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A Note on the Signless Laplacian Spectral Ordering of Graphs with Given Size

Nannan LIU[1,2], **Shuguang GUO**[2,*]
1. *School of Mathematics and Statistics, Qinghai Normal University, Qinghai* 810008*, P. R. China;*
2. *School of Mathematics and Statistics, Yancheng Teachers University, Jiangsu* 224002*, P. R. China*

Abstract For a simple undirected graph G with fixed size $m \geq 2k$ ($k \in \mathbb{Z}^+$) and maximum degree $\Delta(G) \leq m-k$, we give an upper bound on the signless Laplacian spectral radius $q(G)$ of G. For two connected graphs G_1 and G_2 with size $m \geq 8$, employing this upper bound, we prove that $q(G_1) > q(G_2)$ if $\Delta(G_1) > \Delta(G_2)+1$ and $\Delta(G_1) \geq \frac{m}{2}+2$. For triangle-free graphs, we prove two stronger results. As an application, we completely characterize the graph with maximal signless Laplacian spectral radius among all graphs with size m and circumference c for $m \geq \max\{2c, c+9\}$, which partially answers the question proposed by Chen et al. in [Linear Algebra Appl., 2022, 645: 123–136].

Keywords signless Laplacian spectral radius; upper bound; ordering; size; circumference

MR(2020) Subject Classification 05C50

1. Introduction

For a simple undirected graph G, let $A(G)$ denote its adjacency matrix and $D(G)$ denote the diagonal matrix of its degrees. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix (or the Q-matrix) of G. The largest eigenvalues of $A(G)$ and $Q(G)$ are called the spectral radius and the signless Laplacian spectral radius (denoted by $q(G)$) of G, respectively.

The investigation on the extremal problems of the spectral radius and the signless Laplacian spectral radius of graphs is an important topic in the theory of graph spectra. Specially, the problem of characterizing the extremal graph with maximal spectral radius for given size was initiated by Brualdi and Hoffman [1], and completely solved by Rowlinson [2]. From then on, the problem of characterizing the extremal graphs with maximal spectral radius under the constraint of size has been studied extensively (see, e.g., [3–10] and the references therein).

Recently, the problem of characterizing the extremal graph with maximal signless Laplacian spectral radius under the constraint of size has been investigated by researchers. Zhai et al. [11] characterized the graph with maximal signless Laplacian spectral radius among all graphs with given size and among all graphs with given size and clique number (resp., chromatic number). Lou et al. [12] determined the maximal signless Laplacian spectral radius (Laplacian spectral radius) of connected graphs with fixed size and diameter. For more results, one may refer to [13–20].

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* Corresponding author
E-mail address: lnn13666848916@163.com (Nannan LIU); ychgsg@163.com (Shuguang GUO)

Zhang and Guo [17] gave the following upper bound on $q(G)$ for a connected graph G.

Theorem 1.1 ([17]) *Let $k \geq 1$ be an integer, G be a connected graph with fixed size m and maximum degree $\Delta(G) \leq m - k$. If $m \geq 3k$, then $q(G) \leq m - k + 1 + \frac{2k}{m-k}$, and equality holds if and only if $G = K_4$ or K_3.*

Cvetković [21] proposed twelve directions for further research in the theory of graph spectra, one of which is "classifying and ordering graphs". From then on, ordering graph with various properties by their spectra becomes an attractive topic (see, e.g., [16, 22, 23] and the references therein). So far, a simple and general method to ordering graphs according to their spectra has not yet been obtained. Employing Theorem 1.1, Zhang and Guo [17] proved the following theorem on the spectral ordering of graphs.

Theorem 1.2 ([17]) *Let G_1 and G_2 be two connected graphs with fixed size $m \geq 4$. If $\Delta(G_1) > \Delta(G_2)$ and $\Delta(G_1) \geq \frac{2m}{3} + 1$, then $q(G_1) > q(G_2)$.*

In this paper, we weaken the conditions in Theorem 1.1 by proving the following theorem.

Theorem 1.3 *Let $m \geq 8$ and $k \geq 1$ be two integers, and G be a graph with size m and maximum degree $\Delta(G) \leq m - k$. If $m \geq 2k$, then $q(G) \leq m - k + 1 + \frac{2k}{m-k}$, and equality holds if and only if $G = K_5$ possibly with some isolated vertices.*

Employing Theorem 1.3, we prove the following theorem on the signless Laplacian spectral ordering of graphs with given size.

