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Entire Solutions of Some Type of Nonlinear Delay-Differential Equations

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Abstract In this paper, the existence and growth of entire solutions of some type of nonlinear delay-differential equations are studied. Using Cartan's second main theorem and Nevanlinna theory of meromorphic functions, we obtain the exact forms of its entire solutions with hyper-order less than one.

Keywords Nevanlinna theory; differential equation; delay-differential equation; entire solution

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1. Introduction and main results

The solvability and growth of solutions are two important properties in the study of the differential, or difference, or delay-differential equations in complex domain. One of powerful research tools is Nevanlinna theory of meromorphic functions and its difference counterparts [1,2] and references therein. In this paper, we assume that the readers are familiar with the basic notations and results of the above theories such as $m(r,f), N(r,f), T(r,f)$, the first and second main theorems, lemma on the logarithmic derivative etc. [2,3]. In 2010, Yang and Laine [4] used the above tools to obtain certain similarities in solvability between some types of differential equation and difference equation, see Theorems 1.1 and 1.2.

Theorem 1.1 ([4]) *Let p be a nonzero polynomial, and b,c be nonzero constants. If p is nonconstant, then the differential equation*

$$f^3(z)+p(z)f''(z)=c\sin bz \tag{1.1}$$

admits no entire solutions, while if p is constant, then Eq. (1.1) admits three distinct entire solutions, provided $(pb^2/27)^3=\frac{1}{4}c^2$.

Theorem 1.2 ([4]) *The difference equation*

$$f^3(z)+p(z)f(z+1)=c\sin bz, \tag{1.2}$$

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where p is a nonconstant polynomial and b, c are nonzero constants, admits no entire solutions of finite order, while if p is a nonzero constant, then Eq. (1.2) has three distinct entire solutions of finite order, provided $b = 3\pi k$ and $p^3 = (-1)^{k+1}\frac{27}{4}c^2$ for a nonzero integer k.

Observing that the right side of Eq. (1.2) is a linear combination of two exponential functions e^{ibz} and e^{-ibz}, Zhang and Huang [5] studied the growth of solutions of the general difference equation containing m exponential terms

$$f^n(z) + p(z)f(z+\eta) = \beta_1 e^{\omega_1 z} + \cdots + \beta_m e^{\omega_m z} \tag{1.3}$$

and obtained the following result.

Theorem 1.3 ([5]) *Let $n \geq m+2$ be an integer, p be a nonzero polynomial, $\beta_1, \ldots, \beta_m, \eta$ be nonzero constants, and let $\omega_1, \ldots, \omega_m$ be distinct nonzero constants. Assume that $\frac{\omega_i}{\omega_j} \neq n$ for all $i, j \in \{1, \ldots, m\}$, and that $n\omega_k \neq l_{k1}\omega_1 + \cdots + l_{km}\omega_m$ for $5 \leq k \leq m$, where $l_{k1}, \ldots, l_{km} \in \{0, 1, \ldots, n-1\}$ and $l_{k1} + \cdots + l_{km} = n$. Then any meromorphic solution $f(z)$ of Eq. (1.3) satisfies $\sigma_2(f) \geq 1$.*

The above symbol $\sigma_2(f)$ denotes the hyper order of a meromorphic function f. In this paper, we will also use the symbols $\sigma(f)$ and $\lambda(f)$ to denote the order and the exponent of convergence of zeros of f. Please refer to reference [1] for their definitions. Theorem 1.3 has been extended by Li-Hao-Yi [6] and Mao-Liu [7], respectively. The following is a partial result in [7, Theorem 1.1].

Theorem 1.4 ([7]) *Let n, k, m, q be positive integers with $n \geq 2$, η be a nonzero constant, $\omega_1, \ldots, \omega_m$ be distinct nonzero constants, and let $p, H_1, \ldots, H_m$ be nonzero entire functions of orders less than q. If f is a meromorphic solution of the equation*

$$f^n(z) + p(z)f^{(k)}(z+\eta) = H_1(z)e^{\omega_1 z^q} + \cdots + H_m(z)e^{\omega_m z^q} \tag{1.4}$$

satisfying $\sigma_2(f) < 1$ and $N(r, f) = S(r, f)$, then we have two possibilities:

(i) $f(z) = \gamma_j(z)e^{\frac{\omega_j z^q}{n}}$ and $m = 2$, where $\gamma_j^n(z) = H_j(z), \omega_j = n\omega_t$ ($\{j, t\} = \{1, 2\}$).

(ii) $\lambda(f) = \sigma(f) = q$ and $n \leq m+1$.

