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A Modified Tikhonov Regularization Method for a Cauchy Problem of the Biharmonic Equation

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Abstract In this paper, the Cauchy problem of biharmonic equation is considered. This problem is ill-posed, i.e., the solution (if exists) does not depend on the measurable data. Firstly, we give the conditional stability result under the a priori bound assumption for the exact solution. Secondly, a modified Tikhonov regularization method is used to solve this ill-posed problem. Under the a priori and the a posteriori regularization parameter choice rule, the error estimates between the regularization solutions and the exact solution are obtained. Finally, some numerical examples are presented to verify that our method is effective.

Keywords Biharmonic equations; inverse problem; Cauchy problem; Tikhonov regularization method

MR(2020) Subject Classification 35R25; 47A52; 35R30

1. Introduction

Biharmonic equation is a kind of elliptic equation, it can describe some basic equations in plane elasticity and reconstruct geometric curves with given boundary conditions [1]. The boundary value problem of biharmonic equation can also be used to model broadband and low frequency radar imaging in [2].

In this paper, we consider the Cauchy problem of biharmonic equation with nonhomogeneous Dirichlet and Neumann boundary conditions:

$$\begin{cases} u_{xxxx}(x,y)+2u_{xxyy}(x,y)+u_{yyyy}(x,y)=0, & (x,y)\in(0,\pi)\times(0,1),\\ u(x,0)=\varphi_1(x), & x\in[0,\pi],\\ u_y(x,0)=\varphi_2(x), & x\in[0,\pi],\\ \Delta u(x,0)=0, & x\in[0,\pi],\\ \Delta u_y(x,0)=0, & x\in[0,\pi],\\ u(0,y)=u(\pi,y)=\Delta u(0,y)=\Delta u_y(\pi,y)=0, & y\in[0,1]. \end{cases} \tag{1.1}$$

The Cauchy problem of biharmonic equation studied in this paper is to find $u(x,y)$ for $y\in(0,1]$ from the initial data

$$u(x,0)=\varphi_1(x),\ x\in[0,\pi],\quad u_y(x,0)=\varphi_2(x),\ x\in[0,\pi].$$

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Assume the exact data $\varphi_1(x)$, $\varphi_2(x)$ and the measurement data $\varphi_1^\delta(x)$, $\varphi_2^\delta(x)$ satisfy

$$\| \varphi_1^\delta(\cdot) - \varphi_1(\cdot) \| \le \delta, \| \varphi_2^\delta(\cdot) - \varphi_2(\cdot) \| \le \delta,$$

where δ denotes the bound of measured error.

Due to the linear property, we can divide (1.1) into two Cauchy problems as follows:

$$\begin{cases} f_{xxxx}(x,y) + 2f_{xxyy}(x,y) + f_{yyyy}(x,y) = 0, & (x,y) \in (0,\pi) \times (0,1), \\ f(x,0) = \varphi_1(x), & x \in [0,\pi], \\ f_y(x,0) = 0, & x \in [0,\pi], \\ \Delta f(x,0) = 0, & x \in [0,\pi], \\ \Delta f_y(x,0) = 0, & x \in [0,\pi], \\ f(0,y) = f(\pi,y) = \Delta f(0,y) = \Delta f_y(\pi,y) = 0, & y \in [0,1], \end{cases} \tag{1.2}$$

and

$$\begin{cases} g_{xxxx}(x,y) + 2g_{xxyy}(x,y) + g_{yyyy}(x,y) = 0, & (x,y) \in (0,\pi) \times (0,1), \\ g(x,0) = 0, & x \in [0,\pi], \\ g_y(x,0) = \varphi_2(x), & x \in [0,\pi], \\ \Delta g(x,0) = 0, & x \in [0,\pi], \\ \Delta g_y(x,0) = 0, & x \in [0,\pi], \\ g(0,y) = g(\pi,y) = \Delta g(0,y) = \Delta g_y(\pi,y) = 0, & y \in [0,1], \end{cases} \tag{1.3}$$

we know that $u = f + g$ is the solution of problem (1.1). Then we only need to consider (1.2) and (1.3), respectively.

In the sense of Hadamard problems, (1.2) and (1.3) are ill-posed, a small measurement error in the Cauchy data can induce an enormous error in the solution [3]. Thus some regularization techniques are required to overcome the ill-posedness and stabilize numerical computations, please see some regularized strategies in [4]. In the past years, the inverse problem of the biharmonic equation has little research. Kalmenov and Iskakova in [5, 6] studied a mixed boundary value problem for the biharmonic equation where boundary conditions are given on the whole boundary of the domain. However, a regularization method has not yet been mentioned in this study. In [7], Luan et al. used a filter regularization method to transform the ill-posed problem into a well-posed problem for the Cauchy problem of the biharmonic equation. In [8], the authors identified the unknown sources of biharmonic equation by using the Landweber regularization method.

In this paper, we study the inverse problem of biharmonic equations with nonhomogeneous Dirichlet and Neumann boundary conditions. We not only give the a priori regularization parameter choice rule, but also give the a posteriori regularization parameter choice rule. Based on the a priori and the a posteriori regularization parameter choice rules, we give both the error estimates within $0 < y < 1$ and the error estimates at $y = 1$. Moreover, we give the optimal error bound analysis. According to the optimal error bound analysis, we find the error estimations are all order optimal.

The paper is organized as follows. In Section 2, we give some preliminary results. In Section 3, we derive the conditional stability of problems under a priori bound condition for the exact solution. In Section 4, the optimal error bounds for problems (1.2) and (1.3) are given. In Section 5, we propose a modified Tikhonov regularization method. In Section 6, we give the error estimates under the a priori and the a posteriori regularization choice rules. Finally, numerical examples are given in Section 7.

2. Preparation knowledge

In this section, we present some important definitions and lemmas. Firstly, we introduce a function

$$H(\eta)=\begin{cases}\eta^{\eta}(1-\eta)^{1-\eta}, & \eta\in(0,1),\\ 1, & \eta=0,1,\end{cases}\tag{2.1}$$

which was defined in [9] (see formula 2.2), we can see that $H(\eta)\le 1$ clearly.

Lemma 2.1 ([9, Lemma 2 in Section 3]) *If $0\le p\le q<\infty$, $q\neq 0$ and $v>0$, then*

$$\frac{ve^{-p}}{v+e^{-q}}\le H(\frac{p}{q})v^{\frac{p}{q}}.\tag{2.2}$$

Lemma 2.2 *For $0<\alpha<1$ and $p>0$, we obtain*

(a) $\frac{e^s}{2}\le \cos h(s)\le e^s$ *for* $s\ge 0$.

(b) As $s>0$, for $T_2(s):=\frac{\cos h(sy)}{1+\alpha\cos h^2(s)}$, there holds $T_2(s)\le 2\alpha^{-\frac{y}{2}}$.

(c) As $s>0$, for $T_3(s):=\frac{\alpha\cos h(s)\cos h(sy)}{1+\alpha\cos h^2(s)}$, there holds $T_3(s)\le 4\alpha^{\frac{1}{2}-\frac{y}{2}}$.

(d) When $s\ge 1$, for $T_4(s):=\frac{\alpha\cos h(s)}{1+\alpha\cos h^2(s)}e^{-sp}$, there holds

$$T_4(s)\le\begin{cases}2^{1-p}\alpha^{\frac{1}{2}+\frac{p}{2}}, & 0<p<1,\\ \alpha, & p\ge 1.\end{cases}$$

(e) When $s\ge 1$, for $T_5(s):=\frac{\alpha\cos h^2(s)}{1+\alpha\cos h^2(s)}e^{-sp}$, there holds

$$T_5(s)\le\begin{cases}2^{2-p}\alpha^{\frac{p}{2}}, & 0<p<1,\\ \alpha^{\frac{1}{2}}, & p\ge 1.\end{cases}$$

Proof (a) is apparent.

(b) Using Lemma 2.1 and (a), we have $T_2(s)\le\frac{e^{sy}}{1+\alpha\frac{e^{2s}}{4}}=\frac{e^{-2s+sy}}{e^{-2s}+\frac{\alpha}{4}}$.

Let $v=\frac{\alpha}{4}$, $p=2s-sy$, $q=2s$ and use the properties of $H(\eta)$, we obtain

$$T_2(s)\le(\frac{\alpha}{4})^{-1}\frac{\frac{\alpha}{4}e^{-(2s-sy)}}{\frac{\alpha}{4}+e^{-2s}}=v^{-1}\frac{ve^{-p}}{v+e^{-q}}\le v^{-1}H(\frac{p}{q})v^{\frac{p}{q}}\le v^{-1+\frac{p}{q}}\le(\frac{\alpha}{4})^{-\frac{y}{2}}\le 2\alpha^{-\frac{y}{2}}.$$

(c) Using Lemma 2.1 and (a), we have

$$T_3(s)\le\alpha\frac{e^{s+sy}}{1+\alpha\frac{e^{2s}}{4}}=\alpha\frac{e^{-(s-sy)}}{\frac{\alpha}{4}+e^{-2s}}\le\alpha(\frac{\alpha}{4})^{-1}\frac{\frac{\alpha}{4}e^{-(s-sy)}}{\frac{\alpha}{4}+e^{-2s}}\le\alpha(\frac{\alpha}{4})^{-\frac{1}{2}-\frac{y}{2}}\le 4\alpha^{\frac{1}{2}-\frac{y}{2}}.$$

(d) When $0<p<1$, applying Lemma 2.1, we obtain

$$T_4(s)\le\alpha\frac{e^{s-sp}}{1+\frac{\alpha}{4}e^{2s}}\le\alpha(\frac{\alpha}{4})^{-\frac{1}{2}+\frac{p}{2}}\le 2^{1-p}\alpha^{\frac{1}{2}+\frac{p}{2}}.$$

When $p\geq 1$, if $s\geq 1$, we obtain $T_4(s)\leq \alpha\cos h(s)e^{-sp}\leq \alpha e^{(1-p)s}<\alpha$.