Theorem 1.4 *Let G_1 and G_2 be two graphs with size $m \geq 11$. If $\Delta(G_1) > \Delta(G_2) + 1$ and $\Delta(G_1) \geq \frac{m}{2} + 2$, then $q(G_1) > q(G_2)$.*

For triangle-free graphs, we prove two stronger results as follows.

Theorem 1.5 *Let $k \geq 1$ be an integer, G be a triangle-free graph with size $m \geq 7$ and maximum degree $\Delta(G) \leq m - k$. If $m \geq 2k$, then*

$$q(G) \leq m - k + 1 + \frac{k}{m-k},$$

and equality holds if and only if $m = 2k$ and $G = K_{2,k}$.

Theorem 1.6 *Let G_1 and G_2 be two triangle-free graphs with fixed size $m \geq 7$. If G_1 is connected, $\Delta(G_1) > \Delta(G_2)$ and $\Delta(G_1) \geq \frac{m}{2} + 1$, then $q(G_1) > q(G_2)$.*

Let $\mathcal{H}(m, c)$ denote the set of graphs on m edges with circumference c, and $H_{m,c}$ denote the graph obtained from the cycle C_c by linking a vertex of C_c to $c-3$ vertices of C_c and $m-2c+3$ isolated vertices. For $G \in \mathcal{H}(m, c)$ and $m \geq 3c-4$, Chen et al. [14] proved that $q(G) \leq q(H_{m,c})$ with equality if and only if $G = H_{m,c}$, and proposed the following question for further research.

Question 1.7 For $c+1 \leq m \leq 3c-5$, what is the maximum signless Laplacian spectral radius over all graphs in $\mathcal{H}(m, c)$?

Employing Theorem 1.4, we partially answer this question by proving the following theorem.

Theorem 1.8 *Let $G \in \mathcal{H}(m, c)$. If $m \geq \max\{2c, c+9\}$, then $q(G) \leq q(H_{m,c})$ with equality if* *and only if* $G = H_{m,c}$.

The rest of this paper is organized as follows. In Section 2, we recall some useful notions and lemmas used further. In Section 3, we give proofs of Theorems 1.3-1.6 and 1.8, respectively.

2. Preliminaries

Denote by C_n and P_n the cycle and the path of order n, respectively. For a graph G, a pendant vertex of G is a vertex of degree 1, a pendant edge of G is an edge incident with a pendant vertex, and the circumference of G is the maximum length of cycles in G. For $v \in V(G)$, $N_G(v)$ denotes the set of all neighbors of vertex v in G, and $d(v) = |N_G(v)|$ denotes the degree of vertex v in G. Denote by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively, the maximum degree and the minimum degree of G. For a subset S of $V(G)$, $G[S]$ denotes the subgraph of G induced by S and $e(S)$ denotes the number of edges in $G[S]$. For two disjoint subsets S and T of $V(G)$, $e(S, T)$ denotes the number of edges with one endpoint in S and the other in T. Let $G - xy$ denote the graph obtained from G by deleting the edge $xy \in E(G)$. Similarly, $G + xy$ is the graph obtained from G by adding an edge $xy \notin E(G)$, where $x, y \in V(G)$.

In order to complete the proofs of our main results, we need the following lemmas.

Lemma 2.1 ([24]) *If G is a connected graph, and H is a proper subgraph of G, then $q(H) < q(G)$.*

Lemma 2.2 ([25,26]) *Let G be a graph on n vertices. Then*

$$q(G) \le \max\Big\{d(u) + \frac{1}{d(u)} \sum_{uv \in E(G)} d(v)\,|u \in V(G)\Big\},$$

and equality holds if and only if G is either a regular graph or a semi-regular bipartite graph.

Remark 2.3 In 1998, Merris [27] first obtained this type upper bound for the Laplacian spectral radius of a graph.

Lemma 2.4 ([28]) *Let G be a graph with $n \ge 4$ vertices. Then $q(G) \ge \Delta + 1$. If G is connected, then the equality holds if and only if G is the star $K_{1,n-1}$.*

The following lemma is a corollary of [29, Theorem 3.2], and its proof can be found in [19].

