Journal of Mathematical Research with Applications
May, 2024, Vol. 44, No. 3, pp. 337–343
DOI:10.3770/j.issn:2095-2651.2024.03.006
Http://jmre.dlut.edu.cn

(m, n)-Igusa-Todorov Algebras, IT-Dimensions and Triangular Matrix Algebras

Jiyuan TANG, Yuanfeng ZHANG, Hanpeng GAO*
School of Mathematical Sciences, Anhui University, Anhui 230601*, P. R. China*

Abstract Let T, U be two Artin algebras and ${}_UM_T$ be a U-T-bimodule. In this paper, we get a necessary and sufficient condition such that the formal triangular matrix algebra $\Lambda = \left(\begin{smallmatrix} T & 0 \\ M & U \end{smallmatrix}\right)$ is (m, n)-Igusa-Todorov when ${}_UM$, M_T are projective. We also study the Igusa-Todorov dimension of Λ. More specifically, it is proved that

$$\max\{\mathrm{IT.dim}\, T, \mathrm{IT.dim}\, U\} \leqslant \mathrm{IT.dim}\, \Lambda \leqslant \min\{\max\{\mathrm{gl.dim}\, T, \mathrm{IT.dim}\, U\}, \max\{\mathrm{gl.dim}\, U, \mathrm{IT.dim}\, T\}\}.$$

Keywords Igusa-Todorov algebras; IT-dimensions; triangular matrix algebras

MR(2020) Subject Classification 18G20; 16E10; 18E10

1. Introduction

Given an Artin algebra Λ, we denote by $\mathrm{mod}\,\Lambda$ the category of finitely generated left Λ-modules. We recall that the finitistic dimension conjecture states that

$$\mathrm{fin.dim}\,\Lambda := \sup\{\mathrm{pd}\, M \mid \mathrm{pd}\, M < \infty, M \in \mathrm{mod}\,\Lambda\}$$

is finite, for any Artin algebra Λ. It is worth mentioning that the finitistic dimension conjecture is still open and it is one of the main problems in the representation theory of algebras. For more information about the history of the finitistic dimension conjecture, we refer the reader to [1]. Until now, it is known that this conjecture is true for several classes of algebras, among others: algebras with radical cube zero, monomial algebras, left serial algebras, weakly stably hereditary algebras and special biserial algebras. A large class of algebras, containing the mentioned before, is the class of Igusa-Todorov algebras.

The concept of n-Igusa-Todorov algebra was introducted by Wei in [2]. It is proved that the finitistic dimension of Igusa-Todorov algebra is finite . Also, Wei asked the following question: Are all artin algebras Igusa-Todorov? Conde [3] gave a counterexample by pointing out the following fact: Let $\bigwedge(k^m)$ be the exterior algebra of a vector space k^m over an uncountable field k. Then $\bigwedge(k^m)$ is not Igusa-Todorov for $m \geq 3$.

Received July 20, 2023; Accepted December 16, 2023
Supported by the National Natural Science Foundation of China (Grant No. 12301041) and the Science Foundation for Distinguished Young Scholars of Anhui Province (Grant No. 2108085J01).
* Corresponding author
E-mail address: jytang2000@163.com (Jiyuan TANG); yfzhang9804@163.com (Yuanfeng ZHANG); hpgao@ahu.edu.cn (Hanpeng GAO)

As a generalization of n-Igusa-Todorov algebras, Zheng [4] introduced the notion of (m,n)-Igusa-Todorov algebras, where m,n are two nonnegative integers, and then the author proved that all algebras are (m,n)-Igusa-Todorov algebras for some m,n. Moreover, he also gave an upper bound for the derived dimension of (m,n)-Igusa-Todorov algebras. As a consequence, a new upper bound for the derived dimension of n-Igusa-Todorov algebra was given.

In this paper, we will study the (m,n)-Igusa-Todorov triangular matrix algebras and its Igusa-Todorov dimension, the idea comes from [5]. Our main result is as follows.

