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Gorenstein Subcategories and Relative Singularity Categories

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Abstract Let $\mathscr{A}$ be an abelian category, $\mathscr{T}$ a self-orthogonal subcategory of $\mathscr{A}$ and each object in $\mathscr{T}$ admit finite projective and injective dimensions. If the left Gorenstein subcategory $l\mathcal{G}(\mathscr{T})$ equals to the right orthogonal class of $\mathscr{T}$ and the right Gorenstein subcategory $r\mathcal{G}(\mathscr{T})$ equals to the left orthogonal class of $\mathscr{T}$, we prove that the Gorenstein subcategory $\mathcal{G}(\mathscr{T})$ equals to the intersection of the left orthogonal class of $\mathscr{T}$ and the right orthogonal class of $\mathscr{T}$, and prove that their stable categories are triangle equivalent to the relative singularity category of $\mathscr{A}$ with respect to $\mathscr{T}$. As applications, let R be a left Noetherian ring with finite left self-injective dimension and ${}_RC_S$ a semidualizing bimodule, and let the supremum of the flat dimensions of all injective left R-modules be finite. We prove that if ${}_RC$ has finite injective (or flat) dimension and the right orthogonal class of C contains R, then there exists a triangle-equivalence between the intersection of C-Gorenstein projective modules and Bass class with respect to C, and the relative singularity category with respect to C-projective modules. Some classical results are generalized.

Keywords abelian category; self-orthogonal; Gorenstein subcategories; semidualizing bimodules

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1. Introduction

Throughout this paper, $\mathscr{A}$ is an abelian category and a subcategory of $\mathscr{A}$ means a full and additive subcategory closed under isomorphisms and direct summands.

Let $\mathscr{T}$ be a subcategory of $\mathscr{A}$. Sather-Wagstaff, Sharif and White introduced in [1] the Gorenstein subcategory $\mathcal{G}(\mathscr{T})$, which unifies the following notions: modules of Gorenstein dimension zero [2], Gorenstein projective modules, Gorenstein injective modules [3], V-Gorenstein projective modules, V-Gorenstein injective modules [4], W-Gorenstein modules [5], and so on. Recently, Song et al. [6–8] introduced and studied the left Gorenstein subcategory $l\mathcal{G}(\mathscr{T})$ and the right Gorenstein subcategory $r\mathcal{G}(\mathscr{T})$ when $\mathscr{T}$ is self-orthogonal, and get some applications with

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respect to semidualizing bimodules. On the other hand, as a generalization of singularity category of the ring R, Chen [9] introduced and studied the *relative singularity category* of $\mathscr{A}$ with respect to $\mathscr{T}$, which is the Verdier quotient category $D_{\mathscr{T}}(\mathscr{A}) := D^b(\mathscr{A})/K^b(\mathscr{T})$. In this paper, we consider when the equality $\mathcal{G}(\mathscr{T}) = {}^{\perp}\mathscr{T} \cap \mathscr{T}^{\perp}$ holds true. Then we give the relative singularity category $D_{\mathscr{T}}(\mathscr{A})$ under a weaker condition. As applications, we get the relative singularity equivalence of C-version, where ${}_RC_S$ is a semidualizing bimodule. The paper is organized as follows.

In Section 2, we give some terminology and some preliminary results. In Section 3, we give some conditions to get $l\mathcal{G}(\mathscr{T}) = \mathscr{T}^{\perp}$ and $r\mathcal{G}(\mathscr{T}) = {}^{\perp}\mathscr{T}$. In case that $\mathscr{T}$ has finite projective and injective dimensions, then the triangulated subcategories $\underline{{}^{\perp}\mathscr{T} \cap \mathscr{T}^{\perp}}$ and $\mathcal{D}_{\mathscr{T}}(\mathscr{A})$ are equivalent (Theorem 3.6). In Section 4, we apply these results to the category of modules. Let R be a left Noetherian ring with finite left self-injective dimension and the supremum of the flat dimensions of all injective left R-modules finite. We prove that if ${}_RC$ has finite injective (or flat) dimension and ${}_RC^{\perp}$ contains R, then there exists a triangle-equivalence between the intersection of C-Gorenstein projective modules and Bass class $\mathcal{B}_C(R)$, and the relative singularity category with respect to C-projective modules (Theorem 4.8). Finally, some results over a Gorenstein ring are generalized.

2. Preliminaries

We first recall some definitions and then give some basic facts.

Definition 2.1 ([10,11]) *(1) Let $\mathscr{C} \subseteq \mathscr{D}$ be subcategories of $\mathscr{A}$. The morphism $f : C \to D$ in $\mathscr{A}$ with $C \in \mathscr{C}$ and $D \in \mathscr{D}$ is called a right $\mathscr{C}$-approximation of D if for any morphism $g : C' \to D$ in $\mathscr{A}$ with $C' \in \mathscr{C}$, there exists a morphism $h : C' \to C$ such that $g = fh$.*

If each object in $\mathscr{D}$ has a right $\mathscr{C}$-approximation, then $\mathscr{C}$ is called contravariantly finite in $\mathscr{D}$. Dually, the notions of left $\mathscr{C}$-approximations and covariantly finite subcategories are defined. A subcategory $\mathscr{C}$ of $\mathscr{A}$ is called functorially finite if it is contravariantly finite and covariantly finite in $\mathscr{A}$.

(2) A contravariantly finite subcategory $\mathscr{C}$ of $\mathscr{A}$ is called admissible if each right $\mathscr{C}$-approximation is epic. Dually, the notion of coadmissible covariantly finite subcategories is defined.

Let $\mathscr{C}$ be a subcategory of $\mathscr{A}$. Recall that a sequence in $\mathscr{A}$ is called $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact if it is exact after applying the functor $\mathrm{Hom}_{\mathscr{A}}(C, -)$ for any object $C \in \mathscr{C}$. Dually, the notion of a $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact sequence is defined.

