$ and $y_1, y_2 \in 8K_1$. Note that in this case, G is traceable. When they are not adjacent, they must be adjacent to the remaining eight vertices, so they induce a subgraph $K_{2,8}$. Hence $G \cong K_4 \vee K_{2,8}$.

In fact, only $K_5 \vee (K_2 + K_{1,6})$ in above contains a Hamilton path and others are not traceable. Thus, the proof is completed. □

Corollary 2.7 *Let G be a connected graph on $n \geq 14$ vertices of m edges. If $m \geq \binom{n-2}{2} - 3$, then G is traceable unless $G = K_6 \vee 8K_1$.*

Theorem 2.8 *Let G be a connected graph on $n \geq 7$ vertices. Let $a = \max_{uv \in E(G)}\{d(u) + d(v)\}$. If $\rho(G) \leq \frac{a-2}{a} + \frac{a-2}{n^2-5n+10-a}$. Then G is traceable.*

Proof By Lemma 2.4 (1) and hypothesis, we have

$$\frac{2m(a-2)}{a(2m-a)} \leq \rho(G) \leq \frac{a-2}{a} + \frac{a-2}{n^2-5n+10-a},$$

where m is the size of G.

This implies that $m \geq \binom{n-2}{2} + 2$. By Lemma 2.2, G is traceable unless $G \in \{K_1 \vee (K_{n-3} + 2K_1), K_2 \vee (3K_1 + K_2), K_1 \vee K_{2,5}, K_3 \vee 5K_1, K_2 \vee (K_{1,4} + K_1), K_4 \vee 6K_1\}$. By a direct calculation (see Table 1), all graphs except for $G = K_1 \vee (K_{n-3} + 2K_1)$ satisfy

$$\rho(G) > \frac{a-2}{a} + \frac{a-2}{n^2-5n+10-a},$$

and thus, these are not cases.

For $G = K_1 \vee (K_{n-3} + 2K_1)$, let $X = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be the eigenvector corresponding to $\rho(G)$. For convenience, the vertex of degree $n-1$ given by X, say X_1; all vertices of degree $n-3$ have the same values given by X, say X_2; Denote by X_3 the values of degree 1 given by X. Assume $\tilde{X} = (X_1, X_2, X_3)^{\mathrm{T}}$. Then by Eq. (2.3) we have

$$(1-\rho(G))\sqrt{n-1}X_1 = \frac{n-3}{\sqrt{n-3}}X_2 + 2X_3, \tag{2.9}$$

$$(1-\rho(G))\sqrt{n-3}X_2 = \frac{1}{\sqrt{n-1}}X_1 + \frac{n-4}{\sqrt{n-3}}X_2, \tag{2.10}$$

$$(1-\rho(G))X_3 = \frac{1}{\sqrt{n-1}}X_1. \tag{2.11}$$

Transforming Eqs. (2.9)–(2.11) into a matrix equation $(A-\rho(G)I)\tilde{X}=0$, we get

$$A=\begin{pmatrix} 1 & -\frac{n-3}{\sqrt{(n-1)(n-3)}} & -\frac{2}{\sqrt{n-1}} \\ -\frac{1}{\sqrt{(n-1)(n-3)}} & \frac{1}{n-3} & 0 \\ -\frac{1}{\sqrt{n-1}} & 0 & 1 \end{pmatrix}$$

Thus, $\rho(G)$ is the largest root of the following equation:

$$\rho^3-\frac{2n-5}{n-3}\rho^2+\frac{n^2-5n+10}{(n-1)(n-3)}\rho=0.$$

Let $f(x)=x^3-\frac{2n-5}{n-3}x^2+\frac{n^2-5n+10}{(n-1)(n-3)}x$. Setting $f(x)=0$, we have three roots 0, x_1 and x_2 (say), where

$$x_1=\frac{2n-5-\sqrt{\frac{8n^2-55n+95}{n-1}}}{2(n-3)},\quad x_2=\frac{2n-5+\sqrt{\frac{8n^2-55n+95}{n-1}}}{2(n-3)}.$$