Theorems 1.1 and 1.2 show when replacing $p(z)f''(z)$ in Eq. (1.1) by $p(z)f(z+1)$, the existence of entire solutions to Eqs. (1.1) and (1.2) are similar except for order restriction of solutions. It is natural to ask whether there exist some similarities if replacing f^n in the above equations by $f^n f'$. In this paper, we study the delay-differential equation

$$f^n(z)f'(z) + p(z)f^{(k)}(z+\eta) = H_1(z)e^{\omega_1 z^q} + \cdots + H_m(z)e^{\omega_m z^q} \tag{1.5}$$

and obtain some results similar to Theorem 1.4 for entire solutions.

Theorem 1.5 *Let n, k, m, q be positive integers, η be a nonzero constant, $\omega_1, \ldots, \omega_m$ be distinct nonzero constants, and let $p, H_1, \ldots, H_m$ be nonzero entire functions of orders less than q. If $n \geq m+2$ and Eq. (1.5) admits an entire solution $f(z)$ of $\sigma_2(f) < 1$, then we have*

$$m = 2, \quad f(z) = \varphi(z)e^{\alpha z^q},$$

where $\{\alpha, (n+1)\alpha\} = \{\omega_1, \omega_2\}$, $\varphi(z)$ *is an entire function satisfying* $\lambda(\varphi) = \sigma(\varphi) < q$.

Remark 1.6 Theorem 1.5 provides us a method for judging the existence of finite order entire solutions of Eq. (1.5) and finding its exact forms if such solutions exist, see Examples 1.7 and 1.8.

Example 1.7 The equation $f^n(z)f'(z) + p(z)f^{(k)}(z+\eta) = H_1(z)e^z + ze^{(n+1)z}$ has no entire solutions of hyper-order less than 1, where $n \geq 4$ is an integer, p, H_1 are nonzero entire solutions of orders less than 1. In fact, if the above equation admits an entire solution f of $\sigma_2(f) < 1$, then by Theorem 1.5 we have $f(z) = \varphi(z)e^z$, where $\varphi(z)$ is an entire function satisfying $\lambda(\varphi) = \sigma(\varphi) < 1$. Substituting this expression into the above equation, we get

$$\{\varphi^n(z)\varphi'(z) + \varphi^{n+1}(z) - z\}e^{(n+1)z} + \Big\{p(z)e^\eta \sum_{j=0}^{k} C_k^j \varphi^{(j)}(z+\eta) - H_1(z)\Big\}e^z = 0.$$

Since $\max\{\sigma(p), \sigma(H_1), \sigma(\varphi)\} < 1$ and $\sigma(e^{nz}) = 1$, we have

$$\varphi^n(z)\varphi'(z) + \varphi^{n+1}(z) = z. \tag{1.6}$$

If $\varphi(z)$ is transcendental, then by $\lambda(\varphi) = \sigma(\varphi) < 1$, we know that φ has infinitely many zeros. This is impossible by (1.6). If $\varphi(z)$ is a polynomial, then by comparing the degrees of polynomials at both sides of (1.6), we get a contradiction.

Example 1.8 Considering the equation $f^n(z)f'(z) - zf^{(k)}(z+\pi i) = ze^z + e^{(n+1)z}$, where $n \geq 4$ is an integer, by Theorem 1.5, we know that it has only one entire solution $f(z) = e^z$ satisfying $\sigma_2(f) < 1$.

From Theorem 1.5, we know that finite order entire solution f of Eq. (1.5) satisfies $\lambda(f) < \sigma(f)$ provided $n \geq m+2$. But this is not the case for $n \leq m+1$. For example, the equation $f^2(z)f'(z) + f''(z+\pi i) = e^{3z} + 2e^{2z}$ has a solution $f(z) = e^z + 1$ satisfying $\lambda(f) = \sigma(f) = 1$. In fact, for $n \leq m+1$ we have the following result.

Theorem 1.9 *Let* n, k, m, q *be positive integers,* η *be a nonzero constant,* $\omega_1, \ldots, \omega_m$ *be distinct nonzero constants, and let* $p, H_1, \ldots, H_m$ *be nonzero entire functions of orders less than* q. *If* $n \leq m+1$ *and Eq.* (1.5) *admits an entire solution* $f(z)$ *of* $\sigma_2(f) < 1$, *then* $\sigma(f) = q$. *Moreover, if* $f(z)$ *satisfies* $\lambda(f) < \sigma(f)$, *then*

$$(n, m) \in \{(1,2), (2,2), (3,2)\}$$

and $f(z)$ *has the form in Theorem 1.5.*

We give the following examples to illustrate the existence of entire solutions in Theorem 1.9.

Example 1.10 The entire function $f(z) = e^z + e^{-z}$ solves the equation

$$f^2(z)f'(z) + f^{(2k)}(z+\pi i) = e^{3z} - 2e^{-z} - e^{-3z},$$

where k is a positive integer. Here $m = 3$ and $\lambda(f) = \sigma(f) = 1$. This also shows that the condition $m = 2$ is necessary to guarantee that any finite order entire solutions of Eq. (1.5) satisfies $\lambda(f)<\sigma(f)$.