Definition 2.2 ([1]) *Let $\mathscr{C}$ be a subcategory of $\mathscr{A}$. The Gorenstein subcategory $\mathcal{G}(\mathscr{C})$ of $\mathscr{A}$ is defined as $\mathcal{G}(\mathscr{C}) := \{M \in \mathscr{A} |$ there exists an exact sequence*

$$\cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots$$

in $\mathscr{A}$ with all terms in $\mathscr{C}$, which is both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact, such that

$M \cong \mathrm{Im}(C_0 \to C^0)\}$.

The Gorenstein subcategory unifies the following notions: modules of Gorenstein dimension zero [2], Gorenstein projective modules, Gorenstein injective modules [3], V-Gorenstein projective modules, V-Gorenstein injective modules [4], W-Gorenstein modules [5] and so on; see [12] for the details.

Recall from [9] that for every subcategory $\mathscr{T}$ of $\mathscr{A}$, we put

$$\mathscr{T}^{\perp} := \{X \in \mathscr{A} \mid \mathrm{Ext}^i_{\mathscr{A}}(T,X) = 0 \text{ for all } T \in \mathscr{T}, i \geq 1\},$$

$$^{\perp}\mathscr{T} := \{X \in \mathscr{A} \mid \mathrm{Ext}^i_{\mathscr{A}}(X,T) = 0 \text{ for all } T \in \mathscr{T}, i \geq 1\},$$

$$\mathscr{T}^{\perp_1} := \{X \in \mathscr{A} \mid \mathrm{Ext}^1_{\mathscr{A}}(T,X) = 0 \text{ for all } T \in \mathscr{T}\},$$

$$^{\perp_1}\mathscr{T} := \{X \in \mathscr{A} \mid \mathrm{Ext}^1_{\mathscr{A}}(X,T) = 0 \text{ for all } T \in \mathscr{T}\}.$$

$_{\mathscr{T}}\mathcal{X} := \{X \in \mathscr{A} \mid$there exists an exact sequence

$$\cdots \xrightarrow{d^{-2}} T^{-1} \xrightarrow{d^{-1}} T^0 \xrightarrow{d^0} X \to 0, T^i \in \mathscr{T}, \mathrm{Ker}\, d^i \in \mathscr{T}^{\perp}\}.$$

$\mathcal{X}_{\mathscr{T}} := \{X \in \mathscr{A} \mid$there exists an exact sequence

$$0 \to X \xrightarrow{d^{-1}} T^0 \xrightarrow{d^0} T^1 \xrightarrow{d^1} \cdots, T^i \in \mathscr{T}, \mathrm{Coker}\, d^i \in {}^{\perp}\mathscr{T}\}.$$

The subcategory $\mathscr{T}$ is said to be self-orthogonal if $\mathscr{T} \subseteq \mathscr{T}^{\perp}$ (equivelantly $\mathscr{T} \subseteq {}^{\perp}\mathscr{T}$). If $\mathscr{T}$ is self-orthogonal, we obtain that $_{\mathscr{T}}\mathcal{X} \subseteq \mathscr{T}^{\perp}$ and $\mathcal{X}_{\mathscr{T}} \subseteq {}^{\perp}\mathscr{T}$ by the dimension shifting.

Following [1], we write

$$\begin{aligned} \mathrm{res}\, \widetilde{\mathscr{T}} :=\{X \in \mathscr{A} \mid \text{ there exists a } \mathrm{Hom}_{\mathscr{A}}(\mathscr{T}, -)\text{-exact exact sequence} \\ \cdots \to T_{-1} \to T_0 \to X \to 0, T_i \in \mathscr{T}\}. \end{aligned}$$

Dually, cores $\widetilde{\mathscr{T}}$ is defined.

Let $\mathscr{T}$ be self-orthogonal. Recall from [6] that the right Gorenstein subcategory

$$r\mathcal{G}(\mathscr{T}) := {}^{\perp}\mathscr{T} \cap \mathrm{cores}\, \widetilde{\mathscr{T}}$$

and left Gorenstein subcategory

$$l\mathcal{G}(\mathscr{T}) := \mathscr{T}^{\perp} \cap \mathrm{res}\, \widetilde{\mathscr{T}}.$$

Following [12], $\mathcal{G}(\mathscr{T}) = l\mathcal{G}(\mathscr{T}) \cap r\mathcal{G}(\mathscr{T})$.

Lemma 2.3 *Let $\mathscr{T} \subseteq \mathscr{A}$ be a self-orthogonal subcategory. Then $_{\mathscr{T}}\mathcal{X} = l\mathcal{G}(\mathscr{T}), \mathcal{X}_{\mathscr{T}} = r\mathcal{G}(\mathscr{T})$ and $\mathcal{G}(\mathscr{T}) = {}_{\mathscr{T}}\mathcal{X} \cap \mathcal{X}_{\mathscr{T}}$.*

Proof Note that $_{\mathscr{T}}\mathcal{X} \subseteq \mathscr{T}^{\perp}$ and $\mathcal{X}_{\mathscr{T}} \subseteq {}^{\perp}\mathscr{T}$. $_{\mathscr{T}}\mathcal{X} \subseteq \mathrm{res}\, \widetilde{\mathscr{T}}$ is clearly true, hence $_{\mathscr{T}}\mathcal{X} \subseteq l\mathcal{G}(\mathscr{T})$.

Let $X \in l\mathcal{G}(\mathscr{T}) = \mathscr{T}^{\perp} \cap \mathrm{res}\, \widetilde{\mathscr{T}}$. Then there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{T}, -)$-exact exact sequence

$$\cdots \xrightarrow{d^{-2}} T^{-1} \xrightarrow{d^{-1}} T^0 \xrightarrow{d^0} X \to 0$$

with each $T^i \in \mathscr{T}$. Applying the functor $\mathrm{Hom}_{\mathscr{A}}(\mathscr{T}, -)$ to the exact sequence

$$0 \to K_0 \to T^0 \to X \to 0,$$

where $K_0 = \operatorname{Ker} d^0$. We deduce $K_0 \in \mathscr{T}^{\perp}$ by assumption. Similarly, we get $\operatorname{Ker} d^i \in \mathscr{T}^{\perp}$ for any i. It follows that $X \in {}_{\mathscr{T}}\mathcal{X}$. Thus ${}_{\mathscr{T}}\mathcal{X} = l\mathcal{G}(\mathscr{T})$.