(e) When $0<p<1$, using Lemma 2.1, we obtain

$$T_5(s)\leq \frac{\alpha e^{2s-sp}}{1+\alpha\frac{e^{2s}}{4}}=\frac{\alpha e^{-sp}}{e^{-2s}+\frac{\alpha}{4}}\leq \alpha(\frac{\alpha}{4})^{-1+\frac{p}{2}}\leq 2^{2-p}\alpha^{\frac{p}{2}}.$$

When $p\geq 1$, if $s\geq 1$, we obtain $T_5(s)\leq \alpha\frac{\cos h^2(s)}{\alpha^{\frac{1}{2}}\cos h(s)}e^{-sp}\leq \alpha^{\frac{1}{2}}e^{(1-p)s}<\alpha^{\frac{1}{2}}$. □

Lemma 2.3 ([10]) *For $0<\beta<1$ and $p>0$, the following inequalities hold:*

(a) $\frac{\sin h(sy)}{s}\leq e^{sy}$, $\frac{\sin h(s)}{s}\leq e^{s}$ *for* $s>0$.

(b) $\frac{\sin h(sy)}{\sin h(s)}\leq e^{(y-1)s}$ *for* $s>0$.

(c) As $s>0$, *for* $T_6(s):=\frac{\frac{\sin h(sy)}{s}}{1+\beta(\frac{\sin h(s)}{s})^2}$, *there holds* $T_6(s)\leq 2^{-y}\beta^{-\frac{y}{2}}$.

(d) As $s>0$, *for* $T_7(s):=\frac{\beta\frac{\sin h(s)}{s}\frac{\sin h(sy)}{s}}{1+\beta(\frac{\sin h(s)}{s})^2}$, *there holds* $T_7(s)\leq 2^{y-1}\beta^{\frac{1}{2}-\frac{y}{2}}$.

(e) When $s\geq 1$, *for* $T_8(s):=\frac{\beta k_2(1)}{1+\beta k_2^2(1)}e^{-sp}$, *and* $k_2(1)=\frac{\sin h(s)}{s}$, *there holds*

$$T_8(s)\leq \begin{cases}\beta^{\frac{1}{2}+\frac{p}{2}}, & 0<p<1,\\ \beta, & p\geq 1.\end{cases}$$

(f) When $s\geq 1$, *for* $T_9(s):=\frac{\beta k_2^2(1)}{1+\beta k_2^2(1)}e^{-sp}$, *there holds*

$$T_9(s)\leq \begin{cases}2^{-p}\beta^{\frac{p}{2}}, & 0<p<1,\\ \beta^{\frac{1}{2}}, & p\geq 1.\end{cases}$$

Proof The proofs of (a)–(d) were detailed in [10, Lemma 2.2], so they are omitted. We now demonstrate the items (e) and (f).

(e) When $0<p<1$, if $s\geq \ln(\frac{1}{\sqrt{\beta}})$, according to (a) $\frac{\sin h(s)}{s}\leq e^{s}$, we obtain

$$T_8(s)\leq \frac{\beta k_2(1)}{\beta^{\frac{1}{2}}k_2(1)}e^{-sp}\leq \beta^{\frac{1}{2}}e^{-sp}\leq \beta^{\frac{1}{2}+\frac{p}{2}}.$$

If $0<s\leq \ln(\frac{1}{\sqrt{\beta}})$, we obtain $T_8(s)\leq \beta k_2(1)e^{-sp}\leq \beta^{\frac{1}{2}+\frac{p}{2}}$. So, when $s>0$, we obtain $T_8(s)\leq \beta^{\frac{1}{2}+\frac{p}{2}}$. When $p\geq 1$, if $s\geq 1$, we obtain $T_8(s)\leq \beta k_2(1)e^{-sp}\leq \beta e^{(1-p)s}\leq \beta$. To sum up, when $s\geq 1$, we obtain

$$T_8(s)\leq \begin{cases}\beta^{\frac{1}{2}+\frac{p}{2}}, & 0<p<1,\\ \beta, & p\geq 1.\end{cases}$$

(f) When $0<p<1$, if $s\geq \ln(\frac{2}{\sqrt{\beta}})$, we obtain $T_9(s)\leq e^{-sp}\leq 2^{-p}\beta^{\frac{p}{2}}$. If $0<s\leq \ln(\frac{2}{\sqrt{\beta}})$, we obtain $T_9(s)\leq \frac{\beta k_2^2(1)}{\beta^{\frac{1}{2}}k_2(1)}e^{-sp}\leq \frac{\beta^{\frac{1}{2}}}{2}e^{(1-p)s}\leq 2^{-p}\beta^{\frac{p}{2}}$. So, when $s>0$, we obtain $T_9(s)\leq 2^{-p}\beta^{\frac{p}{2}}$. When $p\geq 1$, if $s\geq 1$, we obtain

$$T_9(s)\leq \frac{\beta k_2^2(1)}{\beta^{\frac{1}{2}}k_2(1)}e^{-sp}\leq \beta^{\frac{1}{2}}k_2(1)e^{-sp}\leq \beta^{\frac{1}{2}}e^{(1-p)s}\leq \beta^{\frac{1}{2}}.$$

To sum up, when $s\geq 1$, we obtain

$$T_9(s)\leq \begin{cases}2^{-p}\beta^{\frac{p}{2}}, & 0<p<1,\\ \beta^{\frac{1}{2}}, & p\geq 1.\end{cases} \quad \Box$$

3. The solution, ill-posed analysis, and the results of conditional stability

Using the method of variables separation, the solutions of problems (1.2), (1.3) can be formulated

$$f(x,y)=\sum_{n=1}^{\infty}\cos h(ny)\varphi_{1n}X_n(x),\ \ \varphi_{1n}=\langle\varphi_1,X_n\rangle, \tag{3.1}$$

$$g(x,y)=\sum_{n=1}^{\infty}\frac{\sin h(ny)}{n}\varphi_{2n}X_n(x),\ \ \varphi_{2n}=\langle\varphi_2,X_n\rangle, \tag{3.2}$$

where $X_n:=X_n(x)=\sqrt{\frac{\pi}{2}}\sin(nx)$ is the eigenfunction in $L^2(0,\pi)$, and φ_{1n}, φ_{2n} stand for its Fourier coefficient. Two notations $k_1(y)$, $k_2(y)$ are given to simplify the solution.

$$f(x,y)=\sum_{n=1}^{\infty}k_1(y)\varphi_{1n}X_n(x),\ \ g(x,y)=\sum_{n=1}^{\infty}k_2(y)\varphi_{2n}X_n(x).$$

From formula (3.1), as $n\to\infty$, $\cos h(ny)\to\infty$, the small perturbation of $\varphi_1^\delta(x)$ will cause a great change in the source term $f(x,y)$. This means that problem (1.2) is ill-posed. For the formula (3.2), as $n\to\infty$, $\frac{\sin h(ny)}{n}\to\infty$, the small perturbation of $\varphi_2^\delta(x)$ will cause a great change in the source term $g(x,y)$. This means that problems (1.3) is also ill-posed. So the regularization method is required to solve problem (1.2) and (1.3). Below, we give the a priori bound as follows:

$$\max\{\|f(x,1)\|_{L^2(0,\pi)},\ \|g(x,1)\|_{L^2(0,\pi)}\}\le E_1, \tag{3.3}$$

here $\|f(x,1)\|_{L^2(0,\pi)}=(\sum_{n=1}^{\infty}(\cos h(n)\varphi_{1n})^2)^{\frac12}$, $\|g(x,1)\|_{L^2(0,\pi)}=\big(\sum_{n=1}^{\infty}(\frac{\sin h(n)}{n}\varphi_{2n})^2\big)^{\frac12}$.