Clearly, $\rho(G)=x_2$. And

$$\begin{aligned} f(\frac{a-2}{a}+\frac{a-2}{n^2-5n+10-a}) &= f(\frac{2n-6}{2n-4}+\frac{2n-6}{n^2-7n+14}) \\ &= (\frac{2n-6}{2n-4}+\frac{2n-6}{n^2-7n+14})\times g(n), \end{aligned}$$

where $g(n)=(\frac{2n-6}{2n-4}+\frac{2n-6}{n^2-7n+14})^2-\frac{2n-5}{n-3}\times(\frac{2n-6}{2n-4}+\frac{2n-6}{n^2-7n+14})+\frac{n^2-5n+10}{(n-1)(n-3)}$.

Let

$$\begin{aligned} h(n) &= (\frac{2n-6}{2n-4}+\frac{2n-6}{n^2-7n+14})^2-\frac{2n-5}{n-3}\times(\frac{2n-6}{2n-4}+\frac{2n-6}{n^2-7n+14}) \\ &= \frac{-n^6+17n^5-122n^4+471n^3-1025n^2+1160n-500}{n^6-18n^5+137n^4-560n^3+1288n^2-1568n+784} \\ &= -\frac{n^6-17n^5+122n^4-471n^3+1025n^2-1160n+500}{n^6-18n^5+137n^4-560n^3+1288n^2-1568n+784}. \end{aligned}$$

We first prove $h(n)<-1$. Clearly, $n^6-17n^5+122n^4-471n^3+1025n^2-1160n+500-(n^6-18n^5+137n^4-560n^3+1288n^2-1568n+784)=n^5-15n^4+89n^3-263n^2+408n-284=n^4(n-15)+n^2(89n-263)+408n-284>0$ for $n\geq 15$, together with $n^5-15n^4+89n^3-263n^2+408n-284>0$ for $7\leq n\leq 14$ we have $h(n)<-1$.

Moreover, $\frac{n^2-5n+10}{(n-1)(n-3)}\leq 1$ since $n^2-5n+10\leq (n-1)(n-3)$ for $n\geq 7$. Thus, from the above we have

$$g(n)=h(n)+\frac{n^2-5n+10}{(n-1)(n-3)}<0.$$

It follows that

$$f(\frac{a-2}{a}+\frac{a-2}{n^2-5n+10-a})=(\frac{2n-6}{2n-4}+\frac{2n-6}{n^2-7n+14})\times g(n)<0,$$

and so $\frac{a-2}{a}+\frac{a-2}{n^2-5n+10-a}<x_2$, i.e., $\frac{a-2}{a}+\frac{a-2}{n^2-5n+10-a}<\rho(G)$. Hence this is not a case. So, the proof is completed. $\square$

G	$\frac{a-2}{a}+\frac{a-2}{n^2-5n+10-a}$	$\rho(G)$
$K_2\vee(K_2+3K_1)$	1.6667	1.7284
$K_{2,5}\vee K_1$	1.2879	1.8021
$K_3\vee 5K_1$	1.4571	1.7143
$K_2\vee(K_1+K_{1,4})$	1.4571	1.7024
$K_4\vee 6K_1$	1.2698	1.6667

Table 1 The normalized Laplacian spectral radius of some graphs

Theorem 2.9 *Let G be a connected graph with minimum degree $\delta\ge 2$ on $n\ge 7$ vertices. Let $a=\max_{uv\in E(G)}\{d(u)+d(v)\}$ and $b=\max_{uv\notin E(G)}\{d(u)+d(v)\}$.*

(1) If $\rho(G)<\frac{a-2}{a}+\frac{a-2}{n^2-4n+1-a}$, then G is traceable unless $G=K_2\vee(K_2+3K_1)$.