Lemma 2.5 ([19]) *Let G be a simple connected graph with n vertices and degree sequence $d_1 \ge d_2 \ge \cdots \ge d_n$. If $d_1 \ge s \ge d_2$, then $q(G) \le A(d_1, s)$, where*

$$A(d_1, s) = \frac{1}{2}(d_1 + 2s - 1 + \sqrt{(2s - d_1 + 1)^2 + 8(d_1 - s)}).$$

Let G be a connected graph and v be a vertex of G. Denote by $W_1 \cup W_2$ a non-trivial bipartition of $N_G(v)$. Let G_v denote the graph by splitting a vertex v. Namely, G_v is obtained from $G - v$ by adding two new vertices v_1 and v_2, and adding edges v_1w_1 ($w_1 \in W_1$) and v_2w_2 ($w_2 \in W_2$).

Lemma 2.6 ([30]) *If G_v is obtained from a connected graph G by splitting any vertex v, then $q(G_v) < q(G)$.*

3. Proofs of Theorems

In this section, we will give proofs of Theorems 1.3–1.6 and 1.8, respectively.

Proof of Theorem 1.3 We may assume that G is connected. Otherwise, suppose that G has $s \geq 2$ connected components $G_1, \ldots, G_s$. Then $\Delta(G_i) \leq m-k$ for $i = 1, \ldots, s$. If G has at least two non-trivial connected components, without loss of generality, we may assume that $q(G) = q(G_1)$. Clearly, $m > |E(G_1)|$. We declare that $\delta(G_1) < m-k$. Otherwise, suppose that $\delta(G_1) = m-k$. Then $\Delta(G_1) = m-k$. It follows that $(m-k)(m-k+1) < 2m$. This contradicts the assumptions $m \geq 8$ and $m \geq 2k$. Therefore, $\delta(G_1) < m-k$. Let v be a vertex of G_1 such that $d(v) = \delta(G_1)$, and G^* denote the graph of size m obtained from G_1 by adding a pendant path of length $m - |E(G_1)|$ at v. Clearly, $\Delta(G^*) \leq m-k$. By Lemma 2.1, we have

$$q(G) < q(G^*).$$

If G has only one non-trivial connected component, denoted by G_1, then $q(G) = q(G_1)$. So, in order to complete the proof of Theorem 1.3, we may assume that G is connected.

Let $m \geq 8$, $k \geq 1$, G be a connected graph of size m with $\Delta = \Delta(G) \leq m-k$, and w be a vertex of G such that

$$\max_{u \in V(G)} \left\{ d(u) + \frac{1}{d(u)} \sum_{uv \in E(G)} d(v) \right\} = d(w) + \frac{1}{d(w)} \sum_{wv \in E(G)} d(v),$$

where

$$\sum_{wv \in E(G)} d(v) = 2e(N(w)) + e(N(w), V(G) \setminus N(w)). \tag{3.1}$$

Then $1 \leq d(w) \leq \Delta$. By Lemma 2.2, we have

$$q(G) \leq d(w) + \frac{1}{d(w)} \sum_{wv \in E(G)} d(v). \tag{3.2}$$

If $d(w) = 1$, by (3.2), we have

$$q(G) \leq 1 + d(v) \leq 1 + \Delta \leq m-k+1 < m-k+1 + \frac{2k}{m-k}.$$

If $d(w) = 2$, noting that $e(N(w)) \leq 1$, by (3.1) and (3.2), we have

$$q(G) \leq 2 + \frac{m+1}{2} < m-k+1 + \frac{2k}{m-k}.$$

If $d(w) = 3$, noting that $m \geq 8$ and $e(N(w)) \leq 3$, by (3.1) and (3.2), we have

$$q(G) < 3 + \frac{m+3}{3} \leq m-k+1 + \frac{2k}{m-k}.$$

If $4 \leq d(w) \leq \Delta \leq m-k$, noting that

$$e(N(w), V(G) \setminus N(w)) \leq m - e(N(w)), \quad e(N(w)) \leq m - d(w),$$

by (3.1), we have

$$\sum_{wv \in E(G)} d(v) \leq 2e(N(w)) + m - e(N(w)) = m + e(N(w)) \leq 2m - d(w), \tag{3.3}$$

and equality holds if and only if $e(N(w)) = m - d(w)$. By (3.2) and (3.3), we have

$$q(G) \le d(w) + \frac{2m}{d(w)} - 1.$$

Let $f(x) = x + \frac{2m}{x}$. By mathematical analysis, it is easy to see that the function $f(x)$ is strictly decreasing for $0 < x \le \sqrt{2m}$ and strictly increasing for $x \ge \sqrt{2m}$. It follows that its maximum in any closed interval is attained at one of the ends of this interval. Then we have

$$q(G) \le d(w) + \frac{2m}{d(w)} - 1 \le \max\{f(4), f(\Delta)\} - 1. \tag{3.4}$$