Theorem 3.1 *If $f(x,y)$ and $g(x,y)$ satisfy the priori bound condition (3.3), then we obtain*

$$\|f(x,y)\|_{L^2(0,\pi)}\le 2^yE_1^y\|\varphi_1\|_{L^2(0,\pi)}^{1-y}, \tag{3.4}$$

$$\|g(x,y)\|_{L^2(0,\pi)}\le\frac{2^y}{(1-e^{-2})^y}E_1^y\|\varphi_2\|_{L^2(0,\pi)}^{1-y}. \tag{3.5}$$

Proof According to the formula (3.1), (3.3) and using the Hölder inequality, we have

$$\begin{aligned}
\|f(x,y)\|_{L^2(0,\pi)}^2&=\Big\|\sum_{n=1}^{\infty}\cos h(ny)\varphi_{1n}X_n(x)\Big\|_{L^2(0,\pi)}^2\\
&=\sum_{n=1}^{\infty}\cos h^2(ny)\varphi_{1n}^2=\sum_{n=1}^{\infty}\cos h^2(ny)\varphi_{1n}^{2y}\varphi_{1n}^{2-2y}\\
&\le\Big(\sum_{n=1}^{\infty}\cos h^{\frac2y}(ny)\varphi_{1n}^2\Big)^y\Big(\sum_{n=1}^{\infty}\varphi_{1n}^2\Big)^{1-y}\\
&\le\sup_{n\ge1}|\frac{\cos h^{\frac2y}(ny)}{\cos h^2(n)}|^y\ \Big(\sum_{n=1}^{\infty}\cos h^2(n)\varphi_{1n}^2\Big)^y\|\varphi_1\|_{L^2(0,\pi)}^{2-2y}\\
&\le\sup_{n\ge1}|\frac{e^{2n}}{\frac{e^{2n}}{4}}|^y\ E_1^{2y}\|\varphi_1\|^{2-2y}
\end{aligned}$$

$$\le 4^y E_1^{2y} \|\varphi_1\|_{L^2(0,\pi)}^{2-2y}.$$

Thus

$$\|f(x,y)\|_{L^2(0,\pi)} \le 2^y E_1^y \|\varphi_1\|_{L^2(0,\pi)}^{1-y}.$$

Proof The proof of $g(x,y)$ is the same as that of $f(x,y)$, so it is omitted. □

Remark 3.2 When $y=1$, the error estimate in Theorem 3.1 is only bounded instead of convergence. In order to obtain the convergent error estimate at $y=1$, a stronger a priori hypothesis must be introduced as follows.

The a priori bound in H^p space of functions $f(x,1)$ $g(x,1)$ is defined as follows:

$$\max\{\|f(x,1)\|_{H^p(0,\pi)},\ \|g(x,1)\|_{H^p(0,\pi)}\} \le E_2, \tag{3.6}$$

here

$$\|f(x,1)\|_{H^p(0,\pi)} = \Big(\sum_{n=1}^{\infty}(e^{np}\cos h(n)\varphi_{1n})^2\Big)^{\frac{1}{2}},$$

$$\|g(x,1)\|_{H^p(0,\pi)} = \Big(\sum_{n=1}^{\infty}(e^{np}\frac{\sin h(n)}{n}\varphi_{2n})^2\Big)^{\frac{1}{2}},$$

$E_1,\ E_2$ are positive constants.

Theorem 3.3 *Let* $p>0$, $f(x,1)$ *and* $g(x,1)$ *satisfy the priori bound condition* (3.6). *Then we obtain*

$$\|f(x,1)\|_{L^2(0,\pi)} \le E_2^{\frac{1}{p+1}} \|\varphi_1\|_{L^2(0,\pi)}^{\frac{p}{p+1}}, \tag{3.7}$$

$$\|g(x,1)\|_{L^2(0,\pi)} \le E_2^{\frac{1}{p+1}} \|\varphi_2\|_{L^2(0,\pi)}^{\frac{p}{p+1}}. \tag{3.8}$$

Proof According to the formula (3.1), (3.6) and using the Hölder inequality, we have

$$\begin{aligned}
\|f(x,1)\|_{L^2(0,\pi)}^2 &= \|\sum_{n=1}^{\infty}\cos h(n)\varphi_{1n}X_n(x)\|_{L^2(0,\pi)}^2 \\
&= \sum_{n=1}^{\infty}\cos h^2(n)\varphi_{1n}^2 = \sum_{n=1}^{\infty}\cos h^2(n)\varphi_{1n}^{\frac{2}{p+1}}\varphi_{1n}^{\frac{2p}{p+1}} \\
&\le \Big(\sum_{n=1}^{\infty}\cos h^{2p+2}(n)\varphi_{1n}^2\Big)^{\frac{1}{p+1}}\Big(\sum_{n=1}^{\infty}\varphi_{1n}^2\Big)^{\frac{p}{p+1}} \\
&\le \Big(\sum_{n=1}^{\infty}e^{2np}\cos h^2(n)\varphi_{1n}^2\Big)^{\frac{1}{p+1}}\|\varphi_1\|_{L^2(0,\pi)}^{\frac{2p}{p+1}} \\
&\le E_2^{\frac{2}{p+1}}\|\varphi_1\|_{L^2(0,\pi)}^{\frac{2p}{p+1}}.
\end{aligned}$$

Thus

$$\|f(x,1)\|_{L^2(0,\pi)} \le E_2^{\frac{1}{p+1}} \|\varphi_1\|_{L^2(0,\pi)}^{\frac{p}{p+1}}.$$

The proof of $g(x,1)$ is the same as that of $f(x,1)$, so it is omitted. □

4. Optimal error bounds

In this section, we will give the optimal error bounds of problems (1.2) and (1.3). Now we first give some preliminary conclusions.

4.1. Preliminary

Consider an ill-posed operator equation [11–15]:

$$Kx = y, \tag{4.1}$$

where $K : X \to Y$ is a linear bounded operator between infinite dimensional Hilbert spaces X and Y with non-closed range in Y. We assume that $y^\delta \in Y$ $(\delta > 0)$ is data with measurement error and satisfies

$$\|y^\delta - y\| \leq \delta, \tag{4.2}$$

any operator $R : Y \to X$ can be considered as a useful method for solving (4.1), and the approximate solution of problem (4.1) is given by Ry^δ.

Let $M \subset X$ be a bounded set. Define the worst case error $\Delta(\delta, R)$ for identifying x with y^δ (see [12–14, 16])

$$\Delta(\delta, R) := \sup\{\|Ry^\delta - x\| \mid x \in M, y^\delta \in Y, \|Kx - y^\delta\| \leqslant \delta\}. \tag{4.3}$$

The best possible error bound (or optimal error bound) is defined as the infimum over all mappings $R : Y \to X$,

$$\omega(\delta) := \inf_R \Delta(\delta, R). \tag{4.4}$$

According to [15], the set $M = M_{\varphi,E}$ is a set of elements which satisfy some source condition:

$$M_{\varphi,E} = \{x \in X \mid x = [\varphi(K^*K)]^{\frac{1}{2}} v, \|v\| \leqslant E\}, \tag{4.5}$$

where the operator function $\varphi(K^*K)$ is well defined spectral representation

$$\varphi(K^*K) = \int_0^a \varphi(\lambda) \mathrm{d}E_\lambda, \tag{4.6}$$

where $\{E_\lambda\}$ is the spectral family of the operator K^*K. There exists a constant a so that $\|K^*K\| \leqslant a$. When $K : L^2(R) \to L^2(R)$ is a multiplication operator, $Kx(s) = r(s)x(s)$, the operator function $\varphi(K^*K)$ has the following form:

$$\varphi(K^*K)x(s) = \varphi(|r(s)|^2)x(s). \tag{4.7}$$

There exists a method R_0 which is called [11, 17]

(i) Optimal on the set $M_{\varphi,E}$ if $\Delta(\delta, R) = \omega(\delta, E)$.

(ii) Order optimal on the set $M_{\varphi,E}$ if $\Delta(\delta, R) \leqslant C\omega(\delta, E)$ with $C \geqslant 1$.

Through the assumption in [11, 17], we can derive an explicit (best possible) optimal error bound for the worst case error $\Delta(\delta, R)$ defined in (4.3).

Assumption 4.1 ([11,14,18]) In the formula (4.7), function $\varphi(\lambda) : (0, a] \to (0, \infty)$ is a continuous function, then it has the following properties:

- $\lim_{\lambda\to 0}\varphi(\lambda)=0$.
- $\varphi(\lambda)$ is strictly monotonically increasing on $(0,a]$.
- $\rho(\lambda)=\lambda\varphi^{-1}(\lambda):(0,\varphi(a)]\to(0,a\varphi(a)]$ is convex.

Based on the above assumptions, the next theorem provides us a general formula for the optimal error bound.

Theorem 4.2 ([11,14,16,18]) *Let $M_{\varphi,E}$ be given by formula (4.5). Assumption 4.1 holds and $\frac{\delta^2}{E^2}\in\sigma(K^*K\varphi(K^*K))$, where $\sigma(K^*K)$ represents the spectrum of operator K^*K, then there is*

$$\omega(\delta,M_{\varphi,E})=E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})}. \tag{4.8}$$

We can obtain the optimal error bound from Theorem 4.2. That is a good conclusion, but there also exist two difficulties. One difficulty is that it is hard to check the convexity of ρ, and sometimes it is violated. Another difficulty is that even for very small δ, $\frac{\delta^2}{E^2}$ may not belong to $\sigma(K^*K\varphi(K^*K))$; for example, K is a compact operator. In the next, we present two lemmas to solve the first and the second problems.