(2) If $\rho(G)<\frac{n^2-4n+1}{n^2-4n+1-b}$, then G is traceable.

Proof (1) Suppose that G is not a traceable graph with m edges. By Lemma 2.4 (1) and hypothesis, we have

$$\frac{2m(a-2)}{a(2m-a)}\le\rho(G)<\frac{a-2}{a}+\frac{a-2}{n^2-4n+1-a}.$$

This implies that $m>\frac{n^2-4n+1}{2}$. By Corollary 2.3, $G\in\{K_3\vee 5K_1, K_2\vee(K_1+K_{1,4}), K_{1,2}\vee 5K_1, K_2\vee(K_2+3K_1)\}$. By a direct calculation (see Table 2) we see that all graphs satisfy $\rho(G)>\frac{a-2}{a}+\frac{a-2}{n^2-4n+1-a}$ except for $G=K_2\vee(K_2+3K_1)$ due to $\rho(G)=1.7287<\frac{12-2}{12}+\frac{12-2}{7^2-4\times 7+1-12}\approx 1.8333$.

(2) By Lemma 2.4 (2) and hypothesis, we have

$$\frac{2m}{2m-b}\le\rho(G)<\frac{n^2-4n+1}{n^2-4n+1-b}.$$

This implies that $m>\frac{n^2-4n+1}{2}$. From Corollary 2.3, G is traceable unless $G\in\{K_3\vee 5K_1, K_2\vee(K_1+K_{1,4}), K_{1,2}\vee 5K_1, K_2\vee(K_2+3K_1)\}$. By a direct calculation, it is easy to check that all graphs satisfy $\rho(G)>\frac{n^2-4n+1}{n^2-4n+1-b}$, see Table 2 below. Hence these are not cases. □

G	$\frac{a-2}{a}+\frac{a-2}{n^2-4n+1-a}$	$\frac{n^2-4n+1}{n^2-4n+1-b}$	$\rho(G)$
$K_3\vee 5K_1$	1.4887	1.2222	1.7143
$K_2\vee(K_1+K_{1,4})$	1.4887	1.3200	1.7024
$K_{1,2}\vee 5K_1$	1.3962	1.5714	1.8021
$K_2\vee(K_2+3K_1)$	1.8333	1.2941	1.7287

Table 2 The normalized Laplacian spectral radius of some graphs

Theorem 2.10 *Let G be a connected graph with minimum degree $\delta\ge 3$ on $n\ge 10$ vertices. Let $a=\max_{uv\in E(G)}\{d(u)+d(v)\}$. If $\rho(G)\le\frac{a-2}{a}+\frac{a-2}{n^2-7n+26-a}$. Then G is traceable.*

Proof By Lemma 2.4 (1) and hypothesis, we have

$$\frac{2m(a-2)}{a(2m-a)}\le\rho(G)\le\frac{a-2}{a}+\frac{a-2}{n^2-7n+26-a},$$

where m is the size of G.

This implies that $m \geq \binom{n-3}{2} + 7$. By Theorem 2.6, G is traceable unless $G \in \{H_{10}^5(\ell), H_{11}^5, H_{12}^6(\ell), H_{14}^7, K_5 \vee (K_1 + K_{1,7}), K_4 \vee K_{2,8} | \ell = 0, 1, 2\}$. By a direct calculation, it is easy to check that all graphs in Table 3 satisfy $\rho(G) > \frac{a-2}{a} + \frac{a-2}{n^2-7n+26-a}$. Hence these are not cases.

G	$\frac{a-2}{a} + \frac{a-2}{n^2-7n+26-a}$	$\rho(G)$
H_{10}^5	1.3099	1.6667
H_{11}^5	1.2600	1.6531
H_{12}^6	1.2216	1.6364
H_{14}^7	1.1680	1.6154
$K_5 \vee (K_1 + K_{1,7})$	1.1680	1.6111
$K_4 \vee K_{2,8}$	1.1680	1.6345

Table 3 The normalized Laplacian spectral radius of some graphs

For $G \in H_{10}^5(1)$, that is, G is obtained from the graph H_{10}^5 by deleting an edge, which can have only one of the following degree sequences.