Theorem 1.1 *Let* $\Lambda = \left(\begin{smallmatrix} T & 0 \\ {}_UM_T & U \end{smallmatrix}\right)$ *be a triangular matrix algebra such that* ${}_UM \in \operatorname{add}(U)$, $M_T \in \operatorname{add}(T)$. *Then* Λ *is* (m,n)*-Igusa-Todorov algebra if and only if* T *and* U *are* (m,n)*-Igusa-Todorov algebras. Moreover,*

$$\max\{\mathrm{IT.dim}\,T, \mathrm{IT.dim}\,U\} \leqslant \mathrm{IT.dim}\,\Lambda \leqslant \min\{\max\{\mathrm{gl.dim}\,T, \mathrm{IT.dim}\,U\}, \max\{\mathrm{gl.dim}\,U, \mathrm{IT.dim}\,T\}\}.$$

2. Preliminaries

In this section, we review some facts and definitions of (m,n)-Igusa-Todorov algebras, IT-dimensions and triangular matrix algebras.

2.1. (m,n)-Igusa-Todorov algebras

Let $X \in \operatorname{mod}\Lambda$. Given an epimorphism $f: P \to X$ in $\operatorname{mod}\Lambda$ such that P is a projective cover of X in $\operatorname{mod}\Lambda$, then we write $\Omega^1(X) =: \ker f$. Inductively, for any $n \geq 2$, we write $\Omega^n(X) =: \Omega^1(\Omega^{n-1}(X))$. In particular, we set $\Omega^1(X) := \Omega(X)$ and $\Omega^0(X) := X$.

Definition 2.1 ([2]) *Let* Λ *be an artin algebra and* n *be a nonnegative integer. Then* Λ *is said to be an* n*-Igusa-Todorov algebra if there is a module* $V \in \operatorname{mod}\Lambda$ *such that for any module* M *there exists an exact sequence*

$$0 \to V_1 \to V_0 \to \Omega^n(M) \to 0,$$

where $V_i \in \operatorname{add}V$ *for each* $0 \leq i \leq 1$.

The following definition is a generalization of Definition 2.1.

Definition 2.2 ([4]) *Let* Λ *be an artin algebra and* m,n *be nonnegative integers. Then* Λ *is said to be an* (m,n)*-Igusa-Todorov algebra if there is a module* $V \in \operatorname{mod}\Lambda$ *such that for any module* M *there exists an exact sequence*

$$0 \to V_m \to V_{m-1} \to \cdots \to V_1 \to V_0 \to \Omega^n(M) \to 0$$

where $V_i \in \operatorname{add}V$ *for each* $0 \leq i \leq m$.

For each $i \geq 1$, we denote

$$\begin{aligned}\Omega^i(\operatorname{mod}\Lambda) &:= \{X \mid X = \Omega^i(Y) \oplus P \text{ for some } Y \in \operatorname{mod}\Lambda \text{ and projective module } P \text{ in } \operatorname{mod}\Lambda\}\\ &= \{X \mid \text{ there exists an sequence } 0 \to X \to P_n \to P_{n-1} \to \cdots \to P_1\\ &\qquad \text{with projective module } P_i \text{ in } \operatorname{mod}\Lambda \text{ for each } 0 \leq i \leq n\}.\end{aligned}$$

And $\Omega^0(\operatorname{mod}\Lambda) := \operatorname{mod}\Lambda$. Recall that Λ is said to be n-syzygy-finite if $\Omega^n(\operatorname{mod}\Lambda) := \operatorname{add} M$ for some $M \in \operatorname{mod}\Lambda$. And Λ is said to be syzygy finite if there exists a nonnegative integer n such that $\Omega^n(\operatorname{mod}\Lambda) := \operatorname{add} M$ for some $M \in \operatorname{mod}\Lambda$. In particular, Λ is 0-syzygy-finite if and only if Λ is representation finite type.

Definition 2.3 ([6]) *Let Λ be an artin algebra. We set the Igusa-Todorov dimension of Λ as follows*

$$\mathrm{IT.dim}\,\Lambda := \inf\{m \mid \Lambda \textit{ is an } (m,n)\textit{-Igusa-Todorov algebra}\}.$$

Remark 2.4 (1) An algebra Λ is $(0,n)$-Igusa-Todorov if and only if it is n-syzygy-finite if and only if $\mathrm{IT.dim}\,\Lambda = 0$.