Example 1.11 The entire function $f(z)=e^z+1$ solves the equations

$$f(z)f'(z)-zf^{(k)}(z+\pi i)=e^{2z}+(1+z)e^z,$$

where k is a positive integer. Here $m=2$, but $\lambda(f)=\sigma(f)=1$. This also shows that the condition $m=2$ is not sufficient to guarantee that any finite order entire solution of Eq. (1.5) satisfies $\lambda(f)<\sigma(f)$.

Example 1.12 The entire function $f(z)=ze^{\pi z}$ solves the equation

$$f^3(z)f'(z)+f''(z+2i)=(\pi z^4+z^3)e^{4\pi z}+(\pi^2 z+2\pi+2\pi^2 i)e^{\pi z}.$$

Here $m=2$ and $\lambda(f)<\sigma(f)=1$.

2. Lemmas

In this section, we give some preliminary results for the proof of our results. We will use the symbol E (or E_1) to denote a set of finite logarithmic measure (or finite line measure), not necessary the same at each occurrence. Firstly, we give a double inequality for the growth of entire solutions of Eq. (1.5).

Lemma 2.1 *Under the conditions of Theorem 1.5, if $n\geq 2$ and $f(z)$ is an entire solution of Eq. (1.5) satisfying $\sigma_2(f)<1$, then there exist positive numbers $A_1<A_2$ and a set E of finite logarithmic measure, such that*

$$A_1r^q\leq T(r,f)\leq A_2r^q$$

hold for sufficiently large $r\notin E$.

The proof of Lemma 2.1 needs the following Lemmas.

Lemma 2.2 ([8]) *Let f be a nonconstant meromorphic function of $\sigma_2(f)<1$, and η be a nonzero constant. Then for each $\varepsilon>0$,*

$$m(r,\frac{f(z+\eta)}{f(z)})=o(\frac{T(r,f)}{r^{1-\sigma_2(f)-\varepsilon}}),\quad r\to\infty,\ r\notin E.$$

Remark 2.3 Lemma 2.2 is a version of the difference logarithmic derivative lemma. By this lemma, [8, Lemma 8.3] and the logarithmic derivative lemma, we have the following conclusions.

Let f,η satisfy the conditions of Lemma 2.2. Then

$$T(r,f(z+\eta))=T(r,f(z))+o(T(r,f)),\quad r\to\infty,\ r\notin E,$$

$$N(r,\frac{1}{f(z+\eta)})=N(r,\frac{1}{f(z)})+o(T(r,f)),\quad r\to\infty,\ r\notin E,$$

$$m(r,\frac{f^{(k)}(z+\eta)}{f^{(j)}(z)})=o(T(r,f)),\quad r\to\infty,\ r\notin E,$$

where $f^{(j)}\not\equiv 0$ and $k\geq j\geq 0$.

Lemma 2.4 ([7]) *Let m, q be positive integers, φ_j $(0 \leq j \leq m)$ be meromorphic functions of $\sigma(\varphi_j) < q$, such that $\varphi_j \not\equiv 0$ $(1 \leq j \leq m)$, and let $\omega_1, \ldots, \omega_m$ be distinct nonzero constants. Set $\varphi(z) = \varphi_0(z) + \sum_{j=1}^{m} \varphi_j(z)e^{\omega_j z^q}$, then there exist positive numbers D_1, D_2, such that for sufficiently large r, we have $D_1 r^q \leq T(r, \varphi) \leq D_2 r^q$, and $m(r, \frac{1}{\varphi}) = o(r^q)$ for $\varphi_0 \not\equiv 0$.*

Proof of Lemma 2.1 By Lemma 2.4, Remark 2.3 and Eq. (1.5), there exists a positive number D_1, such that

$$\begin{aligned} D_1 r^q &\leq m(r, \frac{f^n(z)f'(z) + P(z)f^{(k)}(z+\eta)}{f(z)}) + m(r, f(z)) \\ &\leq (n+1)m(r, f) + o(T(r, f)) + o(r^q) \end{aligned}$$

holds for sufficiently large $r \notin E$. This means that $T(r, f) \geq A_1 r^q$ holds for a constant $A_1 > 0$ and sufficiently large $r \notin E$.