Lemma 4.3 ([19]) *If ρ is not necessarily convex, we obtain*

- $E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})}\leqslant\omega(\delta,M_{\varphi,E})\leqslant\sqrt{2}E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})}$ *for* $\frac{\delta^2}{E^2}\in\sigma(K^*K\varphi(K^*K))$.
- $\omega(\delta,M_{\varphi,E})\leqslant\sqrt{2}E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})}$ *for* $\frac{\delta^2}{E^2}\notin\sigma(K^*K\varphi(K^*K))$.

Lemma 4.4 ([19]) *Let K^*K be compact and let $\lambda_1>\lambda_2>\cdots$ be the ordered eigenvalued of K^*K. If there exists a constant $k>0$ such that $\varphi(\lambda_{i+1})\geqslant k\varphi(\lambda_i)$ for all $i\in N$, then*

$$\omega(\delta,M_{\varphi,E})\geqslant\sqrt{k}E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})}$$

for $\delta\in(0,\delta_1]$, where $\delta_1=E\sqrt{\lambda_1\varphi(\lambda_1)}$.

4.2. Optimal error bound for problems (1.2) and (1.3)

In this part, we will present the optimal error bound for problems (1.2) and (1.3). Now, we analyse the optimal error bound for problem (1.2) first. The noise datum $\varphi_1^\delta(x)\in L^2(0,\pi)$ is processed to identify the best possible worst-case error given by formula (4.4) of $f(x,y)$, where $f(x,y)\in M_{p,E}$, $M_{p,E}$ is defined as follows:

$$f(x,y)\in M_{p,E}=\{f(x,y)\in L^2(0,\pi)\,|\,\|f(x,1)\|_{H^p}\leq E_i,\ p\geq 0,\ i=1,2\}, \tag{4.9}$$

when $p=0$, $\|f(x,1)\|_{H^p}$ is L^2-norm, and $\|f(x,1)\|\leq E_1$. When $p\neq 0$, $\|f(x,1)\|_{H^p}$ is Hilbert-norm, thus $\|f(x,1)\|_{H^p}\leq E_2$.

Rewrite Eq. (2.1) as an operator equation:

$$K_1f(x,y)=\varphi_1(x), \tag{4.10}$$

where K_1 is a multiplication operator with parametric variable y and its singular value is as follows:

$$K_{1n}=\frac{1}{\cos h(ny)},\quad K_{1n}^*K_{1n}=\frac{1}{\cos h^2(ny)}. \tag{4.11}$$

Now let us reformulate condition (4.9) into an equivalent one of form (4.5) with a special function $\varphi = \varphi(\lambda)$.

Propositon 4.5 *Consider the operator Eq. (4.10). Then the set $M_{p,E}$ given in (4.9) is equivalent to the general source set $M_{\varphi,E}$ given in (4.5) provided $\varphi = \varphi(\lambda)$ is given (in parameter representation) by*

$$\begin{cases} \lambda(n) = \frac{1}{\cos h^2(ny)}, \\ \varphi(n) = e^{-2np}\frac{\cos h^2(ny)}{\cos h^2(n)}. \end{cases} \tag{4.12}$$

Proof Due to $K_1 f(x,y) = \varphi_1(x)$ for $0 < y \le 1$, we have

$$\varphi_1(x) = \frac{(f(x,y), X_n)}{\cos h(ny)} = \frac{(f(x,1), X_n)}{\cos h(n)}$$

which gives

$$f(x,1) = \frac{\cos h(n)}{\cos h(ny)} f(x,y).$$

Thus, the inequality $\| f(x,1) \|_{H^p} \le E_i$ is equivalent to the inequality

$$\| e^{np}\frac{\cos h(n)}{\cos h(ny)} f(x,y) \| \le E_i,$$

which shows us that the operator function $\varphi(K^*K)$ has the representation

$$\varphi(K^*K) = e^{-2np}\frac{\cos h^2(ny)}{\cos h^2(n)}. \tag{4.13}$$

Together with (4.11), the proposition is proved. □

Proposition 4.6 *The function $\varphi(\lambda)$ defined by (4.12) is continuous and has the following properties:*

Case 1. $p = 0,\ 0 < y < 1$.

(I) $\lim_{\lambda\to 0}\varphi(\lambda) = 0$.

(II) $\varphi(\lambda)$ *is strictly monotonically increasing.*

(III) $\rho(\lambda) = \lambda\varphi^{-1}(\lambda)$ *is strictly monotonic and has the following parameter form:*

$$\begin{cases} \lambda(n) = \frac{\cos h^2(ny)}{\cos h^2(n)}, \\ \rho(n) = \frac{1}{\cos h^2(n)}, \end{cases} \quad 1 \leqslant n < \infty. \tag{4.14}$$

(IV) $\rho^{-1}(\lambda)$ *is strictly monotonically increasing and is represented by the following parameter forms:*

$$\begin{cases} \lambda(n) = \frac{1}{\cos h^2(n)}, \\ \rho^{-1}(n) = \frac{\cos h^2(ny)}{\cos h^2(n)}, \end{cases} \quad 1 \leqslant n < \infty. \tag{4.15}$$

(V) *For the inverse function $\rho^{-1}(\lambda)$, there is*

$$\rho^{-1}(\lambda) = (\frac{\lambda}{4})^{1-y}\big(1 + O(1)\big) \quad \text{for } \lambda \to \infty. \tag{4.16}$$

Case 2. $p > 0$, $y = 1$.

(I) $\lim_{\lambda\to 0}\varphi(\lambda) = 0$.

(II) $\varphi(\lambda)$ *is strictly monotonically increasing.*

(III) $\rho(\lambda) = \lambda\varphi^{-1}(\lambda)$ *is strictly monotonic and has the following parameter form:*

$$\begin{cases} \lambda(n) = e^{-2np}\frac{\cos h^2(ny)}{\cos h^2(n)}, \\ \rho(n) = e^{-2np}\frac{1}{\cos h^2(n)}, \end{cases} \quad 1 \leqslant n < \infty. \tag{4.17}$$

(IV) $\rho^{-1}(\lambda)$ *is strictly monotonically increasing and is represented by the following parameter forms:*

$$\begin{cases} \lambda(n) = e^{-2np}\frac{1}{\cos h^2(n)}, \\ \rho^{-1}(n) = e^{-2np}\frac{\cos h^2(ny)}{\cos h^2(n)}, \end{cases} \quad 1 \leqslant n < \infty. \tag{4.18}$$

(V) For the inverse function $\rho^{-1}(\lambda)$*, there is*

$$\rho^{-1}(\lambda) = (\frac{\lambda}{4})^{\frac{p}{p+1}}(\ln\frac{1}{\sqrt{\lambda}})^{-2p}(1+O(1)) \quad \text{for } \lambda \to \infty. \tag{4.19}$$

Proof For the case 1, we will prove (I) first.

(I) From (4.12), we can see $\lambda(n) = \frac{1}{\cos h^2(ny)}$. When $\lambda \to 0$, that means $n \to \infty$. Therefore,

$$\lim_{\lambda\to 0}\varphi(\lambda) = \lim_{n\to\infty}\frac{\cos h^2(ny)}{\cos h^2(n)} = 0, \quad \text{as } 0 < y < 1.$$

The proof of (II), (III) and (IV) is simple, so we omit the proof.

(V) We only need to prove that $\lim_{\lambda\to 0} F_1(\lambda) = 1$, where

$$F_1(\lambda) := \rho^{-1}(\lambda)/(\frac{\lambda}{4})^{1-y}.$$

According to [17], using (4.15) and noting $\lambda(n)$ is strictly monotonically decreasing with $\lim_{n\to\infty}\lambda(n) = 0$, we have

$$\lim_{\lambda\to 0} F_1(\lambda) = \lim_{n\to\infty}\frac{\cos h^2(ny)}{\cos h^2(n)}[4\cos h^2(n)]^{1-y} = 1.$$

The proof of Case 2 is similar to Case 1, so it is omitted. □

Theorem 4.7 *Assume condition* (4.9) *holds. Then the optimal error bound of the inverse problem* (1.2) *is as follows:*

(i) For $p = 0$ *and* $0 < y < 1$*, we have*

$$\omega(\delta, E) = E^y(\frac{\delta}{2})^{1-y}(1+O(1)). \tag{4.20}$$

(ii) For $p > 0$ *and* $y = 1$*, we have*

$$\omega(\delta, E) = E^{\frac{1}{p+1}}(\frac{\delta}{2})^{\frac{p}{p+1}}\ln(\frac{E}{\delta})^{-2p}. \tag{4.21}$$

Proof Combining formula (4.8), (4.16) with (4.19), for (i), we obtain

$$\omega(\delta, E) = E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})} = E\sqrt{(\frac{\delta^2}{4E^2})^{1-y}} = E^y(\frac{\delta}{2})^{1-y}. \tag{4.22}$$

For (ii), we have

$$\omega(\delta, E) = E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})} = E\sqrt{(\frac{\delta^2}{4E^2})^{\frac{p}{p+1}}\ln(\frac{E}{\delta})^{-2p}} = E^{\frac{1}{p+1}}(\frac{\delta}{2})^{\frac{p}{p+1}}\ln(\frac{E}{\delta})^{-2p}. \quad \square \tag{4.23}$$

Now, we analyse the optimal error bound for problem (1.3). The noise data $\varphi_2^\delta(x) \in L^2(0,\pi)$ is also processed to identify the best possible worst-case error given by formula (4.4) of $g(x,y)$, where $g(x,y) \in M_{p,E}$, $M_{p,E}$ is defined as follows:

$$g(x,y) \in M_{p,E} = \{g(x,y) \in L^2(0,\pi) \mid \| g(x,1) \|_{H^p} \leq E_i, \quad p \geq 0, \ i = 3,4\}, \tag{4.24}$$

when $p = 0$, $\|g(x,1)\|_{H^p}$ is L^2-norm, and $\|g(x,1)\| \leq E_3$. When $p \neq 0$, $\|g(x,1)\|_{H^p}$ is Hilbert-norm, thus $\|g(x,1)\|_{H^p} \leq E_4$. Rewrite Eq. (2.2) as an operator equation:

$$K_2 g(x,y) = \varphi_2(x), \tag{4.25}$$

where K_2 is a multiplication operator with parametric variable y and its singular value is as follows:

$$K_{2n} = \frac{n}{\sin h(ny)}, \quad K_{2n}^* K_{2n} = \frac{n^2}{\sin h^2(ny)}. \tag{4.26}$$

Next up, let us reformulate condition (4.24) into an equivalent one of form (4.5) with a special function $\varphi = \varphi(\lambda)$.