Dual to the above argument, we have $\mathcal{X}_{\mathscr{T}} = r\mathcal{G}(\mathscr{T})$, so

$$\mathcal{G}(\mathscr{T}) = l\mathcal{G}(\mathscr{T}) \cap r\mathcal{G}(\mathscr{T}) = {}_{\mathscr{T}}\mathcal{X} \cap \mathcal{X}_{\mathscr{T}}. \quad \square$$

3. Main results

In this section, assume that $\mathscr{A}$ is an abelian category with enough projective objects and enough injective objects. Denote by $\mathscr{P}$ (resp., $\mathscr{I}$) the subcategory of $\mathscr{A}$ consisting of projective (resp., injective) objects. We always suppose that $\mathscr{T}$ is a self-orthogonal subcategory of $\mathscr{A}$. We will give the main results in this paper and some applications.

Lemma 3.1 *(1) $l\mathcal{G}(\mathscr{T}) = \mathscr{T}^{\perp}$ if and only if X admits an epic right $\mathscr{T}$-approximation for every $X \in \mathscr{T}^{\perp}$.*

(2) $r\mathcal{G}(\mathscr{T}) = {}^{\perp}\mathscr{T}$ if and only if Y admits a monic left $\mathscr{T}$-approximation for every $Y \in {}^{\perp}\mathscr{T}$.

Proof We only prove (1), and (2) is its dual. Assume that $l\mathcal{G}(\mathscr{T}) = \mathscr{T}^{\perp}$. Let $X \in \mathscr{T}^{\perp} = l\mathcal{G}(\mathscr{T})$, we have the following $\operatorname{Hom}_{\mathscr{A}}(\mathscr{T}, -)$-exact exact sequence

$$\cdots \xrightarrow{d^{-2}} T^{-1} \xrightarrow{d^{-1}} T^0 \xrightarrow{d^0} X \to 0$$

with $T^i \in \mathscr{T}$. Then $T^0 \xrightarrow{d^0} X \to 0$ is an epic right $\mathscr{T}$-approximation of X.

Conversely, note that $l\mathcal{G}(\mathscr{T}) \subseteq \mathscr{T}^{\perp}$. Assume that $X \in \mathscr{T}^{\perp}$, we have an epic right $\mathscr{T}$-approximation $T^0 \xrightarrow{d^0} X \to 0$ with $T^0 \in \mathscr{T}$. Consider the exact sequence

$$0 \to \operatorname{Ker} d^0 \to T^0 \xrightarrow{d^0} X \to 0.$$

By applying the functor $\operatorname{Hom}_{\mathscr{A}}(\mathscr{T}, -)$, we deduce $\operatorname{Ker} d^0 \in \mathscr{T}^{\perp}$ by assumption. Iterating this process, we have $X \in l\mathcal{G}(\mathscr{T})$, and then $l\mathcal{G}(\mathscr{T}) = \mathscr{T}^{\perp}$. This completes the proof. $\square$

For an object M in $\mathscr{A}$, the $\mathscr{X}$-dimension of M, denoted by $\mathscr{X}$-dim M, is defined as $\inf\{n \geq 0|$ there exists an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ in $\mathscr{A}$ with all X_i in $\mathscr{X}\}$. We set $\mathscr{X}$-dim M infinity if no such integer exists. The $\mathscr{X}$-dimension $\mathscr{X}$-dim $\mathscr{C}$ of $\mathscr{C}$ is defined to be the supremum of the $\mathscr{X}$-dimensions of all the objects in $\mathscr{C}$. Dually, the notions of the $\mathscr{X}$-codimension $\mathscr{X}$-codim M of M and the $\mathscr{X}$-codimension $\mathscr{X}$-codim $\mathscr{C}$ of $\mathscr{C}$ are defined. In particular, if $\mathscr{X} = \mathscr{P}$ (resp., $\mathscr{I}$), then $\mathscr{X}$-dim $M = \operatorname{pd} M$ (resp., $\mathscr{X}$-codim $M = \operatorname{id} M$), $\mathscr{X}$-dim $\mathscr{C} = \operatorname{pd} \mathscr{C}$ (resp., $\mathscr{X}$-codim $\mathscr{C} = \operatorname{id} \mathscr{C}$).

Proposition 3.2 *(1) Let $\mathscr{C}$ be an admissible contravariantly finite subcategory and $\mathscr{T}$-codim $\mathscr{C} < \infty$. If $\mathscr{T}$ is contravariantly finite or $\mathscr{T} \subseteq \mathscr{C}$, then $l\mathcal{G}(\mathscr{T}) = \mathscr{T}^{\perp}$.*

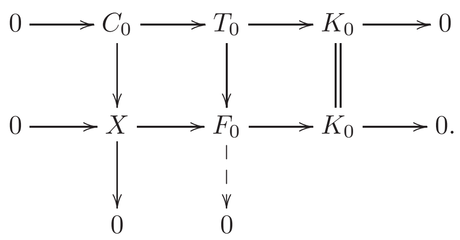
(2) Let $\mathscr{D}$ be a coadmissible covariantly finite subcategory and $\mathscr{T}$-dim $\mathscr{D} < \infty$. If $\mathscr{T}$ covariantly finite or $\mathscr{T} \subseteq \mathscr{D}$, then $r\mathcal{G}(\mathscr{T}) = {}^{\perp}\mathscr{T}$.

Proof We only prove (1), and (2) is its dual.