Let $G(z) = \sum_{j=1}^{m} H_j(z)e^{\omega_j z^q}$. For a fixed $r > 0$, let

$$\Lambda_1 = \{\theta \in [0, 2\pi) : |f(re^{i\theta})| > 1\}, \quad \Lambda_2 = [0, 2\pi) - \Lambda_1,$$

then by the definition of $m(r, f)$, Lemma 2.4, Remark 2.3 and Eq. (1.5), we obtain

$$\begin{aligned} m(r, f^2(z)) &= \frac{1}{2\pi}\int_{\Lambda_1} \log^+ |f^2(re^{i\theta})| \mathrm{d}\theta \\ &\leq \frac{1}{2\pi}\int_{\Lambda_1} \{\log^+ |\frac{G(re^{i\theta})f(re^{i\theta})}{f'(re^{i\theta})}| + \log^+ |\frac{P(re^{i\theta})f^{(k)}(re^{i\theta}+\eta)}{f'(re^{i\theta})}| + O(1)\}\mathrm{d}\theta \\ &\leq m(r, G(z)) + m(r, \frac{f(z)}{f'(z)}) + m(r, \frac{f^{(k)}(z+\eta)}{f'(z)}) + o(r^q) \\ &\leq T(r, \frac{f'(z)}{f(z)}) + D_2 r^q + o(T(r, f)) \\ &\leq N(r, \frac{f'(z)}{f(z)}) + D_2 r^q + o(T(r, f)) \\ &\leq T(r, f(z)) + D_2 r^q + o(T(r, f)) \end{aligned}$$

holds for a constant $D_2 > 0$ and sufficiently large $r \notin E$. This means that $T(r, f) \leq A_2 r^q$ holds for a constant $A_2 > 0$ and sufficiently large $r \notin E$. Lemma 2.1 is proved. □

The following three lemmas will be used to estimate the zero distribution of entire solutions of Eq. (1.5), in which Lemma 2.5 is a simple version of Cartan's second main theorem. The symbol $N_k(r, \frac{1}{f})$ below denotes the integrated counting function corresponding to $n_k(r, \frac{1}{f})$, where each zero of $f(z)$ of multiplicity l is counted $\min\{l, k\}$ times.

Lemma 2.5 ([9, 10]) *Let $f_1, f_2, \ldots, f_k$ be linearly independent entire functions. Assume that for each complex number z, $\max\{|f_1(z)|, \ldots, |f_k(z)|\} > 0$. For $r > 0$, set*

$$T(r) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\mathrm{d}\theta - u(0), \quad u(z) = \sup_{1\leq j\leq k} \log |f_j(z)|.$$

Set $f_{k+1}=f_1+\cdots+f_k$. *Then*

$$T(r)\le\sum_{j=1}^{k+1}N_{k-1}(r,\frac{1}{f_j})+S(r)\le(k-1)\sum_{j=1}^{k+1}\overline{N}(r,\frac{1}{f_j})+S(r),$$

where $S(r)$ *is a quantity satisfying* $S(r)=O(\log T(r))+O(\log r)$ $(r\to\infty, r\notin E_1)$. *If at least one of the quotients* f_j/f_m *is transcendental, then* $S(r)=o(T(r))$ $(r\to\infty, r\notin E_1)$, *while if all the quotients* f_j/f_m *are rational functions, then* $S(r)\le-\frac{1}{2}k(k-1)\log r+O(1)$ $(r\to\infty, r\notin E_1)$.

Lemma 2.6 ([9, 10]) *Assume that the hypotheses of Lemma 2.5 hold. Then for any* j *and* m, *we have*

$$T(r,f_j/f_m)=T(r)+O(1),\quad r\to\infty,$$

and for any j, *we have*

$$N(r,1/f_j)=T(r)+O(1),\quad r\to\infty.$$

Lemma 2.7 ([11, p. 39]) *Let* f *be a nonconstant meromorphic function, and* k *be a positive integer. Then*

$$N(r,\frac{1}{f^{(k)}})\le N(r,\frac{1}{f})+k\overline{N}(r,f)+o(T(r,f)),\quad r\to\infty,\ r\notin E_1.$$

The next lemma will be used to prove Theorem 1.5, which play an important role in judging the linear dependence of entire functions over complex domain.

Lemma 2.8 ([11, p. 70]) *Let* $f_1,f_2,\ldots,f_n$ *be linearly independent meromorphic functions satisfying* $\sum_{j=1}^{n}f_j\equiv1$. *Then for* $1\le j\le n$, *we have*

$$\begin{aligned}T(r,f_j)&\le\sum_{k=1}^{n}N(r,\frac{1}{f_k})+N(r,f_j)+N(r,D)-\sum_{k=1}^{n}N(r,f_k)-N(r,\frac{1}{D})+o\Big(\max_{1\le k\le n}\{T(r,f_k)\}\Big)\\&\le\sum_{k=1}^{n}N(r,\frac{1}{f_k})+(n-1)\sum_{k=1}^{n}\overline{N}(r,f_k)-N(r,\frac{1}{D})+o\Big(\max_{1\le k\le n}\{T(r,f_k)\}\Big)\end{aligned}$$

as $r\to\infty$, $r\notin E_1$, *where* D *is the Wronskian determinant* $W(f_1,f_2,\ldots,f_n)$.

The following is Borel's theorem on the combinations of entire functions, which will be used to judge the existence of entire solutions satisfying $\lambda(f)<\sigma(f)$ in the proof of Theorem 1.9.