(2) An algebra Λ is $(1,n)$-Igusa-Todorov if and only if it is n-Igusa-Todorov if and only if $\mathrm{IT.dim}\,\Lambda \leqslant 1$.

(3) (m,n)-Igusa-Todorv algebras are $(m+i,n-i)$-Igusa-Todorov algebras for $i \leqslant n$.

(4) If $\mathrm{gl.dim}\,\Lambda < \infty$, then Λ is a $(\mathrm{gl.dim}\,\Lambda, 0)$-Igusa-Todorov algebra.

Let T, U be Artin R-algebras and M be a U-T-bimodule. Then the triangular matrix algebra

$$\Lambda := \begin{pmatrix} T & 0 \\ M & U \end{pmatrix}$$

can be defined by the ordinary operation on matrices. Let $\mathcal{C}_\Lambda$ be the category whose objects are the triples (A,B,f), where A is a T-module, B is a U-module and $f \in \operatorname{Hom}_U(M \otimes_T A, B)$. The morphisms from (A,B,f) to (A',B',f') are pairs of (α,β) such that the following diagram

$$\begin{array}{ccc} M \otimes_T A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle M\otimes\alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ M \otimes_T A' & \xrightarrow{\quad f' \quad} & B' \end{array}$$

Diagram 1 The morphism from (A,B,f) to (A',B',f')

commutes, where $\alpha \in \operatorname{Hom}_T(A,A')$ and $\beta \in \operatorname{Hom}_U(B,B')$.

It is well known that there exists an equivalence of categories between $\operatorname{mod}\Lambda$ and $\mathcal{C}_\Lambda$. Hence we can view a Λ-module as a triple (A,B,f) with $A \in \operatorname{mod} T$ and $B \in \operatorname{mod} U$. Moreover, a sequence

$$0 \to (A_1,B_1,f_1) \xrightarrow{(\alpha_1,\beta_1)} (A_2,B_2,f_2) \xrightarrow{(\alpha_2,\beta_2)} (A_3,B_3,f_3) \to 0$$

in $\operatorname{mod}\Lambda$ is exact if and only if

$$0 \to A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \to 0$$

is exact in $\operatorname{mod} T$ and

$$0 \to B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \to 0$$

is exact in $\operatorname{mod} U$. All indecomposable projective modules in $\operatorname{mod}\Lambda$ are exactly of the forms $(P, M \otimes P, 1)$ and $(0,Q,0)$, where P is an indecomposable projective T-module and Q is an indecomposable projective U-module.

Lemma 2.5 ([5]) *Let $\Lambda = \begin{pmatrix} T & 0 \\ {}_UM_T & U \end{pmatrix}$ be a triangular matrix Artin R-algebra such that ${}_UM \in \operatorname{proj}(U)$ and $M_T \in \operatorname{proj}(T)$. Then, the n-th syzygy of the Λ-module (A, B, f) is*

$$\Omega^n(A, B, f) = (\Omega^n A, M \otimes P^A_{n-1}, M \otimes i^A_n) \oplus (0, \Omega^n B, 0).$$

3. Main result

Now, we can prove our main theorem.

Theorem 3.1 *Let $\Lambda = \begin{pmatrix} T & 0 \\ {}_UM_T & U \end{pmatrix}$ be a triangular matrix Artin R-algebra such that ${}_UM \in \operatorname{add}(U)$, $M_T \in \operatorname{proj}(T)$. Then Λ is (m, n)-Igusa-Todorov algebra if and only if both T and U are (m, n)-Igusa-Todorov algebras.*

Proof Assume that T and U are (m, n)-Igusa-Todorov algebras. Indeed, let $(A, B, f) \in \operatorname{mod}\Lambda$. Hence we have the following two exact sequences

$$0 \to V_m \to V_{m-1} \to \cdots \to V_0 \to \Omega^n A \to 0$$

and

$$0 \to W_m \to W_{m-1} \to \cdots \to W_0 \to \Omega^n B \to 0$$

with $V_i \in \operatorname{add}(V)$ and $W_i \in \operatorname{add}(W)$, $i = 0, 1, 2, \ldots, m$.