(a) If G_1 has degree sequence $\pi = (3, 4, 4, 4, 4, 4, 8, 9, 9, 9)$, i.e., $G_1 = K_3 \vee (K_1 + K_{1,5})$, then $\rho(G_1) = 1.6587 > \frac{a-2}{a} + \frac{a-2}{n^2-7n+26-a} = 1.3099$.

(b) If G_2 has degree sequence $\pi = (4, 4, 4, 4, 4, 4, 8, 8, 9, 9)$, i.e., $G_2 = K_2 \vee K_{2,6}$, then $\rho(G_2) = 1.7138 > \frac{a-2}{a} + \frac{a-2}{n^2-7n+26-a} = 1.3099$.

For $G \in H_{10}^5(2)$, that is, G is obtained from the graph H_{10}^5 by removing two edges, which can have degree sequences $\pi = (3, 3, 4, 4, 4, 4, 7, 9, 9, 9), (3, 3, 4, 4, 4, 4, 8, 8, 9, 9), (3, 4, 4, 4, 4, 4, 7, 8, 9, 9), (3, 4, 4, 4, 4, 4, 8, 8, 8, 9), (4, 4, 4, 4, 4, 4, 7, 8, 8, 9)$ and $(4, 4, 4, 4, 4, 4, 8, 8, 8, 8)$. By a direct calculation, we can obtain that the spectral radius of the graph determined by above degree sequences are $1.6554, 1.6492, 1.7027, 1.7108, 1.7665$ and 1.7500, respectively, which are all greater than 1.3099.

For $G \in H_{12}^6(1)$, that is, G is obtained from the graph H_{12}^6 by removing an edge, which can have only one of the following degree sequences.

(i) If G_1 has degree sequence $\pi = (5, 5, 5, 5, 5, 5, 5, 10, 10, 11, 11, 11)$, i.e., $G_1 = K_3 \vee K_{2,7}$, then $\rho(G_1) = 1.6651 > \frac{a-2}{a} + \frac{a-2}{n^2-7n+26-a} = 1.2216$.

(ii) If G_2 has degree sequence $\pi = (4, 5, 5, 5, 5, 5, 5, 10, 11, 11, 11, 11)$, i.e., $G_2 = K_4 \vee (K_1 + K_{1,6})$, then $\rho(G_2) = 1.6307 > \frac{a-2}{a} + \frac{a-2}{n^2-7n+26-a} = 1.2216$.

For $G \in H_{12}^6(2)$, that is, G is obtained from the graph H_{12}^6 by deleting two edges, which can have degree sequences $\pi = (3, 5, 5, 5, 5, 5, 5, 10, 10, 11, 11, 11), (4, 4, 5, 5, 5, 5, 5, 10, 10, 11, 11, 11), (4, 4, 5, 5, 5, 5, 5, 9, 11, 11, 11, 11), (5, 5, 5, 5, 5, 5, 5, 9, 10, 10, 11, 11), (5, 5, 5, 5, 5, 5, 5, 10, 10, 10, 10, 11), (4, 5, 5, 5, 5, 5, 5, 9, 10, 11, 11, 11)$ and $(4, 5, 5, 5, 5, 5, 5, 10, 10, 10, 11, 11)$. By a direct calculation, we can obtain that the spectral radius of the graph determined by above degree sequences are $1.6244, 1.6242, 1.6277, 1.6979, 1.6892, 1.6570$ and 1.6615, respectively, which are all greater than 1.2216.

Thus, graphs $G \in \{H_{10}^5(\ell), H_{12}^6(\ell) | \ell = 1, 2\}$ do not satisfy the assumption. So, the proof is completed. $\square$

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