Lemma 2.9 ([11, p. 77]) *Let* f_j,g_j $(j=1,\ldots,n)$ $(n\ge2)$ *be entire functions satisfying the following conditions.*

(1) $\sum\limits_{j=1}^{n}f_j(z)e^{g_j(z)}\equiv0$;

(2) The orders of f_j *are less than that of* $e^{g_t(z)-g_k(z)}$ *for* $1\le j\le n$, $1\le t<k\le n$.

Then $f_j(z)\equiv0$ $(j=1,\ldots,n)$.

3. Proofs of results

This section is devoted to proving Theorems 1.5 and 1.9.

Proof of Theorem 1.5 Let $f(z)$ be an entire solution of $\sigma_2(f) < 1$ of Eq. (1.5), by Lemma 2.1 we have

$$\sigma(f) = q. \tag{3.1}$$

On the other hand, by Lemma 2.1 and Remark 2.3, there exists a constant $D_1 > 0$ such that for sufficiently large $r \notin E$,

$$\begin{aligned} T(r, \frac{p(z)f^{(k)}(z+\eta)}{f^n(z)f'(z)}) \geq & T(r, \frac{f^n(z)f'(z)}{f^{(k)}(z+\eta)}) - o(r^q) \\ \geq & T(r, f^n(z)) - T(r, \frac{f^{(k)}(z+\eta)}{f'(z)}) - o(r^q) \\ \geq & T(r, f^n(z)) - N(r, \frac{f^{(k)}(z+\eta)}{f'(z)}) - o(T(r,f)) - o(r^q) \\ \geq & T(r, f^n(z)) - N(r, \frac{1}{f'(z)}) - o(r^q) \\ \geq & T(r, f^n(z)) - T(r, f'(z)) - o(r^q) \\ \geq & (n - 1 - o(1))D_1 r^q. \end{aligned} \tag{3.2}$$

Now we discuss the following two cases.

Case 1. Suppose that $-p(z)f^{(k)}(z+\eta), H_1(z)e^{\omega_1 z^q}, \ldots, H_m(z)e^{\omega_m z^q}$ are linearly independent. Let $\phi_1(z)$ denote the canonical product (or polynomial) generated by the common zeros of $-p(z)f^{(k)}(z+\eta), H_1(z), \ldots, H_m(z)$, each common zero is counted the minimum of its multiplicity. Then $-\frac{p(z)f^{(k)}(z+\eta)}{\phi_1(z)}, \frac{H_1(z)e^{\omega_1 z^q}}{\phi_1(z)}, \ldots, \frac{H_m(z)e^{\omega_m z^q}}{\phi_1(z)}$ are entire functions without common zeros, and

$$N(r, \frac{1}{\phi_1(z)}) = o(r^q), \quad r \to \infty. \tag{3.3}$$

Rewrite Eq. (1.5) in the form

$$\frac{f^n(z)f'(z)}{\phi_1(z)} = \sum_{j=1}^{m} \frac{H_j(z)e^{w_j z^q}}{\phi_1(z)} - \frac{p(z)f^{(k)}(z+\eta)}{\phi_1(z)}. \tag{3.4}$$

It follows from (3.2) that $\frac{p(z)f^{(k)}(z+\eta)}{\phi_1(z)} / \frac{f^n(z)f'(z)}{\phi_1(z)}$ is transcendental. So by (3.3), (3.4), Lemmas 2.5–2.7, Remark 2.3 and Lemma 2.1, we obtain

$$\begin{aligned} N(r, \frac{1}{f^n(z)f'(z)}) \leq & N(r, \frac{\phi_1(z)}{f^n(z)f'(z)}) + o(r^q) \\ \leq & T_1(r) + o(r^q) \\ \leq & \sum_{j=1}^{m} N_m(r, \frac{\phi_1(z)}{H_j(z)e^{w_j z^q}}) + N_m(r, \frac{\phi_1(z)}{p(z)f^{(k)}(z+\eta)}) + \\ & N_m(r, \frac{\phi_1(z)}{f^n(z)f'(z)}) + o(T_1(r)) + o(r^q) \\ \leq & N(r, \frac{1}{f^{(k)}(z+\eta)}) + N_m(r, \frac{1}{f^n(z)f'(z)}) + o(T_1(r)) + o(r^q) \\ \leq & N(r, \frac{1}{f(z)}) + N_m(r, \frac{1}{f^n(z)f'(z)}) + o(T_1(r)) + o(r^q) \end{aligned} \tag{3.5}$$

as $r \to \infty$ and $r \notin E$, where

$$T_1(r) = \frac{1}{2\pi}\int_0^{2\pi} u_1(re^{i\theta})\mathrm{d}\theta - u_1(0),$$

$$u_1(z) = \sup\{\log|\frac{p(z)f^{(k)}(z+\eta)}{\phi_1(z)}|, \log|\frac{H_j(z)e^{\omega_j z^q}}{\phi_1(z)}| : 1 \le j \le m\}.$$