Now, set $K_i := \operatorname{Ker}(V_i \to V_{i-1})$ and $L_i := \operatorname{Ker}(W_i \to W_{i-1}), 0 < i \leqslant m-1$, $K_{m-1} := V_m$ and $L_{m-1} := W_m$.

Consider the exact sequences,

$$0 \longrightarrow K_0 \xrightarrow{l_0^A} V_0 \xrightarrow{\rho_0^A} \Omega^n A \longrightarrow 0, \quad 0 \longrightarrow L_0 \xrightarrow{l_0^B} W_0 \xrightarrow{\rho_0^B} \Omega^n B \longrightarrow 0.$$

We have the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & M \otimes K_0 \arrow[r, "M\otimes l_0^A"] \arrow[d, dotted, "\kappa_0"] & M \otimes V_0 \arrow[r, "M\otimes \rho_0^A"] \arrow[d, "\begin{pmatrix} 0 \\ M\otimes i_n^A \rho_0^A \end{pmatrix}"] & M \otimes \Omega^n A \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ M\otimes i_n^A \end{pmatrix}"] & 0 \\
0 \arrow[r] & L_0 \arrow[r, "\begin{pmatrix} l_0^B \\ 0 \end{pmatrix}"'] & W_0 \oplus (M \otimes P^A_{n-1}) \arrow[r, "\begin{pmatrix} \rho_0^B & 0 \\ 0 & 1 \end{pmatrix}"'] & \Omega^n B \oplus (M \otimes P^A_{n-1}) \arrow[r] & 0
\end{tikzcd}$$

Diagram 2 The construction of morphism κ_0

where κ_0 is induced by the universal property of kernel. It implies $l_0^B \kappa_0 = 0$ and hence $\kappa_0 = 0$. Thus, we construct the following exact and commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \otimes K_0 \arrow[r, "M\otimes l_0^A"] \arrow[d, "\begin{pmatrix} 0 \\ M\otimes l_0^A \end{pmatrix}"] & M \otimes V_0 \arrow[r, "M\otimes \rho_0^A"] \arrow[d, "\begin{pmatrix} 0 \\ M\otimes i_n^A \rho_0^A \\ 1 \end{pmatrix}"] & M \otimes \Omega^n A \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ M\otimes i_n^A \end{pmatrix}"] & 0 \\
0 \arrow[r] & L_0 \oplus (M \otimes V_0) \arrow[r, "\begin{pmatrix} l_0^B & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}"'] & W_0 \oplus (M \otimes P^A_{n-1}) \oplus (M \otimes V_0) \arrow[r, "\begin{pmatrix} \rho_0^B & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}"'] & \Omega^n B \oplus (M \otimes P^A_{n-1}) \arrow[r] & 0
\end{tikzcd}$$

Diagram 3 The construction of morphism $\begin{pmatrix} 0 \\ M\otimes l_0^A \end{pmatrix}$

we obtain the exact sequence of Λ-modules

$$0 \longrightarrow \left(K_0, L_0 \oplus (M\otimes V_0), \begin{pmatrix} 0 \\ M\otimes l_0^A \end{pmatrix}\right) \longrightarrow \left(V_0, W_0 \oplus (M\otimes P_{n-1}^A) \oplus (M\otimes V_0), \begin{pmatrix} 0 \\ M\otimes i_n^A \rho_0^A \\ 1 \end{pmatrix}\right)$$
$$\longrightarrow \left(\Omega^n A, \Omega^n B \oplus (M\otimes P_{n-1}^A), \begin{pmatrix} 0 \\ M\otimes i_n^A \end{pmatrix}\right) \longrightarrow 0.$$

From the preceding exact sequence, we consider the exact sequences,

$$0 \longrightarrow K_1 \xrightarrow{l_1^A} V_1 \xrightarrow{\pi_1^A} K_0 \longrightarrow 0$$

and

$$0 \longrightarrow L_1 \xrightarrow{l_1^B} W_1 \xrightarrow{\pi_1^B} L_0 \longrightarrow 0.$$