Case 1. $\Delta \ge \frac{m}{2}$. Then $f(4) \le f(\Delta)$. Noting that

$$\sqrt{2m} \le \frac{m}{2} \le \Delta \le m - k,$$

we have that Δ and $m-k$ are in the same monotonic interval of $f(x)$. By (3.4), we have

$$q(G) \le f(\Delta) - 1 \le f(m-k) - 1 = m - k + 1 + \frac{2k}{m-k}.$$

If the equality in Theorem 1.3 holds, then $\Delta = m-k$ and the equalities in (3.2)–(3.4) all hold. It follows that G is a regular graph and $d(w) = m-k$. This implies that $(m-k)(m-k+1) = 2m$. Namely,

$$m^2 - (2k+1)m + k^2 - k = 0.$$

This contradicts $m \ge 8$ and $m \ge 2k$. Therefore, the equality cannot hold in this case.

Case 2. $4 \le \Delta < \frac{m}{2}$. In this case, we have

$$f(4) \ge f(\Delta), \quad 4 \le \sqrt{2m} \le \frac{m}{2} \le m - k.$$

It follows that $\frac{m}{2}$ and $m-k$ are in the same monotonic interval of $f(x)$. Noting that

$$f(4) = f(\frac{m}{2}) = \frac{m}{2} + 4,$$

by (3.4), we have

$$q(G) \le f(4) - 1 = f(\frac{m}{2}) - 1 \le f(m-k) - 1 = m - k + 1 + \frac{2k}{m-k}.$$

If the equality in Theorem 1.3 holds, then $m = 2k$ and the equalities in (3.2)–(3.4) all hold. It follows that G is a regular graph and $d(w) = 4$. This implies that $G = K_5$. Conversely, if $G = K_5$, the equality holds clearly.

Combining the above arguments, we complete the proof. □

Proof of Theorem 1.4 Let $\Delta(G_1) = m - k_1$ and $\Delta(G_2) = m - k_2$. Since $\Delta(G_1) > \Delta(G_2) + 1$, it follows that $k_2 \ge k_1 + 2$.

If $k_1 = 0$, then $G_1 = K_{1,m}$ and $k_2 \ge 2$. From [30], we know that $q(G_1) = m+1$. Noting that $m \ge 11$ and $\Delta(G_2) \le m - 2$, by Theorem 1.3, we have

$$q(G_2) < m - 1 + \frac{4}{m-2} < m + 1 = q(G_1).$$

If $k_1 \geq 1$ and $m \geq 2k_2$, by Lemma 2.4, we have $q(G_1) \geq m - k_1 + 1$. Recalling that $m \geq 11$ and $k_2 \geq k_1 + 2$, by Theorem 1.3, we have

$$q(G_2) < m - k_2 + 1 + \frac{2k_2}{m - k_2} \leq m - k_1 - 2 + 1 + 2 = m - k_1 + 1 \leq q(G_1).$$

In the case when $k_1 \geq 1$ and $m < 2k_2$, let $l = \lfloor \frac{m}{2} \rfloor$. Then $m \geq 2l$ and $k_2 > l$. Recalling that $m \geq 11$ and $\Delta(G_1) = m - k_1 \geq \frac{m}{2} + 2$, we have $k_1 + 2 \leq \frac{m}{2}$. It follows that $k_1 + 2 \leq l$. By Theorem 1.3 and Lemma 2.4, we have

$$q(G_2) < m - l + 1 + \frac{2l}{m - l} \leq m - k_1 - 2 + 1 + 2 = m - k_1 + 1 \leq q(G_1). \quad \square$$

Proof of Theorem 1.5 Similar to the proof of Theorem 1.3, we may assume that G is a connected graph with size $m \geq 7$ and $\Delta = \Delta(G) \leq m - k$. Let w be a vertex of G such that

$$\max_{u \in V(G)} \{d(u) + \frac{1}{d(u)} \sum_{uv \in E(G)} d(v)\} = d(w) + \frac{1}{d(w)} \sum_{wv \in E(G)} d(v).$$