Let $X \in \mathscr{T}^{\perp}$. Consider an epic right $\mathscr{C}$-approximation $p : C_0 \to X \to 0$. Since $\mathscr{T}$-codim $\mathscr{C} < \infty$, there exists a long exact sequence

$$0 \to C_0 \xrightarrow{e} T_0 \to T_1 \xrightarrow{f} \cdots \to T_r \to 0$$

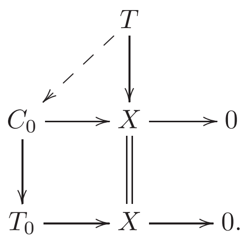
with $T_i \in \mathscr{T}$. Notice that $T_i \in {}^{\perp}(\mathscr{T}^{\perp})$, then $K_0 = \operatorname{Ker} f \in {}^{\perp}(\mathscr{T}^{\perp})$. Consider the following pushout diagram:



Dragram 1 Pushout of p and e

Since $X \in \mathscr{T}^{\perp}$, the second row splits. So X is a direct summand of F_0, and then we have an epimorphism $T_0 \xrightarrow{h} X$.

Assume $\mathscr{T}$ is contravariantly finite, take a right $\mathscr{T}$-approximation of X: $T^0 \xrightarrow{d^0} X$, there exists $\theta : T_0 \to T^0$, such that $h = d^0\theta$. Then d^0 is epic since h is epic, and hence $T^0 \xrightarrow{d^0} X$ is an epic right $\mathscr{T}$-approximation of X. Assume $\mathscr{T} \subseteq \mathscr{C}$. For any object $T \in \mathscr{T} \subseteq \mathscr{C}$ and any morphism $T \to X$, we get the following commutative diagram:



Dragram 2 Approximation property of p

Then $T_0 \to X$ is an epic right $\mathscr{T}$-approximation of X. It follows from Lemma 3.1 (1), $l\mathcal{G}(\mathscr{T}) = \mathscr{T}^{\perp}$. □

Proposition 3.3 *(1) Let $\mathscr{C}$ be an admissible contravariantly finite subcategory and* $\operatorname{id}\mathscr{C} < \infty$. *If $\mathscr{I}$ is contravariantly finite or $\mathscr{I} \subseteq \mathscr{C}$, then $\mathcal{G}(\mathscr{I}) = \mathscr{I}^{\perp}$.*

(2) Let $\mathscr{D}$ be a coadmissible covariantly finite subcategory and $\operatorname{pd}\mathscr{D} < \infty$. *If $\mathscr{P}$ is covariantly finite or $\mathscr{P} \subseteq \mathscr{D}$, then $\mathcal{G}(\mathscr{P}) = {}^{\perp}\mathscr{P}$.*

Proof We only prove (1), and (2) is its dual. Putting $\mathscr{T} = \mathscr{I}$. Note that $l\mathcal{G}(\mathscr{I}) = \mathcal{G}(\mathscr{I})$. The assertion follows from Proposition 3.2 (1). □

Recall that a subcategory is said to be a Frobebnius category if it is an exact category in which there are enough (relative) projective objects and (relative) injective objects, such that the projective objects coincide with the injective objects. In this case, the stable category of

Frobebnius category becomes a triangulated category.

Lemma 3.4 ([9]) *$\mathcal{G}(\mathscr{T})$ is a Frobenius category.*

Proof Since $\mathscr{T}$ is self-orthogonal, the subcategories ${}_{\mathscr{T}}\mathcal{X}$ and $\mathcal{X}_{\mathscr{T}}$ are closed under extensions by [10], and therefore so is $\mathcal{G}(\mathscr{T}) = {}_{\mathscr{T}}\mathcal{X} \cap \mathcal{X}_{\mathscr{T}}$ (by Lemma 2.3). Hence $\mathcal{G}(\mathscr{T})$ is en exact category whose conflations are just short exact sequences with all terms in $\mathcal{G}(\mathscr{T})$. Note that $\mathcal{G}(\mathscr{T}) \subseteq {}^{\perp}\mathscr{T} \cap \mathscr{T}^{\perp}$. Then

$$\mathrm{Ext}^1_{\mathscr{A}}(T, X) = 0 = \mathrm{Ext}^1_{\mathscr{A}}(X, T)$$

for every $T \in \mathscr{T}$ and $X \in \mathcal{G}(\mathscr{T})$. We infer that objects in $\mathscr{T}$ are (relatively) projective and injective in $\mathcal{G}(\mathscr{T})$, that is, the objects of the additive closure $\mathrm{add}\mathscr{T}$ of $\mathscr{T}$ are (relatively) projective and injective in $\mathcal{G}(\mathscr{T})$.

On the other hand, let $X \in \mathcal{G}(\mathscr{T}) \subseteq \mathcal{X}_{\mathscr{T}}$, we get an exact sequence

$$(\star): \ 0 \to X \to T \to X' \to 0$$

with $T \in \mathscr{T}, X' \in \mathcal{X}_{\mathscr{T}}$. Note that $X, T \in {}_{\mathscr{T}}\mathcal{X}$, then $X' \in {}_{\mathscr{T}}\mathcal{X}$, and thus $X' \in \mathcal{G}(\mathscr{T})$. So the exact sequence $(\star)$ is in $\mathcal{G}(\mathscr{T})$, and $\mathcal{G}(\mathscr{T})$ has enough injective objects. Dually, we get that $\mathcal{G}(\mathscr{T})$ has enough projective objects.

So $\mathcal{G}(\mathscr{T})$ is a Frobenius category, whose projective-injective subcategory are precisely $\mathrm{add}\mathscr{T}$. □

In the following, we consider the stable category $\underline{\mathcal{G}(\mathscr{T})}$ of $\mathcal{G}(\mathscr{T})$ modulo $\mathscr{T}$. Then it is not hard to see that the stable category $\underline{\mathcal{G}(\mathscr{T})}$ is a triangulated category by [13]. As a generalization of singularity category of the ring R, Chen [9] introduced and studied the relative singularity category of $\mathscr{A}$ with respect to $\mathscr{T}$, which is the Verdier quotient category $D_{\mathscr{T}}(\mathscr{A}) := D^b(\mathscr{A})/K^b(\mathscr{T})$. The following lemma comes from [9, Theorem 2.1].