Proposition 4.8 *Consider the operator Eq. (4.25). Then the set $M_{p,E}$ given in (4.24) is equivalent to the general source set $M_{\varphi,E}$ given in (4.5) provided $\varphi = \varphi(\lambda)$ is given (in parameter representation) by*

$$\begin{cases} \lambda(n) = \frac{1}{\sin h^2(ny)}, \\ \varphi(n) = e^{-2np} \frac{\sin h^2(ny)}{\sin h^2(n)}. \end{cases} \tag{4.27}$$

Proof This proof is the same as Proposition 4.5 and the proof is omitted. □

Proposition 4.9 *The function $\varphi(\lambda)$ defined by (4.27) is continuous and has the following properties:*

Case 1. $p = 0$, $0 < y < 1$.

(I) $\lim_{\lambda \to 0} \varphi(\lambda) = 0$.

(II) $\varphi(\lambda)$ is strictly monotonically increasing.

(III) $\rho(\lambda) = \lambda \varphi^{-1}(\lambda)$ is strictly monotonic and has the following parameter form:

$$\begin{cases} \lambda(n) = \frac{\sin h^2(ny)}{\sin h^2(n)}, \\ \rho(n) = \frac{n^2}{\sin h^2(n)}, \end{cases} \quad 1 \leqslant n < \infty. \tag{4.28}$$

(IV) $\rho^{-1}(\lambda)$ is strictly monotonically increasing and is represented by the following parameter forms:

$$\begin{cases} \lambda(n) = \frac{n^2}{\sin h^2(n)}, \\ \rho^{-1}(n) = \frac{\sin h^2(ny)}{\sin h^2(n)}, \end{cases} \quad 1 \leqslant n < \infty. \tag{4.29}$$

(V) For the inverse function $\rho^{-1}(\lambda)$, there is

$$\rho^{-1}(\lambda) = \lambda^{1-y} (\ln \frac{1}{\sqrt{\lambda}})^{2(y-1)} (1 + O(1)) \quad \text{for } \lambda \to \infty. \tag{4.30}$$

Case 2. $p > 0$, $y = 1$.

(I) $\lim_{\lambda \to 0} \varphi(\lambda) = 0$.

(II) $\varphi(\lambda)$ *is strictly monotonically increasing.*

(III) $\rho(\lambda) = \lambda\varphi^{-1}(\lambda)$ *is strictly monotonic and has the following parameter form:*

$$\begin{cases} \lambda(n) = e^{-2np\frac{\sin h^2(ny)}{\sin h^2(n)}}, \\ \rho(n) = e^{-2np\frac{n^2}{\sin h^2(n)}}, \end{cases} \quad 1 \leqslant n < \infty. \tag{4.31}$$

(IV) $\rho^{-1}(\lambda)$ *is strictly monotonically increasing and is represented by the following parameter forms:*

$$\begin{cases} \lambda(n) = e^{-2np\frac{n^2}{\sin h^2(n)}}, \\ \rho^{-1}(n) = e^{-2np\frac{\sin h^2(ny)}{\sin h^2(n)}}, \end{cases} \quad 1 \leqslant n < \infty. \tag{4.32}$$

(V) For the inverse function $\rho^{-1}(\lambda)$*, there is*

$$\rho^{-1}(\lambda) = \lambda^{\frac{p}{p+1}}(\ln\frac{1}{\sqrt{\lambda}})^{\frac{2}{p+1}}(1+O(1)) \quad \text{for } \lambda \to \infty. \tag{4.33}$$

Proof The proofs of (I)–(IV) are obvious, we only give the proof of (V). We only need to prove that $\lim_{\lambda\to 0} F_3(\lambda) = 1$, where

$$F_3(\lambda) := \rho^{-1}(\lambda)/\lambda^{1-y}(\ln\frac{1}{\sqrt{\lambda}})^{2(y-1)}.$$

Using (4.29) and noting $\lambda(n)$ is strictly monotonically decreasing with $\lim_{n\to\infty}\lambda(n) = 0$, we have

$$\begin{aligned} \lim_{\lambda\to 0} F_3(\lambda) &= \lim_{n\to\infty} \frac{\sin h^2(ny)}{\sin h^2(n)} \Big/ (\frac{n^2}{\sin h^2(n)})^{1-y}(\ln\frac{1}{\sqrt{\frac{n^2}{\sin^2(n)}}})^{2(y-1)} \\ &= \lim_{n\to\infty} \frac{\sin h^2(ny)\sin h^{2-2y}(n)}{\sin h^2(n)} \Big/ n^{2-2y}(\ln\frac{\sin h(n)}{n})^{2(y-1)} \\ &= 1. \end{aligned}$$

The proof of Case 2 is similar to Case 1, so it is omitted. □

Theorem 4.10 *Assume condition (4.9) holds. Then the optimal error bound of the inverse problem (1.3) is as follows:*

(i) For $p = 0$ *and* $0 < y < 1$*, we have*

$$\omega(\delta, E) = E^y\delta^{1-y}(\ln\frac{\delta}{E})^{y-1}(1+O(1)). \tag{4.34}$$

(ii) For $p > 0$ *and* $y = 1$*, we have*

$$\omega(\delta, E) = E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}(\ln\frac{E}{\delta})^{\frac{1}{p+1}}(1+O(1)). \tag{4.35}$$

Proof Combining formula (4.8), (4.30) with (4.33), for (i), we obtain

$$\omega(\delta, E) = E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})} = E\sqrt{(\frac{\delta^2}{E^2})^{1-y}(\ln\frac{1}{\sqrt{\frac{\delta^2}{E^2}}})^{2(y-1)}} = E^y\delta^{1-y}(\ln\frac{E}{\delta})^{y-1}.$$

For (ii), we have

$$\omega(\delta, E) = E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})} = E\sqrt{(\frac{\delta^2}{E^2})^{\frac{p}{p+1}}(\ln\frac{1}{\sqrt{\frac{\delta^2}{E^2}}})^{\frac{2}{p+1}}} = E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}(\ln\frac{E}{\delta})^{\frac{1}{p+1}}. \quad \square$$

5. The regularization method

From the formula of the solutions, we can see that $\cos h(ny)$, $\frac{\sin h(ny)}{n}$ are unbounded as $n \to \infty$, so problems (1.2) and (1.3) are ill-posed. If we want to restore the stability of solutions, we need to use the regularization method. In this section, we use the modified Tikhonov regularization method to obtain regularization solutions for (1.2) and (1.3).

Define an operator $K_1(\cdot) : L^2(0, \pi) \to L^2(0, \pi)$ for $0 < y \le 1$, so problem (1.2) can be formulated as the following operator equation:

$$K_1(y) f(x, y) = \varphi_1(x), \quad 0 < y < 1, \tag{5.1}$$

$$K_1(1) f(x, 1) = \varphi_1(x), \quad y = 1. \tag{5.2}$$

Define $f^\delta(x, 0) = \varphi_1^\delta$, and we seek Tikhonov regularization solutions $f^\delta_{\alpha_1}(x, y)$ and $f^\delta_{\alpha_2}(x, 1)$ by solving the minimization problems,

$$\min_{f \in L^2(0,\pi)} J_{\alpha_1}(f), \ J_{\alpha_1}(f) := \| K_1(y) f(x, y) - \varphi_1^\delta \|^2 + \alpha_1 \| f(x, y) \|^2, \tag{5.3}$$

$$\min_{f \in L^2(0,\pi)} J_{\alpha_2}(f), \ J_{\alpha_2}(f) := \| K_1(1) f(x, 1) - \varphi_1^\delta \|^2 + \alpha_2 \| f(x, 1) \|^2 . \tag{5.4}$$

Hence, $f^\delta_{\alpha_1}(x, y)$, $f^\delta_{\alpha_2}(x, 1)$ are the solutions of Euler equations respectively

$$(\frac{1}{k_1^2(y)} + \alpha_1) f^\delta_{\alpha_1}(x, y) = \frac{1}{k_1(y)} \varphi_1^\delta(x), \tag{5.5}$$

$$(\frac{1}{k_1^2(1)} + \alpha_2) f^\delta_{\alpha_2}(x, 1) = \frac{1}{k_1(1)} \varphi_1^\delta(x). \tag{5.6}$$

From (5.5) and (5.6), we can derive that

$$f^\delta_{\alpha_1}(x, y) = \sum_{n=1}^{\infty} \frac{k_1(y) \varphi^\delta_{1n} X_n(x)}{1 + \alpha_1 k_1^2(y)}, \tag{5.7}$$

$$f^\delta_{\alpha_2}(x, 1) = \sum_{n=1}^{\infty} \frac{k_1(1) \varphi^\delta_{1n} X_n(x)}{1 + \alpha_2 k_1^2(1)}, \tag{5.8}$$

where $\varphi^\delta_{1n} = \langle \varphi_1^\delta, X_n \rangle$, the error data $\varphi_1^\delta(x)$ satisfies

$$\| \varphi_1^\delta(\cdot) - \varphi_1(\cdot) \| \le \delta, \tag{5.9}$$

δ denotes the bound of measured error, α is the regularization parameter.