Then $1 \leq d(w) \leq \Delta$. By Lemma 2.2, we have

$$q(G) \leq d(w) + \frac{1}{d(w)} \sum_{wv \in E(G)} d(v). \tag{3.5}$$

If $d(w) = 1$, by (3.5), we have

$$q(G) \leq 1 + d(v) \leq 1 + \Delta \leq m - k + 1 < m - k + 1 + \frac{k}{m - k}.$$

If $2 \leq d(w) \leq \Delta \leq m - k$, noting that G is triangle-free, we have $e(N(w)) = 0$ and

$$\sum_{wv \in E(G)} d(v) = 2e(N(w)) + e(N(w), V(G) \setminus N(w)) = e(N(w), V(G) \setminus N(w)) \leq m. \tag{3.6}$$

By (3.5) and (3.6), we have

$$q(G) \leq d(w) + \frac{m}{d(w)}.$$

Let $f(x) = x + \frac{m}{x}$. By mathematical analysis, it is easy to see that the function $f(x)$ is strictly decreasing for $0 < x \leq \sqrt{m}$ and strictly increasing for $x \geq \sqrt{m}$. It follows that its maximum in any closed interval is attained at one of the ends of this interval. Then we have

$$q(G) \leq d(w) + \frac{m}{d(w)} \leq \max\{f(2),\ f(\Delta)\}. \tag{3.7}$$

Case 1. $\Delta \geq \frac{m}{2}$. In this case, we have $f(2) \leq f(\Delta)$ and $\sqrt{m} < \frac{m}{2} \leq \Delta \leq m - k$. Clearly, Δ and $m - k$ are in the same monotonic interval of $f(x)$. By (3.7), we have

$$q(G) \leq f(\Delta) \leq f(m - k) = m - k + 1 + \frac{k}{m - k}.$$

If the equality in Theorem 1.5 holds, then $\Delta = m - k$ and the equalities in (3.5)–(3.7) all hold. By Lemma 2.2 and Equality (3.7), G is a regular graph or a semi-regular bipartite graph, and $d(w) = m - k$. If G is a regular graph, then $(m - k)(m - k + 1) < 2m$. Namely,

$$m^2 - (2k + 1)m + k^2 - k < 0.$$

This contradicts $m \geq 7$ and $m \geq 2k$. If G is a semi-regular bipartite graph, recalling that $d(w) = m - k$ and $m \geq 2k$, by Equality (3.6), we have $m = 2k$ and $G = K_{2,k}$. Conversely, if $G = K_{2,k}$, then the equality holds clearly.

Case 2. $2 \leq \Delta < \frac{m}{2}$. In this case, we have $f(2) \geq f(\Delta)$ and $2 < \sqrt{m} < \frac{m}{2} \leq m - k$. It follows that $\frac{m}{2}$ and $m - k$ are in the same monotonic interval of $f(x)$. Noting that

$$f(2) = f(\frac{m}{2}) = \frac{m}{2} + 2,$$

by (3.7), we have

$$q(G) \leq f(2) = f(\frac{m}{2}) \leq f(m-k) = m - k + 1 + \frac{k}{m-k}.$$

If the equality in Theorem 1.5 holds, then $m = 2k$ and the equalities in (3.5)–(3.7) all hold. By Lemma 2.2 and Equality (3.7), G is a regular graph or a semi-regular bipartite graph, and $d(w) = 2$. If G is a regular graph, by Equality (3.6), we have $G = C_4$. This contradicts $m \geq 7$. If G is a semi-regular bipartite graph, recalling that $d(w) = 2$ and $m = 2k$, by Equality (3.6), we have $G = K_{2,k}$. Conversely, if $G = K_{2,k}$, then the equality holds clearly.

Combining the above arguments, we complete the proof. □

Proof of Theorem 1.6 Let $\Delta(G_1) = m - k_1$ and $\Delta(G_2) = m - k_2$. Since $\Delta(G_1) > \Delta(G_2)$ and $\Delta(G_1) \geq \frac{m}{2} + 1$, it follows that $k_1 < k_2$ and $k_1 + 1 \leq \frac{1}{2}m$.