Lemma 3.5 *If $\mathcal{X}_{\mathscr{T}}$-$\dim \mathscr{A} < \infty$ and ${}_{\mathscr{T}}\mathcal{X}$-$\mathrm{codim}\, \mathscr{A} < \infty$, then $\underline{\mathcal{G}(\mathscr{T})} \xrightarrow{\simeq} \mathcal{D}_{\mathscr{T}}(\mathscr{A})$ is a triangle-equivalence.*

We have the following main result in this section.

Theorem 3.6 *Let $\mathrm{pd}\,\mathscr{T} < \infty$ and $\mathrm{id}\,\mathscr{T} < \infty$. If $l\mathcal{G}(\mathscr{T}) = \mathscr{T}^{\perp}$ and $r\mathcal{G}(\mathscr{T}) = {}^{\perp}\mathscr{T}$, then $\underline{\mathcal{G}(\mathscr{T})} = \underline{{}^{\perp}\mathscr{T} \cap \mathscr{T}^{\perp}} \xrightarrow{\simeq} \mathcal{D}_{\mathscr{T}}(\mathscr{A})$ is a triangle-equivalence.*

Proof Observe that $\mathcal{G}(\mathscr{T}) = {}^{\perp}\mathscr{T} \cap \mathscr{T}^{\perp}$.

For any $M \in \mathscr{A}$, there exists an exact sequence

$$\cdots \to P^n \to P^{n-1} \xrightarrow{d^{n-1}} \cdots \to P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \to 0$$

with each $P^i \in \mathscr{P}$. It follows that $P^i \in {}^{\perp}\mathscr{T}$. Denote by $K^{i+1} = \mathrm{Ker}\, d^i$ for any $i \geq 0$. Then

$$\mathrm{Ext}^{n+i}_{\mathscr{A}}(M, \mathscr{T}) \cong \mathrm{Ext}^{i}_{\mathscr{A}}(K^n, \mathscr{T})$$

for any $n, i \geq 1$ by dimension shifting. Since $\mathrm{id}\,\mathscr{T} < \infty$, if $\mathrm{id}\,\mathscr{T} \leq n_0$ for some n_0, then $\mathrm{Ext}^{n_0+i}_{\mathscr{A}}(M, \mathscr{T}) = 0$ for any $i \geq 1$, and hence $K^{n_0} \in {}^{\perp}\mathscr{T}$. Note that ${}^{\perp}\mathscr{T} = r\mathcal{G}(\mathscr{T}) = \mathcal{X}_{\mathscr{T}}$ by Lemma 2.3. Therefore, $\mathcal{X}_{\mathscr{T}}$-$\dim \mathscr{A} < \infty$.

Dual to the above argument, we get ${}_{\mathscr{T}}\mathcal{X}\text{-codim}\,\mathscr{A} < \infty$. Thus the assertion follows from Lemma 3.5. □

Corollary 3.7 *Let* $\operatorname{pd}\mathscr{T} < \infty$ *and* $\operatorname{id}\mathscr{T} < \infty$. *If* $\mathscr{T}$ *satisfies conditions (1) and (2) of Proposition 3.2, then* $\underline{\mathcal{G}(\mathscr{T})} = \underline{{}^{\perp}\mathscr{T} \cap \mathscr{T}^{\perp}} \xrightarrow{\simeq} \mathcal{D}_{\mathscr{T}}(\mathscr{A})$ *is a triangle-equivalence.*

Proof The assertion follows from Proposition 3.2 and Theorem 3.6. □

A subcategory $\mathscr{T}$ of $\mathscr{A}$ is said to be tilting if $\operatorname{pd}\mathscr{T} < \infty$ and $\mathscr{T}\text{-codim}\,\mathscr{P} < \infty$. Dually, $\mathscr{T}$ is said to be cotilting if $\operatorname{id}\mathscr{T} < \infty$ and $\mathscr{T}\text{-dim}\,\mathscr{I} < \infty$.

Proposition 3.8 *Let* $\mathscr{T} \subseteq \mathscr{A}$ *be a tilting-cotilting and functorially finite subcategory. Then* $\underline{\mathcal{G}(\mathscr{T})} = \underline{{}^{\perp}\mathscr{T} \cap \mathscr{T}^{\perp}} \xrightarrow{\simeq} \mathcal{D}_{\mathscr{T}}(\mathscr{A})$ *is a triangle-equivalence.*

Proof Set $\mathscr{D} = \mathscr{I}$ and $\mathscr{C} = \mathscr{P}$. Thus the assertion follows from Corollary 3.7. □

Proposition 3.9 *(1) Let* $\mathscr{C}$ *be an admissible contravariantly finite subcategory and* $\operatorname{id}\mathscr{C} < \infty, \operatorname{pd}\mathscr{I} < \infty$. *If* $\mathscr{I}$ *is contravariantly finite or* $\mathscr{I} \subseteq \mathscr{C}$, *then* $\underline{\mathcal{G}(\mathscr{I})} = \underline{\mathscr{I}^{\perp}} \xrightarrow{\simeq} \mathcal{D}_{\mathscr{I}}(\mathscr{A})$ *is a triangle-equivalence.*

(2) Let $\mathscr{D}$ *be a coadmissible covariantly finite subcategory and* $\operatorname{pd}\mathscr{D} < \infty, \operatorname{id}\mathscr{P} < \infty$. *If* $\mathscr{P}$ *is covariantly finite or* $\mathscr{P} \subseteq \mathscr{D}$, *then* $\underline{\mathcal{G}(\mathscr{P})} = \underline{{}^{\perp}\mathscr{P}} \xrightarrow{\simeq} \mathcal{D}_{\mathscr{P}}(\mathscr{A})$ *is a triangle-equivalence.*