We have the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & M\otimes K_1 \arrow[r, "M\otimes l_1^A"] \arrow[d, "\begin{pmatrix} 0 \\ M\otimes l_1^A \end{pmatrix}"] & M\otimes V_1 \arrow[r, "M\otimes \pi_1^A"] \arrow[d, "\begin{pmatrix} 0 \\ M\otimes \pi_1^A l_0^A \\ 1 \end{pmatrix}"] & M\otimes K_0 \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ M\otimes l_0^A \end{pmatrix}"] & 0 \\
0 \arrow[r] & L_1 \oplus (M\otimes V_1) \arrow[r, "\begin{pmatrix} l_1^B & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}"'] & W_1 \oplus (M\otimes V_0) \oplus (M\otimes V_1) \arrow[r, "\begin{pmatrix} \pi_1^B & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}"'] & L_0 \oplus (M\otimes V_0) \arrow[r] & 0
\end{tikzcd}$$

Diagram 4 The construction of morphism $\begin{pmatrix} 0 \\ M\otimes l_1^A \end{pmatrix}$

In order to prove the result, we proceed by induction. Therefore, we have the following exact and commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M\otimes K_i \arrow[r, "M\otimes l_i^A"] \arrow[d, "\begin{pmatrix} 0 \\ M\otimes l_i^A \end{pmatrix}"] & M\otimes V_i \arrow[r, "M\otimes \pi_i^A"] \arrow[d, "\begin{pmatrix} 0 \\ M\otimes \pi_i^A l_{i-1}^A \\ 1 \end{pmatrix}"] & M\otimes K_{i-1} \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ M\otimes l_{i-1}^A \end{pmatrix}"] & 0 \\
0 \arrow[r] & L_i \oplus (M\otimes V_i) \arrow[r, "\begin{pmatrix} l_i^B & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}"'] & W_i \oplus (M\otimes V_{i-1}) \oplus (M\otimes V_i) \arrow[r, "\begin{pmatrix} \pi_i^B & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}"'] & L_{i-1} \oplus (M\otimes V_{i-1}) \arrow[r] & 0
\end{tikzcd}$$

Diagram 5 The construction of morphism $\begin{pmatrix} 0 \\ M\otimes l_i^A \end{pmatrix}$

And we have the following exact sequence

$$0 \longrightarrow \left(K_i, L_i \oplus (M\otimes V_i), \begin{pmatrix} 0 \\ M\otimes l_i^A \end{pmatrix}\right) \longrightarrow \left(V_i, W_i \oplus (M\otimes V_{i-1}) \oplus (M\otimes V_i), \begin{pmatrix} 0 \\ M\otimes \pi_i^A l_{i-1}^A \\ 1 \end{pmatrix}\right)$$
$$\longrightarrow \left(K_{i-1}, L_{i-1} \oplus (M\otimes V_{i-1}), \begin{pmatrix} 0 \\ M\otimes l_{i-1}^A \end{pmatrix}\right) \longrightarrow 0.$$

Thus, we have a long exact sequence

$$\begin{aligned}
0 &\longrightarrow \left(V_m, W_m \oplus (M\otimes V_{m-1}), \begin{pmatrix} 0 \\ M\otimes l_{m-1}^A \end{pmatrix}\right) \\
&\longrightarrow \left(V_{m-1}, W_{m-1} \oplus (M\otimes V_{m-2}) \oplus (M\otimes V_{m-1}), \begin{pmatrix} 0 \\ M\otimes \pi_{m-1}^A l_{m-2}^A \\ 1 \end{pmatrix}\right) \\
&\longrightarrow \left(V_{m-2}, W_{m-2} \oplus (M\otimes V_{m-3}) \oplus (M\otimes V_{m-2}), \begin{pmatrix} 0 \\ M\otimes \pi_{m-2}^A l_{m-3}^A \\ 1 \end{pmatrix}\right) \longrightarrow \cdots
\end{aligned}$$

$$\begin{aligned}&\longrightarrow \left(V_0, W_0 \oplus (M \otimes P_{n-1}^A) \oplus (M \otimes V_0), \left(\begin{smallmatrix} 0 \\ M\otimes i_n^A \rho_0^A \\ 1 \end{smallmatrix}\right)\right)\\ &\longrightarrow \left(\Omega^n A, \Omega^n B \oplus (M \otimes P_{n-1}^A), \left(\begin{smallmatrix} 0 \\ M\otimes i_n^A \end{smallmatrix}\right)\right) \longrightarrow 0.\end{aligned}$$