If $k_1 = 0$, then $G_1 = K_{1,m}$ and $k_2 \geq 1$. From [30], we know that $q(G_1) = m + 1$. Noting that $\Delta(G_2) \leq m - 1$, by Theorem 1.5, we have

$$q(G_2) \leq m + \frac{1}{m-1} < m + 1 = q(G_1).$$

If $k_1 \geq 1$ and $m \geq 2k_2$, noting that G_1 is connected and by Lemma 2.4, we have $q(G_1) > m - k_1 + 1$. Recalling that $k_2 > k_1$, by Theorem 1.5, we have

$$q(G_2) \leq m - k_2 + 1 + \frac{k_2}{m - k_2} \leq m - k_1 - 1 + 1 + 1 = m - k_1 + 1 < q(G_1).$$

In the case when $k_1 \geq 1$ and $m < 2k_2$, let $l = \lfloor \frac{m}{2} \rfloor$. Then $m \geq 2l$ and $k_2 > l$. Recalling that $k_1 + 1 \leq \frac{m}{2}$, we have $k_1 + 1 \leq \lfloor \frac{m}{2} \rfloor = l$. By Theorem 1.5 and Lemma 2.4, we have

$$q(G_2) \leq m - l + 1 + \frac{l}{m-l} \leq m - k_1 - 1 + 1 + 1 = m - k_1 + 1 < q(G_1). \quad \square$$

Proof of Theorem 1.8 Let $m \geq \max\{2c, c+9\}$, $0 \leq k \leq c-3$, $3 \leq i_1 < \cdots < i_k \leq c-1$ be k integers, and $H_{m,c}^{i_1\cdots i_k}$ denote the graph obtained from a cycle $C_c = v_1v_2\cdots v_cv_1$ by linking the vertex v_1 to the vertices $v_{i_1}, \ldots, v_{i_k}$ and $m - c - k$ isolated vertices. In particular, when $k = c - 3$, $H_{m,c}^{3\cdots(c-1)} = H_{m,c}$. Clearly,

$$\Delta(H_{m,c}^{i_1\cdots i_k}) = m - c + 2 \geq \frac{m}{2} + 2.$$

Let

$$\mathcal{A} = \{H_{m,c}^{i_1\cdots i_k} \,|\, 3 \leq i_1 < \cdots < i_k \leq c-1,\ 0 \leq k \leq c-3\}.$$

Let G^* denote an extremal graph with maximal signless Laplacian spectral radius among all graphs in $\mathcal{H}(m, c)$, and $v_1 \in V(G)$ such that $d(v_1) = \Delta(G)$. We declare $\Delta(G^*) = m - c + 2$.

Otherwise, suppose that $\Delta(G^*) \leq m-c+1$. If $\Delta(G^*) \leq m-c$, then $\Delta(H_{m,c}) > \Delta(G^*)+1$. Recalling that $\Delta(H_{m,c}) \geq \frac{m}{2}+2$, by Theorem 1.4, we have

$$q(G^*) < q(H_{m,c}),$$

a contradiction. If $\Delta(G^*) = m-c+1$, then v_1 is contained in a cycle of length c in G^*. It follows that

$$G^* = H_{m-1,c}^{i_1\cdots i_k} \cup K_2,$$

where

$$|V(K_2) \cap V(H_{m-1,c}^{i_1\cdots i_k})| \geq 1, \quad v_1 \notin V(K_2).$$

This implies that the second largest degree $d_2 = d_2(G^*) \leq 4$. By Lemma 2.5, we have

$$q(G^*) \leq A(m-c+1, 4) \leq m-c+3 < q(H_{m,c})$$

for $m \geq c+9$, a contradiction. Therefore, $\Delta(G^*) = m-c+2$.

We declare that $G^* = H_{m,c}$. Otherwise, suppose that $G^* \neq H_{m,c}$. Recalling that $\Delta(G^*) = m-c+2$, we know that $G^* \in \mathcal{A}$ and there exists $3 \leq j \leq c-1$ such that $d(v_j) = 2$. Since $m \geq 2c$, there exists a pendent edge at v_1, denoted by v_1v_s. Let $G' = G^* - v_1v_s + v_1v_j$. Clearly, $G' \in \mathcal{H}(m,c)$. By Lemma 2.6, we have $q(G^*) < q(G')$, a contradiction. Therefore, $G^* = H_{m,c}$. $\square$

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