In this paper, we replace the kernel $\frac{k_1(y)}{1+\alpha_1 k_1^2(y)}$ by the modified kernel $\frac{k_1(y)}{1+\alpha_1 k_1^2(1)}$ and obtain a modified regularization solution

$$f^\delta_{1,\alpha_1}(x, y) = \sum_{n=1}^{\infty} \frac{k_1(y) \varphi^\delta_{1n} X_n(x)}{1 + \alpha_1 k_1^2(1)}. \tag{5.10}$$

For the endpoint, let $f^\delta_{\alpha_2}(x, 1) = f^\delta_{2,\alpha_2}(x, 1)$, we can obtain

$$f^\delta_{2,\alpha_2}(x, 1) = \sum_{n=1}^{\infty} \frac{k_1(1) \varphi^\delta_{1n} X_n(x)}{1 + \alpha_2 k_1^2(1)}. \tag{5.11}$$

Similarly, we define an operator $K_2(\cdot): L^2(0,\pi) \to L^2(0,\pi)$ for $0 < y \le 1$, so problem (1.3) can be formulated as the following operator equation:

$$K_2(y)g(x,y) = \varphi_2(x), \quad 0 < y < 1,$$

$$K_2(1)g(x,1) = \varphi_2(x), \quad y = 1.$$

Using the same method as above, we can derive the regular solution of $g(x,y)$ at interval $0 < y < 1$ and endpoint $y = 1$:

$$g^\delta_{1,\beta_1}(x,y) = \sum_{n=1}^{\infty} \frac{k_2(y)\varphi^\delta_{2n} X_n(x)}{1+\beta_1 k_2^2(1)}, \quad g^\delta_{2,\beta_2}(x,1) = \sum_{n=1}^{\infty} \frac{k_2(1)\varphi^\delta_{2n} X_n(x)}{1+\beta_1 k_2^2(1)}, \tag{5.12}$$

and the error data $\varphi^\delta_2(x)$ satisfies

$$\|\varphi^\delta_2(\cdot) - \varphi_2(\cdot)\| \le \delta. \tag{5.13}$$

6. The error estimation

In this section, under the a priori and a posteriori rules, we are going to present the convergence error estimations for problems (1.2) and (1.3). Since the derivation of the convergence error estimate of problem (1.3) is more difficult than that of problem (1.2), in the following parts, we focus on the derivation process of problem (1.3).

6.1. The priori convergence error estimation in interval $0 < y < 1$

In this section, under the priori regularization parameter choice rule, we first give the priori error estimation between the regularization solution and the exact solution.

Theorem 6.1 *$g(x,y)$ is the exact solution of problem (1.3). The regularization solution $g^\delta_{1,\beta_1}(x,y)$ is given by (5.12) and the measured data $\varphi^\delta_2(x)$ satisfies (5.13). When $0 < y < 1$, if the priori bound condition (3.3) holds, and the regularization parameter β_1 is selected as*

$$\beta_1 = (\frac{\delta}{E_1})^2, \tag{6.1}$$

we have the error estimate

$$\| g^\delta_{1,\beta_1}(x,y) - g(x,y) \| \le C_1 E_1^y \delta^{1-y}, \tag{6.2}$$

where $C_1 := 2^{-y} + 2^{y-1}$.

Proof Using the triangle inequality, we have

$$\| g^\delta_{1,\beta_1}(x,y) - g(x,y) \| \le \| g^\delta_{1,\beta_1}(x,y) - g_{1,\beta_1}(x,y) \| + \| g_{1,\beta_1}(x,y) - g(x,y) \|, \tag{6.3}$$

where $g_{1,\beta_1}(x,y)$ is the regularization solution with no error. From Lemma 2.3 (c), (5.12), and (5.13), we have

$$\| g^\delta_{1,\beta_1}(x,y) - g_{1,\beta_1}(x,y) \| = \Big\| \sum_{n=1}^{\infty} \frac{k_2(y)}{1+\beta_1 k_2^2(1)} (\varphi^\delta_{2n} - \varphi_{2n}) X_n(x) \Big\|$$

$$= \Big(\sum_{n=1}^{\infty}(\frac{k_2(y)}{1+\beta_1 k_2^2(1)})^2(\varphi_{2n}^{\delta}-\varphi_{2n})^2\Big)^{\frac{1}{2}}$$

$$\le \sup_{n\ge 1} \mid \frac{k_2(y)}{1+\beta_1 k_2^2(1)} \mid \Big(\sum_{n=1}^{\infty}(\varphi_{2n}^{\delta}-\varphi_{2n})^2\Big)^{\frac{1}{2}} \le 2^{-y}\beta_1^{-\frac{y}{2}}\delta.$$

So

$$\parallel g_{1,\beta_1}^{\delta}(x,y) - g_{1,\beta_1}(x,y) \parallel \le 2^{-y}\beta_1^{-\frac{y}{2}}\delta. \tag{6.4}$$

By (3.2), (3.3), (5.12), and Lemma 2.3 (d), we have

$$\parallel g_{1,\beta_1}(x,y) - g(x,y) \parallel = \Big\| \sum_{n=1}^{\infty} \frac{\beta_1 k_2^2(1) k_2(y)}{1+\beta_1 k_2^2(1)}\varphi_{2n}X_n(x) \Big\|$$

$$= \Big(\sum_{n=1}^{\infty}\Big(\frac{\beta_1 k_2^2(1)k_2(y)}{1+\beta_1 k_2^2(1)}\Big)^2\varphi_{2n}^2\Big)^{\frac{1}{2}}$$

$$\le \sup_{n\ge 1} \mid \frac{\beta_1 k_2(1) k_2(y)}{1+\beta_1 k_2^2(1)} \mid \Big(\sum_{n=1}^{\infty} k_2^2(1)\varphi_{2n}^2\Big)^{\frac{1}{2}}$$

$$\le \sup_{n\ge 1} \mid \frac{\beta_1 k_2(1) k_2(y)}{1+\beta_1 k_2^2(1)} \mid \cdot E_1 \le 2^{y-1}\beta_1^{\frac{1}{2}-\frac{y}{2}}E_1.$$

Thus

$$\parallel g_{1,\beta_1}(x,y) - g(x,y) \parallel \le 2^{y-1}\beta_1^{\frac{1}{2}-\frac{y}{2}}E_1. \tag{6.5}$$

Combining (6.3), (6.4) with (6.5), if the regularization parameter $\beta_1 = (\frac{\delta}{E_1})^2$ is selected, then

$$\parallel g_{1,\beta_1}^{\delta}(x,y) - g(x,y) \parallel \le C_1\delta^{1-y}E_1^y, \tag{6.6}$$

where $C_1 := 2^{-y} + 2^{y-1}$. □

Theorem 6.2 *$f(x,y)$ is the exact solution of problem (1.2). The regularization solution $f_{1,\alpha_1}^{\delta}(x,y)$ is given by (5.10), the measured data $\varphi_1^{\delta}(x)$ satisfies (5.9). When $0 < y < 1$, if the priori bound condition (3.3) holds, and the regularization parameter α_1 is selected as*

$$\alpha_1 = (\frac{\delta}{E_1})^2, \tag{6.7}$$

we have the error estimate

$$\parallel f_{1,\alpha_1}^{\delta}(x,y) - f(x,y) \parallel \le 6\delta^{1-y}E_1^y. \tag{6.8}$$

Proof The proof of Theorem 6.2 is similar to Theorem 6.1, so it is omitted. □

Remark 6.3 From Theorems 6.1, 6.2, 4.7 and 4.10, we can deduce that the error estimate obtained by the priori regularization parameter choice rule is order optimal for $0 < y < 1$.

6.2. The posteriori convergence error estimation in interval $0 < y < 1$

The priori parameter choice is based on the priori bound E_1 of the exact solution. However, in practice the priori bound E_1 generally can not be known easily. In this condition, we choose the regularization parameter by adopting the posteriori rule. We consider an a posteriori regularization choice rule which is called Morozov's discrepancy principle. When $0 < y < 1$, we select the regularization parameter β_1 by the following equation

$$\|K_2(y)g^{\delta}_{1,\beta_1}(x,y) - \varphi^{\delta}_2(x)\| = \tau\delta, \tag{6.9}$$

where $K_2(y) = \frac{1}{k_2(y)}$, $\tau > 1$ is a positive constant, and $\| \varphi^{\delta}_2(x) \| \geq \tau\delta$.