Define $C := \bigoplus_{i=1}^r (V, W\oplus M\oplus M\otimes V, h_i)$, where $h_1, h_2, \dots, h_r$ are R-generators of $\mathrm{Hom}_U(M\otimes V, W\oplus M)$. It follows that Λ is (m,n)-Igusa-Todorov algebra.

Conversely, suppose that Λ is (m,n)-Igusa-Todorov algebra. Let us prove that T is an (m,n)-Igusa-Todorov algebra. Indeed, there exists a module $(X,Y,f) \in \mathrm{mod}\,\Lambda$ such that for any $A \in \mathrm{mod}\,T$, since Λ is an (m,n)-Igusa-Todorov algebra, there is an exact sequence of Λ-modules

$$\begin{aligned}&0 \to (X_m, Y_m, f_m) \to (X_{m-1}, Y_{m-1}, f_{m-1}) \to \cdots \to (X_0, Y_0, f_0)\\ &\to (\Omega^n A, M\otimes P_{n-1}^A, M \otimes i_n^A) \to 0\end{aligned}$$

with $(X_i, Y_i, f_i) \in \mathrm{add}(X,Y,f)$. In particular, we get the exact sequence

$$0 \to X_m \to X_{m-1} \to \cdots \to X_0 \to \Omega^n A \to 0$$

of T-modules with $X_i \in \mathrm{add}\,X$. Then T is (m,n)-Igusa-Todorov algebra.

We assert that U is an (m,n)-Igusa-Todorov algebra. There exists a module $(X,Y,f) \in \mathrm{mod}\,\Lambda$ such that for any $B \in \mathrm{mod}\,U$, there is an exact sequence

$$0 \to (0, Y_m, 0) \to (0, Y_{m-1}, 0) \to \cdots \to (0, Y_0, 0) \to (0, \Omega^n B, 0) \to 0$$

with $(0, Y_j, 0) \in \mathrm{add}(X,Y,f)$. Hence, we get the exact sequence

$$0 \to Y_m \to Y_{m-1} \to \cdots \to Y_0 \to \Omega^n B \to 0$$

with $Y_j \in \mathrm{add}\,Y$. Then U is (m,n)-Igusa-Todorov algebra. $\Box$

Lemma 3.2 *Let $\Lambda = \left(\begin{smallmatrix} T & 0 \\ {}_UM_T & U \end{smallmatrix}\right)$ be a triangular matrix Artin R-algebra such that ${}_UM \in \mathrm{add}(U)$, $M_T \in \mathrm{proj}(T)$. Then*

$$\max\{\mathrm{IT.dim}\,T, \mathrm{IT.dim}\,U\} \leqslant \mathrm{IT.dim}\,\Lambda \leqslant \max\{\mathrm{gl.dim}\,T, \mathrm{IT.dim}\,U\}.$$

Proof Let $t = \max\{\mathrm{gl.dim}\,T, \mathrm{IT.dim}\,U\}$. We can assume $t < \infty$, $\mathrm{IT.dim}\,U = m \leqslant t$ and U is an (m,n)-Igusa-Todorov algebra. By Remark 2.4, T is $(t,0)$-Igusa-Todorov. Therefore, T and U are (t,n)-Igusa-Todorov. Theorem 3.1 implies Λ is a (t,n)-Igusa-Todorov algebra. So, $\mathrm{IT.dim}\,\Lambda \leqslant \max\{\mathrm{gl.dim}\,T, \mathrm{IT.dim}\,U\}$. $\max\{\mathrm{IT.dim}\,T, \mathrm{IT.dim}\,U\} \leqslant \mathrm{IT.dim}\,\Lambda$ is obvious. $\Box$