Let z_0 be a zero of $f(z)$ with multiplicity l. Then z_0 is a zero of $f^n(z)f'(z)$ with multiplicity $(n+1)l-1$. Since $n \ge m+2$, the contribution of z_0 to $n(r, \frac{1}{f^n(z)f'(z)}) - n_m(r, \frac{1}{f^n(z)f'(z)})$ is $(n+1)l-1-m$ $(\ge 2l)$ when $|z_0| \le r$. This means that

$$N(r, \frac{1}{f^n(z)f'(z)}) - N_m(r, \frac{1}{f^n(z)f'(z)}) \ge 2N(r, \frac{1}{f(z)}). \tag{3.6}$$

So by (3.5) and (3.6), we obtain

$$N(r, \frac{1}{f(z)}) \le o(T_1(r)) + o(r^q), \quad r \to \infty, \ r \notin E. \tag{3.7}$$

On the other hand, by (3.5) and Lemma 2.1, there exists a constant $D_2 > 0$ such that

$$\begin{aligned}(1-o(1))T_1(r) &\le N(r, \frac{1}{f(z)}) + N_m(r, \frac{1}{f^n(z)f'(z)}) + o(r^q)\\ &\le T(r, f(z)) + T(r, f^n(z)f'(z)) + o(r^q)\\ &\le (n+2)T(r,f) + o(r^q)\\ &\le D_2 r^q, \quad r \to \infty, \ r \notin E,\end{aligned}$$

which means that

$$o(T_1(r)) \subset o(r^q), \quad r \to \infty, \ r \notin E. \tag{3.8}$$

So combining (3.7) and (3.8), we obtain

$$N(r, \frac{1}{f(z)}) = o(r^q), \quad r \to \infty, \ r \notin E. \tag{3.9}$$

Then by (3.9), Lemma 2.7 and Remark 2.3, we obtain

$$\sum_{j=1}^{m} N(r, \frac{f^n(z)f'(z)}{H_j(z)e^{w_j z^q}}) + N(r, \frac{f^n(z)f'(z)}{p(z)f^{(k)}(z+\eta)}) = o(r^q), \quad r \to \infty, \ r \notin E \tag{3.10}$$

and

$$\sum_{j=1}^{m} \overline{N}(r, \frac{H_j(z)e^{w_j z^q}}{f^n(z)f'(z)}) + \overline{N}(r, \frac{p(z)f^{(k)}(z+\eta)}{f^n(z)f'(z)}) = o(r^q), \quad r \to \infty, \ r \notin E. \tag{3.11}$$

Let

$$T_f(r) = \max\{T(r, \frac{p(z)f^{(k)}(z+\eta)}{f^n(z)f'(z)}), T(r, \frac{H_j(z)e^{w_j z^q}}{f^n(z)f'(z)}) : 1 \le j \le m\}.$$

Note that (1.5) implies

$$\sum_{j=1}^{m} \frac{H_j(z)e^{w_j z^q}}{f^n(z)f'(z)} - \frac{p(z)f^{(k)}(z+\eta)}{f^n(z)f'(z)} = 1,$$

then by Lemma 2.8, (3.10) and (3.11), we obtain

$$T_f(r) = o(r^q), \quad r \to \infty, \ r \notin E,$$

which implies that

$$T(r, \frac{p(z)f^{(k)}(z+\eta)}{f^n(z)f'(z)}) = o(r^q), \quad r \to \infty, \ r \notin E.$$

This contradicts (3.2).

Case 2. Suppose that $-p(z)f^{(k)}(z+\eta), H_1(z)e^{\omega_1 z^q}, \dots, H_m(z)e^{\omega_m z^q}$ are linearly dependent. Since $H_1(z)e^{\omega_1 z^q}, \dots, H_m(z)e^{\omega_m z^q}$ are linearly independent, there exist constants $c_1, \dots, c_m$ that are not all zero, such that

$$p(z)f^{(k)}(z+\eta) = \sum_{j=1}^{m} c_j H_j(z)e^{\omega_j z^q}. \tag{3.12}$$

Substituting (3.12) into (1.5), we obtain

$$f^n(z)f'(z) = \sum_{j=1}^{m} (1-c_j)H_j(z)e^{\omega_j z^q}. \tag{3.13}$$

Now we discuss the following two subcases.