Proof The assertions follow from Proposition 3.3 and Theorem 3.6. □

4. Applications to module categories

In this section, all rings are associative rings with identity. For a ring R, denote by $\operatorname{Mod} R$ the category of left R-modules. By an R-module we mean a left R-module; right R-modules are considered as modules over the opposite ring R^{op}. For an R-module M, we denote the projective, injective and flat dimensions of M by $\operatorname{pd}{}_RM$, $\operatorname{id}{}_RM$ and $\operatorname{fd}{}_RM$, respectively. Denote by $\mathcal{P}(R)$ and $\mathcal{I}(R)$ the subcategories of $\operatorname{Mod} R$ consisting of projective and injective modules, respectively. Denote by $\mathcal{P}^{<\infty}(R)$ and $\mathcal{I}^{<\infty}(R)$ the subcategories of $\operatorname{Mod} R$ consisting of modules with finite projective and injective dimensions, respectively. Denote by $\operatorname{sfli}(R)$ the supremum of the flat dimensions of all injective R-modules.

The following result is a generalization of [14, Theorem 9.1.10] and it is used frequently below.

Lemma 4.1 *Let* R *be a left Noetherian ring with* $\operatorname{id}{}_RR = n(< \infty)$ *and* $\operatorname{sfli}(R) < \infty$. *Then the following are equivalent for an* R*-module* M.

(1) $\operatorname{id}{}_RM < \infty$. *(2)* $\operatorname{pd}{}_RM < \infty$. *(3)* $\operatorname{fd}{}_RM < \infty$. *(4)* $\operatorname{id}{}_RM \leq n$. *(5)* $\operatorname{pd}{}_RM \leq n$. *(6)* $\operatorname{fd}{}_RM \leq n$.

Proof The implications (5) ⇒ (2) ⇒ (3) and (5) ⇒ (6) ⇒ (3) are trivial. It follows from [14, Proposition 9.1.2] that (3) ⇒ (5). Thus (2) ⇔ (3) ⇔ (5) ⇔ (6).

The implication (3) ⇒ (4) follows from [15, Theorem 3.8] and (4) ⇒ (1) is trivial.

(1) ⇒ (3). Let $\mathrm{id}\,_RM<\infty$. There exists an exact sequence

$$0\to M\to I^0\to I^1\to\cdots\to I^r\to 0$$

with each $I^i\in\mathcal{I}(R)$. Since $\mathrm{sfli}(R)<\infty$, it is not hard to get $\mathrm{fd}\,_RM<\infty$ by [12, Corollary 3.3]. □

Proposition 4.2 *Let R be a left Noetherian ring with $\mathrm{id}\,_RR<\infty$ and $\mathrm{sfli}(R)<\infty$. Then $\mathcal{I}^{<\infty}(R)$ is coadmissible covariantly finite.*

Proof It follows from Lemma 4.1 that $\mathcal{I}^{<\infty}(R)=\mathcal{P}^{<\infty}(R)$. The assertion follows from a similar proof of [14, Lemma 10.2.13]. □

Definition 4.3 ([16]) *Let R and S be rings. An (R,S)-bimodule $_RC_S$ is called semidualizing if the following conditions are satisfied.*

(a1) $_RC$ *admits a degreewise finite R-projective resolution.*

(a2) C_S *admits a degreewise finite S-projective resolution.*

(b1) The homothety map $_RR_R\overset{R\gamma}{\to}\mathrm{Hom}_{S^{op}}(C,C)$ *is an isomorphism.*

(b2) The homothety map $_SS_S\overset{\gamma S}{\to}\mathrm{Hom}_R(C,C)$ *is an isomorphism.*

(c1) $\mathrm{Ext}_R^{\geq 1}(C,C)=0$.

(c2) $\mathrm{Ext}_{S^{op}}^{\geq 1}(C,C)=0$.

Wakamatsu [17] introduced and studied the so-called generalized tilting modules, which are usually called Wakamatsu tilting modules [18,19]. Note that an (R,S)-bimodule $_RC_S$ is semidualizing if and only if $_RC$ (resp., C_S) is Wakamatsu tilting with $S=\mathrm{End}(_RC)$ (resp., $R=\mathrm{End}(C_S)$), and if and only if both $_RC$ and C_S are Wakamatsu tilting with $S=\mathrm{End}(_RC)$ and $R=\mathrm{End}(C_S)$ (see [20, Corollary 3.2]). Examples of semidualizing bimodules are referred to [16,21]. In particular, $_RR_R$ is a semidualizing (R,R)-bimodule.

From now on, R and S are arbitrary rings and we fix a semidualizing bimodule $_RC_S$. By $\mathrm{Add}_R\,C$ we denote the subcategory of $\mathrm{Mod}\,R$ consisting of direct summands of direct sums of copies of C, and write

$$\mathcal{P}_C(R):=\{C\otimes_S P\mid P\text{ is projective in}\,\mathrm{Mod}\,S\},$$
$$\mathcal{I}_C(S):=\{\mathrm{Hom}_R(C,I)\mid I\text{ is injective in}\,\mathrm{Mod}\,R\}.$$

Then $\mathrm{Add}_R\,C=\mathcal{P}_C(R)$ (see [22, Proposition 2.4(1)]). The modules in $\mathcal{P}_C(R)$ and $\mathcal{I}_C(S)$ are called C-projective and C-injective, respectively. When $_RC_S={}_RR_R$, C-projective and C-injective modules are exactly projective and injective modules, respectively.

Definition 4.4 ([22]) *An R-module $M\in\mathrm{Mod}\,R$ is called C-Gorenstein projective if $M\in{}^{\perp}\mathcal{P}_C(R)$ and there exists a $\mathrm{Hom}_R(-,\mathcal{P}_C(R))$-exact exact sequence*

$$0\to M\to G^0\to G^1\to\cdots\to G^i\to\cdots$$

in $\mathrm{Mod}\,R$ with all G^i in $\mathcal{P}_C(R)$. Dually, the notion of C-Gorenstein injective modules in $\mathrm{Mod}\,S$ is defined.