Similarly, we can prove $\mathrm{IT.dim}\,\Lambda \leqslant \max\{\mathrm{gl.dim}\,U, \mathrm{IT.dim}\,T\}$. Thus we have

Theorem 3.3 *Let $\Lambda = \left(\begin{smallmatrix} T & 0 \\ {}_UM_T & U \end{smallmatrix}\right)$ be a triangular matrix Artin R-algebra such that ${}_UM \in \mathrm{add}(U)$, $M_T \in \mathrm{proj}(T)$. Then*

$$\max\{\mathrm{IT.dim}\,T, \mathrm{IT.dim}\,U\} \leqslant \mathrm{IT.dim}\,\Lambda \leqslant \min\{\max\{\mathrm{gl.dim}\,T, \mathrm{IT.dim}\,U\}, \max\{\mathrm{gl.dim}\,U, \mathrm{IT.dim}\,T\}\}.$$

Remark 3.4 Let $\Lambda = \left(\begin{smallmatrix} T & 0 \\ {}_UM_T & U \end{smallmatrix}\right)$ be a triangular matrix Artin R-algebra such that ${}_UM \in \mathrm{add}(U)$ and $M_T \in \mathrm{add}(T)$. By [5], $\mathrm{IT.dim}\,T = \mathrm{IT.dim}\,U = 0$ if and only if $\mathrm{IT.dim}\,\Lambda = 0$. Moreover, $\max\{\mathrm{IT.dim}\,T, \mathrm{IT.dim}\,U\} = 1$ if and only if $\mathrm{IT.dim}\,\Lambda = 1$.

Putting $U = M = T$ in Theorem 3.3, we have the following Corollary.

Corollary 3.5 *Let T be an Artin algebra and $\Lambda = \left(\begin{smallmatrix} T & 0 \\ T & T \end{smallmatrix}\right)$. Then*

$$\mathrm{IT.dim}\, T \leqslant \mathrm{IT.dim}\, \Lambda \leqslant \mathrm{gl.dim}\, T.$$

Corollary 3.6 *Let $\Lambda = \left(\begin{smallmatrix} T & 0 \\ {}_U M_T & U \end{smallmatrix}\right)$ be a triangular matrix Artin R-algebra such that ${}_U M \in \mathrm{add}(U)$ and $M_T \in \mathrm{add}(T)$. If T is semisimple, then*

$$\mathrm{IT.dim}\, \Lambda = \mathrm{IT.dim}\, U.$$

Let k be a field. Given a finite dimensional k-algebra U and a U-module ${}_U M$. Recall that the special matrix algebra $\Lambda = \left(\begin{smallmatrix} k & 0 \\ M & U \end{smallmatrix}\right)$ is said to be the one-point extension of U by M.

Corollary 3.7 *Let U be a finite dimensional k-algebra and Λ be the one-point extension of U by a projective U-module M. Then*

$$\mathrm{IT.dim}\, \Lambda = \mathrm{IT.dim}\, U.$$

Acknowledgements The authors would like to thank Professor Yanhong Bao for helpful discussions. They also thank the referee for the useful and detailed suggestions.

References

[1] Changchang XI. *On the finitistic dimension conjecture (II). Related to finite global dimension.* Adv. Math., 2006, **201**(1): 116–142.

[2] Jiaqun WEI. *Finitistic dimension and Igusa–Todorov algebras.* Adv. Math., 2009, **222**(6): 2215–2226.

[3] T. CONDE. *On certain strongly quasihereditary algebras.* University of Oxford., 2015, PhD Thesis.

[4] Junling ZHENG. *The derived dimensions of (m,n)-Igusa-Iodorov algebras.* J. Algebra, 2022, **612**: 227–247.

[5] D. BRAVO, M. LANZILOTTA, O. MENDOZA. *Pullback diagrams, syzygy finite classes and Igusa-Todorov algebras.* J. Pure Appl. Algebra, 2019, **223**(10): 4494–4508.

[6] Junling ZHENG. *Igusa-Todorov dimensions and derived dimensions of artin algebras.* arxiv:2211.00544.