Subcase 2.1. If at least two among $1-c_1, \dots, 1-c_m$ are not zero, without loss of generality, we assume that $1-c_1, \dots, 1-c_t$ $(2 \le t \le m)$ are not zero, then by (3.13) and Lemma 2.4 there exists a constant $D_3 > 0$ such that

$$\begin{aligned} N(r, \frac{1}{f^n(z)f'(z)}) &= N\big(r, \frac{1}{\sum_{j=1}^{t}(1-c_j)H_j(z)e^{(\omega_j-\omega_1)z^q}}\big) \\ &= T\Big(r, \sum_{j=1}^{t}(1-c_j)H_j(z)e^{(\omega_j-\omega_1)z^q}\Big) - o(r^q) \\ &\ge D_3 r^q, \quad r \to \infty. \end{aligned} \tag{3.14}$$

On the other hand, using the argument similar to that of Case 1, there exists an entire function $\phi_2(z)$ such that $\frac{(1-c_1)H_1(z)e^{\omega_1 z^q}}{\phi_2(z)}, \dots, \frac{(1-c_t)H_t(z)e^{\omega_t z^q}}{\phi_2(z)}$ are entire functions without common zeros, and

$$N(r, \frac{1}{\phi_2(z)}) = o(r^q), \quad r \to \infty. \tag{3.15}$$

Rewrite (3.13) in the form

$$\frac{f^n(z)f'(z)}{\phi_2(z)} = \sum_{j=1}^{t} \frac{(1-c_j)H_j(z)e^{\omega_j z^q}}{\phi_2(z)}. \tag{3.16}$$

Since $\frac{(1-c_1)H_1(z)e^{\omega_1 z^q}}{\phi_2(z)} / \frac{(1-c_2)H_2(z)e^{\omega_2 z^q}}{\phi_2(z)}$ is transcendental, by Lemmas 2.5 and 2.6, (3.16) and (3.15), we obtain

$$\begin{aligned} N(r, \frac{1}{f^n(z)f'(z)}) \le& N(r, \frac{\phi_2(z)}{f^n(z)f'(z)}) + o(r^q) \\ \le& T_2(r) + o(r^q) \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{t} N_{t-1}(r, \frac{\phi_2(z)}{(1-c_j)H_j(z)e^{\omega_j z^q}}) + N_{t-1}(r, \frac{\phi_2(z)}{f^n(z)f'(z)}) + \\
&\quad o(T_2(r)) + o(r^q) \\
&\le N_{t-1}(r, \frac{1}{f^n(z)f'(z)}) + o(T_2(r)) + o(r^q) \\
&\le T(r, f^n(z)f'(z)) + o(T_2(r)) + o(r^q) \\
&\le (n+1)T(r, f(z)) + o(T_2(r)) + o(r^q), \quad r \to \infty, \ r \notin E,
\end{aligned} \tag{3.17}
$$

where

$$
T_2(r) = \frac{1}{2\pi}\int_0^{2\pi} u_2(re^{i\theta})\mathrm{d}\theta - u_2(0),
$$

$$
u_2(z) = \sup\{\log|\frac{(1-c_j)H_j(z)e^{\omega_j z^q}}{\phi_2(z)}| : 1 \le j \le t\}.
$$

Combining (3.17) and Lemma 2.1, we have

$$
o(T_2(r)) \subset o(r^q), \quad r \to \infty, \ r \notin E. \tag{3.18}
$$

Then by (3.17), (3.18), (3.6) and $N_{t-1}(r, \frac{1}{f^n(z)f'(z)}) \le N_m(r, \frac{1}{f^n(z)f'(z)})$, we obtain

$$
N(r, \frac{1}{f(z)}) = o(r^q), \quad r \to \infty, \ r \notin E.
$$

From this, Lemmas 2.7 and 2.1, we obtain

$$
N(r, \frac{1}{f^n(z)f'(z)}) \le (n+1)N(r, \frac{1}{f(z)}) + o(T(r,f)) = o(r^q), \quad r \to \infty, \ r \notin E,
$$

which contradicts (3.14).

Subcase 2.2. If only one among $1-c_1, \dots, 1-c_m$ is not zero, without loss of generality, we assume that $1-c_1 \ne 0$, then (3.13) reduces to the form

$$
f^n(z)f'(z) = (1-c_1)H_1(z)e^{\omega_1 z^q}. \tag{3.19}
$$

We claim that only one among $c_1, \dots, c_m$ is not zero. If not, then by (3.12) and Lemma 2.4, there exists a constant $D_4 > 0$, such that for sufficiently large r,

$$
\begin{aligned}
N(r, \frac{1}{f^{(k)}(z+\eta)}) &\ge N(r, \frac{1}{\sum_{j=1}^{m} c_j H_j(z)e^{\omega_j z^q}}) - N(r, \frac{1}{p(z)}) \\
&= N(r, \frac{1}{\sum_{j=1}^{m} c_j H_j(z)e^{(\omega_j - \omega_{j_0})z^q}}) - o(r^q) \\
&= T(r, \sum_{j=1}^{m} c_j H_j(z)e^{(\omega_j - \omega_{j_0})z^q}) - o(r^q) \\
&\ge D_4 r^q,
\end{aligned} \tag{3.20}
$$

where $j_0 \in \{1, \dots, m\}$ such that $c_{j_0} \ne 0$. By (3.20), (3.19), Lemma 2.7, Remark 2.3 and Lemma