We use $\mathcal{GP}_C(R)$ (resp., $\mathcal{GI}_C(S)$) to denote the subcategory of $\operatorname{Mod} R$ (resp., $\operatorname{Mod} S$) consisting of C-Gorenstein projective (resp., injective) modules. When ${}_RC_S = {}_RR_R$, C-Gorenstein projective and C-Gorenstein injective modules are exactly Gorenstein projective and Gorenstein injective modules, respectively.

Definition 4.5 ([16]) *The Bass class $\mathcal{B}_C(R)$ with respect to C consists of all modules M in* $\operatorname{Mod} R$ *satisfying the following conditions.*

(B1) $\operatorname{Ext}^i_R(C, M) = 0$ *for any* $i \geq 1$;

(B2) $\operatorname{Tor}^S_i(C, \operatorname{Hom}_R(C, M)) = 0$ *for any* $i \geq 1$; *and*

(B3) The natural evaluation homomorphism $\nu_M : C \otimes_S \operatorname{Hom}_R(C, M) \to M$ *is an isomorphism (of R-modules).*

The following lemma is crucial for the proof of the main result in this section.

Lemma 4.6 *Let R be a left Noetherian ring with* $\operatorname{id}{}_RR < \infty$ *and* $\operatorname{sfli}(R) < \infty, R \in {}_RC^{\perp}$.

(1) $\mathcal{P}_C(R)$ *is admissible contravariantly finite.*

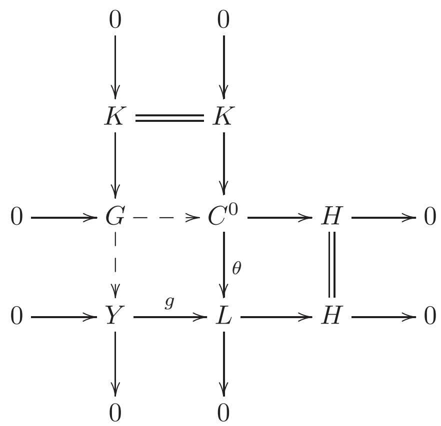
(2) If $\operatorname{id}_R C < \infty$ *or* $\operatorname{fd}_R C < \infty$, *then Y admits a monic left $\mathcal{P}_C(R)$-approximation for every* $Y \in {}^{\perp}\mathcal{P}_C(R)$.

Proof (1) Note that $\mathcal{P}_C(R)$ is contravariantly finite by [16, Proposition 5.3(b)]. It follows from [8, Proposition 4.6(1)] that $\mathcal{P}_C(R)$ contains all projective R-modules. Thus $\mathcal{P}_C(R)$ is admissible.

(2) Let $Y \in {}^{\perp}\mathcal{P}_C(R)$. By Proposition 4.2, there exists a monic left $\mathcal{I}^{<\infty}(R)$-approximation of Y: $Y \xrightarrow{g} L$ with $L \in \mathcal{I}^{<\infty}(R)$. By (1), we have

$$0 \to K \to C^0 \xrightarrow{\theta} L \to 0$$

with $C^0 \in \mathcal{P}_C(R)$, where $K = \operatorname{Ker}\theta$. Consider the following pullback diagram:



Dragram 3 Pullback of g and θ

where $H = \operatorname{Coker} g$. By Lemma 4.1, one always has $\operatorname{id}_R C < \infty$, it follows that $\operatorname{id}_R \mathcal{P}_C(R) < \infty$. Then $K \in \mathcal{I}^{<\infty}(R)$, and hence $K \in \mathcal{P}^{<\infty}(R)$ by Lemma 4.1. It is not hard to see that

$\mathrm{Ext}^1_R(Y,K)=0$ as $Y\in {}^{\perp}\mathcal{P}_C(R)\subseteq {}^{\perp}\mathcal{P}(R)$. It yields that the leftmost column

$$0\to K\to G\to Y\to 0$$

in the above diagram splits, then there exists a monomorphism $h:Y\to C^0$ such that $g=\theta h$. We claim that $Y\xrightarrow{h} C^0$ is a monic left $\mathcal{P}_C(R)$-approximation of Y. Indeed, for any $P\in\mathcal{P}_C(R)$ and morphism $j:Y\to P$. Notice that $P\in\mathcal{I}^{<\infty}(R)$, then there exists a morphism $i:L\to P$ such that $j=ig$ since g is a left $\mathcal{I}^{<\infty}(R)$-approximation, and thus $j=ig=i\theta h$, that is, j factors through h, the claim is proved. □

Proposition 4.7 *Let R be a left Noetherian ring with $\mathrm{id}\,{}_RR<\infty$ and $\mathrm{sfli}(R)<\infty$, $R\in {}_RC^{\perp}$.*

(1) $\mathcal{B}_C(R)=\mathcal{P}_C(R)^{\perp}={}_RC^{\perp}$.

(2) If $\mathrm{id}_R\,C<\infty$ or $\mathrm{fd}_R\,C<\infty$, then $\mathcal{GP}_C(R)={}^{\perp}\mathcal{P}_C(R)$.

Proof It follows from [23, Lemma 2.5(1)] that $\mathcal{P}_C(R)$ is self-orthogonal. Putting $\mathscr{T}=\mathcal{P}_C(R)$ $(=\mathrm{Add}_R\,C)$. Then $\mathcal{P}_C(R)^{\perp}={}_RC^{\perp}$.

(1) It follows from Lemmas 4.6 (1) and 3.1 (1) that $l\mathcal{G}(\mathcal{P}_C(R))=\mathcal{P}_C(R)^{\perp}$. Thus the assertion follows from [7, Lemma 4.14].