Lemma 6.4 *Let $\varrho(\beta_1) = \| K_2(y)g^{\delta}_{1,\beta_1}(x,y) - \varphi^{\delta}_2(x) \|$. If $\| \varphi^{\delta}_2(x) \| \geq \tau\delta$, we have*

(a) $\varrho(\beta_1)$ is a continuous function;

(b) $\lim_{\beta_1\to 0} \varrho(\beta_1) = 0$;

(c) $\lim_{\beta_1\to\infty} \varrho(\beta_1) = \| \varphi^{\delta}_2 \|$;

(d) For $\beta_1 \in (0,\infty)$, $\varrho(\beta_1)$ is a strictly increasing function.

Proof Lemma 6.4 can be easily proven with expression

$$\varrho(\beta_1) = \Big(\sum_{n=1}^{\infty}\big(\frac{\beta_1 k_2^2(1)}{1+\beta_1 k_2^2(1)}\big)^2(\varphi^{\delta}_{2n})^2\Big)^{\frac{1}{2}}. \ \square \tag{6.10}$$

Lemma 6.4 indicates that there exists a unique solution for (6.9).

Lemma 6.5 *For fixed $\tau > 1$, let the regularization parameter β_1 satisfy (6.9) and $g(x,y)$ satisfy (3.2). Then we obtain $\beta_1^{-1} \leq (\frac{E_1}{(\tau-1)\delta})^2$.*

Proof According to (6.9) and basic inequality, we have

$$\begin{aligned}
\tau\delta &= \Big\|\sum_{n=1}^{\infty}\frac{\beta_1 k_2^2(1)}{1+\beta_1 k_2^2(1)}\varphi^{\delta}_{2n}X_n(x)\Big\| \\
&= \Big\|\sum_{n=1}^{\infty}\frac{\beta_1 k_2^2(1)}{1+\beta_1 k_2^2(1)}(\varphi^{\delta}_{2n}-\varphi_{2n})X_n(x) + \sum_{n=1}^{\infty}\frac{\beta_1 k_2^2(1)}{1+\beta_1 k_2^2(1)}\varphi_{2n}X_n(x)\Big\| \\
&\leq \Big\|\sum_{n=1}^{\infty}\frac{\beta_1 k_2^2(1)}{1+\beta_1 k_2^2(1)}(\varphi^{\delta}_{2n}-\varphi_{2n})X_n(x)\Big\| + \Big\|\sum_{n=1}^{\infty}\frac{\beta_1 k_2^2(1)}{1+\beta_1 k_2^2(1)}\varphi_{2n}X_n(x)\Big\| \\
&\leq \delta + \Big\|\sum_{n=1}^{\infty}\frac{\beta_1 k_2^2(1)}{1+\beta_1 k_2^2(1)}\varphi_{2n}X_n(x)\Big\|
\end{aligned}$$

and

$$\begin{aligned}
&\Big\|\sum_{n=1}^{\infty}\frac{\beta_1 k_2^2(1)}{1+\beta_1 k_2^2(1)}\varphi_{2n}X_n(x)\Big\| = \Big(\sum_{n=1}^{\infty}\big(\frac{\beta_1 k_2^2(1)}{1+\beta_1 k_2^2(1)}\big)^2\varphi^2_{2n}\Big)^{\frac{1}{2}} \\
&\leq \sup_{n\geq 1}\Big|\frac{\beta_1 k_2(1)}{1+\beta_1 k_2^2(1)}\Big|\Big(\sum_{n=1}^{\infty}k_2^2(1)\varphi^2_{2n}\Big)^{\frac{1}{2}} \\
&\leq \sup_{n\geq 1}\Big|\frac{\beta_1 k_2(1)}{1+\beta_1 k_2^2(1)}\Big| E_1 \leq \beta_1^{\frac{1}{2}}E_1.
\end{aligned}$$

According to the proofs above, we obtain that $(\tau-1)\delta \leq \sqrt{\beta_1}E_1$. Then Lemma 6.5 is proved. $\square$

Theorem 6.6 *$g(x,y)$ is the exact solution of problem (1.3). The regularization solution*

$g^{\delta}_{1,\beta_1}(x,y)$ is given by (5.12), the measured data $\varphi^{\delta}_2(x)$ satisfies (5.13). When $0 < y < 1$, if the priori condition (3.3) holds, and the regularization parameter β_1 is selected by (6.9), we have the error estimate

$$\| g^{\delta}_{1,\beta_1}(x,y) - g(x,y) \| \leq C_2\delta^{1-y}E_1^y, \tag{6.11}$$

where $C_2 := 2^{-y}(\frac{1}{(\tau-1)})^y + 2^y(1-e^{-2})^{-y}(\tau+1)^{1-y}$.

Proof Using the triangle inequality, we obtain

$$\| g^{\delta}_{1,\beta_1}(x,y) - g(x,y) \| \leq \| g^{\delta}_{1,\beta_1}(x,y) - g_{1,\beta_1}(x,y) \| + \| g_{1,\beta_1}(x,y) - g(x,y) \| . \tag{6.12}$$

By (6.4)) and Lemma 6.5, we have

$$\| g^{\delta}_{1,\beta_1}(x,y) - g_{1,\beta_1}(x,y) \| \leq 2^{-y}\beta_1^{-\frac{y}{2}}\delta \leq 2^{-y}(\frac{1}{\tau-1})^y\delta^{1-y}E_1^y. \tag{6.13}$$

According to the priori bound condition (3.3), we have

$$\| g_{1,\beta_1}(x,y) - g(x,y) \|_{L^2(0,\pi)} = \| \sum_{n=}^{\infty}\frac{\beta_1k_2^2(1)k_2(y)}{1+\beta_1k_2^2(1)}\varphi_{2n}X_n(x) \|$$
$$= \Big(\sum_{n=1}^{\infty}(\frac{\beta_1k_2^2(1)k_2(y)}{1+\beta_1k_2^2(1)})^2\varphi_{2n}^2\Big)^{\frac{1}{2}}$$
$$\leq \Big(\sum_{n=1}^{\infty}k_2^2(y)\varphi_{2n}^2\Big)^{\frac{1}{2}} \leq \Big(\sum_{n=1}^{\infty}k_2^2(1)\varphi_{2n}^2\Big)^{\frac{1}{2}} \leq E_1.$$

By the condition stability result (3.5), we have

$$\| g_{1,\beta_1}(x,y) - g(x,y) \| \leq 2^y(1-e^{-2})^{-y}E_1^y\|K_2(y)g_{1,\beta_1}(x,y) - K_2(y)g(x,y)\|^{1-y}, \tag{6.14}$$

here

$$\|K_2(y)g_{1,\beta_1}(x,y) - K_2(y)g(x,y)\| = \Big\|\sum_{n=1}^{\infty}\frac{1}{1+\beta_1k_2^2(1)}\varphi_{2n}X_n(x) - \sum_{n=1}^{\infty}\varphi_{2n}X_n(x)\Big\|$$
$$= \Big\|\sum_{n=1}^{\infty}\frac{-\beta_1k_2^2(1)}{1+\beta_1k_2^2(1)}\varphi_{2n}X_n(x)\Big\|$$
$$= \Big\|\sum_{n=1}^{\infty}\frac{\beta_1k_2^2(1)}{1+\beta_1k_2^2(1)}(\varphi^{\delta}_{2n}-\varphi_{2n})X_n(x) + \sum_{n=1}^{\infty}\frac{-\beta_1k_2^2(1)}{1+\beta_1k_2^2(1)}\varphi^{\delta}_{2n}X_n(x)\Big\|$$
$$\leq \Big\|\sum_{n=1}^{\infty}\frac{\beta_1k_2^2(1)}{1+\beta_1k_2^2(1)}(\varphi^{\delta}_{2n}-\varphi_{2n})X_n(x)\Big\| + \Big\|\sum_{n=1}^{\infty}\frac{\beta_1k_2^2(1)}{1+\beta_1k_2^2(1)}\varphi^{\delta}_{2n}X_n(x)\Big\|$$
$$\leq \delta + \Big\|\sum_{n=1}^{\infty}\frac{\beta_1k_2^2(1)}{1+\beta_1k_2^2(1)}\varphi^{\delta}_{2n}X_n(x)\Big\| \leq \delta + \tau\delta.$$

From (6.14), we have

$$\| g_{1,\beta_1}(x,y) - g(x,y) \| \leq 2^y(1-e^{-2})^{-y}(\tau+1)^{1-y}\delta^{1-y}E_1^y. \tag{6.15}$$

Finally, combining (6.13) with (6.15), we can obtain the error estimate (6.11). □

Next, the posteriori convergence error estimate for problem (1.2) is given.