2.1, we obtain for sufficiently large $r \notin E$,

$$D_4 r^q \leq N(r, \frac{1}{f^{(k)}(z+\eta)}) \leq N(r, \frac{1}{f(z)}) + o(T(r,f)) \leq N(r, \frac{1}{H_1(z)}) + o(r^q) = o(r^q).$$

This is a contradiction. So we have only $c_{j_0} \neq 0$, and (3.12) gives

$$p(z)f^{(k)}(z+\eta) = c_{j_0} H_{j_0}(z) e^{\omega_{j_0} z^q}. \tag{3.21}$$

By (3.19) and (3.1), we have $\lambda(f) < \sigma(f) = q$. So by Hadamard's factorization theorem, we obtain

$$f(z) = \varphi(z) e^{\alpha z^q}, \tag{3.22}$$

where $\varphi(z)$ is an entire function satisfying $\sigma(\varphi) = \lambda(\varphi) = \lambda(f) < q$, α is a nonzero constant. From (3.22), we obtain

$$f^{(j)}(z) = \varphi_j(z) e^{\alpha z^q}, \quad j = 1, \ldots, k, \tag{3.23}$$

where $\varphi_j(z) = \varphi'_{j-1}(z) + q\alpha z^{q-1}\varphi_{j-1}(z) (\not\equiv 0)$ $(j = 1, \ldots, k)$, $\varphi_0 = \varphi$. Substituting (3.22) and (3.23) into (3.19) and (3.21), respectively, we have $\alpha = \frac{\omega_1}{n+1} = \omega_{j_0}$. So $j_0 \neq 1$. From this, (3.19), (3.21) and (1.5), we get $m = 2$. Theorem 1.5 is thus proved. □

Proof of Theorem 1.9 Let f be an entire solution of Eq. (1.5) satisfying $\sigma_2(f) < 1$. If $n \geq 2$, then by Lemma 2.1 we have $\sigma(f) = q$. Now we discuss the case $n = 1$. Let $G(z), \Lambda_1, \Lambda_2$ be defined as the proof of Lemma 2.1. By (1.5) we know that

$$|f'(re^{i\theta})| \leq |G(re^{i\theta})| + |\frac{p(re^{i\theta}) f^{(k)}(re^{i\theta}+\eta)}{f(re^{i\theta})}|$$

holds for $\theta \in \Lambda_1$. From this, Lemma 2.1 and Remark 2.3, there exists a constant $D_5 > 0$, such that for sufficiently large $r \notin E$,

$$\begin{aligned} m(r, f'(z)) &\leq \frac{1}{2\pi}\int_{\Lambda_1} \log^+ |f'(re^{i\theta})| d\theta + \frac{1}{2\pi}\int_{\Lambda_2} \log^+ |\frac{f'(re^{i\theta})}{f(re^{i\theta})}| d\theta \\ &\leq m(r, G(z)) + m(r, \frac{f^{(k)}(z+\eta)}{f(z)}) + m(r, p(z)) + o(T(r,f)) \\ &\leq D_5 r^q. \end{aligned} \tag{3.24}$$

Since $\sigma(f) = \sigma(f')$, by (3.24) we have $\sigma(f) \leq q$. If $\sigma(f) < q$, then by (1.5), we obtain that

$$q = \sigma\Big(\sum_{j=1}^{m} H_j(z) e^{\omega_j z^q}\Big) < q.$$

This is a contradiction. So we have $\sigma(f) = q$. The first result of Theorem 1.9 is thus proved.

Next we consider the case $\lambda(f) < \sigma(f)$. By $\sigma(f) = q$ and Hadamard's factorization theorem, we know that $f(z)$ and $f^{(k)}(z)$ have the form of (3.22) and (3.23), respectively. Substituting (3.22) and (3.23) into (1.5), we obtain

$$\varphi^n(z)\varphi_1(z) e^{(n+1)\alpha z^q} + p(z)\varphi_k(z+\eta) e^{\alpha((z+\eta)^q - z^q)} e^{\alpha z^q} = \sum_{j=1}^{m} H_j(z) e^{\omega_j z^q}. \tag{3.25}$$

Since the orders of $\varphi^n(z)\varphi_1(z)$, $p(z)\varphi_k(z+\eta)e^{\alpha((z+\eta)^q-z^q)}$, $H_j(z)$ $(j=1,\ldots,m)$ are less than q, by (3.25) and Lemma 2.9, we obtain that $m=2, \{\alpha,(n+1)\alpha\}=\{\omega_1,\omega_2\}$. Then by $n\leq m+1$, we have $(n,m)\in\{(1,2),(2,2),(3,2)\}$. Theorem 1.9 is thus proved. $\square$

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