(2) It follows from Lemmas 4.6 (2) and 3.1 (2) that $r\mathcal{G}(\mathcal{P}_C(R))={}^{\perp}\mathcal{P}_C(R)$. Thus the assertion follows from [7, Lemma 4.7]. □

Finally, we get the following singularity equivalence with respect to a semidualizing bimodule ${}_RC_S$, which is the main result in this section.

Theorem 4.8 *Let R be a left Noetherian ring with $\mathrm{id}\,{}_RR<\infty$ and $\mathrm{sfli}(R)<\infty$, $R\in {}_RC^{\perp}$. If $\mathrm{id}_R\,C<\infty$ or $\mathrm{fd}_R\,C<\infty$, then*

$$\underline{\mathcal{G}(\mathcal{P}_C(R))}=\underline{\mathcal{GP}_C(R)\cap\mathcal{B}_C(R)}=\underline{{}^{\perp}\mathcal{P}_C(R)\cap {}_RC^{\perp}}\xrightarrow{\simeq} D^b(\mathrm{Mod}\,R)/K^b(\mathcal{P}_C(R))$$

is a triangle-equivalence.

Proof Putting $\mathscr{T}=\mathcal{P}_C(R)$ $(=\mathrm{Add}_R\,C)$. Since $\mathrm{id}_R\,C<\infty$, it follows from Lemma 4.1 that $\mathrm{pd}_R\,C<\infty$. Then $\mathrm{id}_R\,\mathcal{P}_C(R)<\infty$ and $\mathrm{pd}_R\,\mathcal{P}_C(R)<\infty$. It follows from Lemmas 4.6 and 3.1 that $l\mathcal{G}(\mathcal{P}_C(R))=\mathcal{P}_C(R)^{\perp}$ and $r\mathcal{G}(\mathcal{P}_C(R))={}^{\perp}\mathcal{P}_C(R)$. Thus the assertion follows from Proposition 4.7 and Theorem 3.6. □

The following result is a generalization of [9, Theorem 3.3], which says that for a Gorenstein ring, the big singularity category is triangle-equivalent to the stable category of Gorenstein-projective modules.

Proposition 4.9 *Let R be a left Noetherian ring with $\mathrm{id}\,{}_RR<\infty$ and $\mathrm{sfli}(R)<\infty$. We have*

(1) $\underline{\mathcal{GI}(R)}=\underline{\mathcal{I}(R)^{\perp}}\xrightarrow{\simeq} D^b(\mathrm{Mod}\,R)/K^b(\mathcal{I}(R))$ is a triangle-equivalence;

(2) $\underline{\mathcal{GP}(R)}=\underline{{}^{\perp}\mathcal{P}(R)}\xrightarrow{\simeq} D^b(\mathrm{Mod}\,R)/K^b(\mathcal{P}(R))$ is a triangle-equivalence;

(3) $D^b(\mathrm{Mod}\,R)/K^b(\mathcal{I}(R))=D^b(\mathrm{Mod}\,R)/K^b(\mathcal{P}(R))$.

Proof It follows from Lemma 4.1 that $\mathrm{id}\,\mathcal{P}(R)<\infty,\mathrm{pd}\,\mathcal{I}(R)<\infty$.

(1) Putting $\mathscr{C}=\mathcal{P}(R)$. Note that $\mathcal{I}(R)$ is contravariantly finite. So the assertion follows

from Proposition 3.9 (1).

(2) Putting ${}_RC_S = {}_RR_R$. The assertion follows from Theorem 4.8.

(3) The assertion follows from [24, Theorem 4.13]. □

Recall from [14] that a ring R is said to be Gorenstein if R is two-sided noetherian, and the regular module R has finite injective dimension both as a left and right module.

Proposition 4.10 *Let R be a Gorenstein ring and $R \in {}_RC^{\perp}$. If $\mathrm{id}_R C < \infty$ or $\mathrm{fd}_R C < \infty$, then*

$$\underline{\mathcal{G}(\mathcal{P}_C(R))} = \underline{\mathcal{G}\mathcal{P}_C(R) \cap \mathcal{B}_C(R)} = \underline{{}^{\perp}\mathcal{P}_C(R) \cap {}_RC^{\perp}} \xrightarrow{\simeq} D^b(\mathrm{Mod}\, R)/K^b(\mathcal{P}_C(R))$$

is a triangle-equivalence.

Let A be a finite-dimensional associative algebra over a field k. We denote by $\mathrm{mod}\, A$ the category of finitely generated left A-modules, and by $\mathrm{proj}A$ and $\mathrm{inj}A$ the subcategories of finitely generated projective and injective left A-modules respectively. For $\mathrm{mod}\, A$, a self-orthogonal module T is called generalized tilting if $\mathrm{pd}\, T < \infty$ and $\mathrm{add}\, T$-codim $\mathrm{proj}A < \infty$. A self-orthogonal module T is called generalized cotilting if $\mathrm{id}\, T < \infty$ and $\mathrm{add}\, T$-dim $\mathrm{inj}A < \infty$.

Proposition 4.11 ([25, Theorem 2.5]) *Let A be a finite-dimensional Gorenstein algebra and T a generalized tilting A-module. Then the natural functor induces a triangle-equivalence*

$$\underline{\mathcal{G}(\mathrm{add}\, T)} = \underline{{}^{\perp}T \cap T^{\perp}} \xrightarrow{\simeq} D^b(\mathrm{mod}\, A)/K^b(\mathrm{add}\, T).$$

Proof Putting $\mathscr{T} = \mathrm{add}\, T$. Notice that generalized tilting modules coincide with generalized cotilting modules by [26, Lemma 1.3], then $\mathscr{T}$ is self-orthogonal functorially finite in $\mathrm{mod}\, A$, and hence $\mathcal{G}(\mathrm{add}\, T) = {}^{\perp}T \cap T^{\perp}$. Thus the assertion follows from Proposition 3.8. □

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