When $0 < y < 1$, we select the regularization parameter α_1 by the following equation

$$\| K_1(y)f^{\delta}_{1,\alpha_1}(x,y) - \varphi^{\delta}_1(x) \| = \tau\delta, \tag{6.16}$$

where $\tau > 1$ is a positive constant, and $\| \varphi^{\delta}_1(x) \| \geq \tau\delta$.

Lemma 6.7 *Let $\rho(\alpha_1) = \| K_1(y)f(x,y) - \varphi^{\delta}_1(x) \|$. If $\| \varphi^{\delta}_1(x) \| \geq \tau\delta$, we have*

(a) *$\rho(\alpha_1)$ is a continuous function;*

(b) *$\lim_{\alpha_1 \to 0} \rho(\alpha_1) = 0$;*

(c) *$\lim_{\alpha_1 \to \infty} \rho(\alpha_1) = \| \varphi^{\delta}_1 \|$;*

(d) *For $\alpha_1 \in (0, \infty)$, $\rho(\alpha_1)$ is a strictly increasing function.*

Proof Lemma 6.7 can be easily proven with expression

$$\rho(\alpha_1) = \Big(\sum_{n=1}^{\infty} \big(\frac{\alpha_1 k_1^2(1)}{1 + \alpha_1 k_1^2(1)} \big)^2 (\varphi^{\delta}_{1n})^2 \Big)^{\frac{1}{2}}. \quad \square \tag{6.17}$$

Lemma 6.7 indicates that there exists a unique solution for (6.16).

Lemma 6.8 *For fixed $\tau > 1$, let the regularization parameter α_1 satisfy (6.16) and $f(x,y)$ satisfy (3.1). Then we obtain $\alpha_1^{-1} \leq (\frac{E_1}{(\tau-1)\delta})^2$.*

Proof The proof of Lemma 6.8 is similar to Lemma 6.5, so it is omitted. $\square$

Theorem 6.9 *$f(x,y)$ is the exact solution of problem (1.2). The regularization solution $f^{\delta}_{1,\alpha_1}(x,y)$ is given by (5.10), the measured data $\varphi^{\delta}_1(x)$ satisfies (5.9). When $0 < y < 1$, the regularization parameter α_1 is selected by (6.16), we have the error estimate*

$$\| f^{\delta}_{1,\alpha_1}(x,y) - f(x,y) \| \leq C_3 \delta^{1-y} E_1^y, \tag{6.18}$$

where $C_3 := 2(\frac{1}{\tau-1})^y + 2^y(\tau+1)^{1-y}$.

Proof The proof of Theorem 6.9 is similar to Theorem 6.6, so it is omitted. $\square$

Remark 6.10 From Theorems 6.6, 6.9, 4.7 and 4.10, we can deduce that the error estimate obtained by the posteriori regularization parameter choice rule is order optimal for $0 < y < 1$.

6.3. The priori convergence error estimation at endpoint $y = 1$

From Theorems 6.9 and 6.6, we cannot obtain the error estimation at $y = 1$. So in this section, we will give the error estimation between the regularization solution and the exact solution at $y = 1$.

Theorem 6.11 *$g(x,1)$ is the exact solution of problem (1.3). The regularization solution $g^{\delta}_{2,\beta_2}(x,1)$ is given by (5.12), the measured data $\varphi^{\delta}_2(x)$ satisfies (5.13). When $y = 1$, if the priori condition (3.6) holds, and the regularization parameter β_2 is selected as*

$$\beta_2 = \begin{cases} (\frac{\delta}{E_2})^{\frac{2}{p+1}}, & 0 < p < 1, \\ \frac{\delta}{E_2}, & p \geq 1, \end{cases} \tag{6.19}$$

we have the error estimate

$$\| g^{\delta}_{2,\beta_2}(x,1) - g(x,1) \| \le \begin{cases} (2^{-p}+1)\delta^{\frac{p}{p+1}} E_2^{\frac{1}{p+1}}, & 0<p<1, \\ 2\delta^{\frac{1}{2}} E_2^{\frac{1}{2}}, & p \ge 1. \end{cases} \tag{6.20}$$

Proof Using the triangle inequalities, we have

$$\| g^{\delta}_{2,\beta_2}(x,1) - g(x,1) \| \le \| g^{\delta}_{2,\beta_2}(x,1) - g_{2,\beta_2}(x,1) \| + \| g_{2,\beta_2}(x,1) - g(x,1) \|, \tag{6.21}$$

where $g_{2,\beta_2}(x,1)$ is the regularization solution with no error.

From (5.12), (5.13) and basic inequality, we have

$$\begin{aligned} &\| g^{\delta}_{2,\beta_2}(x,1) - g_{2,\beta_2}(x,1) \| = \Big\| \sum_{n=1}^{\infty} \frac{k_2(1)}{1+\beta_1 k_2^2(1)} (\varphi^{\delta}_{2n} - \varphi_{2n}) X_n(x) \Big\| \\ &= \Big(\sum_{n=1}^{\infty} (\frac{k_2(1)}{1+\beta_1 k_2^2(1)})^2 (\varphi^{\delta}_{2n} - \varphi_{2n})^2 \Big)^{\frac{1}{2}} \\ &\le \sup_{n\ge 1} \Big| \frac{k_2(1)}{1+\beta_1 k_2^2(1)} \Big| \Big(\sum_{n=1}^{\infty} (\varphi^{\delta}_{2n} - \varphi_{2n})^2 \Big)^{\frac{1}{2}} \le \beta_2^{-\frac{1}{2}} \delta. \end{aligned}$$

Thus

$$\| g^{\delta}_{2,\beta_2}(x,1) - g_{2,\beta_2}(x,1) \| \le \beta_2^{-\frac{1}{2}} \delta. \tag{6.22}$$

Applying Lemma 2.3 (f) and formula (3.2), (3.6), (5.12), we have

$$\begin{aligned} &\| g_{2,\beta_2}(x,1) - g(x,1) \| = \Big\| \sum_{n=1}^{\infty} \frac{\beta_1 k_2^3(1)}{1+\beta_1 k_2^2(1)} \varphi_{2n} X_n(x) \Big\| \\ &= \Big(\sum_{n=1}^{\infty} (\frac{\beta_1 k_2^3(1)}{1+\beta_1 k_2^2(1)})^2 \varphi_{2n}^2 \Big)^{\frac{1}{2}} \le \sup_{n\ge 1} \Big| \frac{\beta_1 k_2^2(1)}{1+\beta_1 k_2^2(1)} e^{-np} \Big| \Big(\sum_{n=1}^{\infty} e^{2np} k_2^2(1) \varphi_{2n}^2 \Big)^{\frac{1}{2}} \\ &\le \sup_{n\ge 1} \Big| \frac{\beta_1 k_2^2(1)}{1+\beta_1 k_2^2(1)} e^{-np} \Big| \cdot E_2 \\ &\le \begin{cases} 2^{-p} \beta_2^{\frac{p}{2}} E_2, & 0<p<1, \\ \beta_2^{\frac{1}{2}} E_2, & p \ge 1. \end{cases} \end{aligned}$$

To sum up,

$$\| g_{2,\beta_2}(x,1) - g(x,1) \| \le \begin{cases} 2^{-p} \beta_2^{\frac{p}{2}} E_2, & 0<p<1, \\ \beta_2^{\frac{1}{2}} E_2, & p \ge 1. \end{cases} \tag{6.23}$$

By (6.22) and (6.23), the regularization parameter β_2 is chosen as

$$\beta_2 = \begin{cases} (\frac{\delta}{E_2})^{\frac{2}{p+1}}, & 0<p<1, \\ \frac{\delta}{E_2}, & p \ge 1. \end{cases} \tag{6.24}$$

From (6.21)–(6.24), we have

$$\| g^{\delta}_{2,\beta_2}(x,1) - g(x,1) \| \le \begin{cases} (2^{-p}+1)\delta^{\frac{p}{p+1}} E_2^{\frac{1}{p+1}}, & 0<p<1, \\ 2\delta^{\frac{1}{2}} E_2^{\frac{1}{2}}, & p \ge 1. \end{cases} \quad \square \tag{6.25}$$

Theorem 6.12 *$f(x,1)$ is the exact solution of problem (1.2). The regularization solution $f^{\delta}_{2,\alpha_2}(x,1)$ is given by (5.11), the measured data $\varphi^{\delta}_1(x)$ satisfies (5.9). When $y=1$, if the priori* *condition (3.6) holds, and the regularization parameter α_2 is selected as*

$$\alpha_2 = \begin{cases} (\frac{\delta}{E_2})^{\frac{2}{p+1}}, & 0 < p < 1; \\ \frac{\delta}{E_2}, & p \geq 1, \end{cases} \tag{6.26}$$

we have the error estimate

$$\| f^{\delta}_{2,\alpha_2}(x,1) - f(x,1) \| \leq \begin{cases} (2^{2-p}+1)\delta^{\frac{p}{p+1}} E_2^{\frac{1}{p+1}}, & 0 < p < 1, \\ 2\delta^{\frac{1}{2}} E_2^{\frac{1}{2}}, & p \geq 1. \end{cases} \tag{6.27}$$

Proof The proof of Theorem 6.12 is similar to Theorem 6.11, so it is omitted. □

Remark 6.13 From Theorems 6.11, 6.12, 4.7 and 4.10, we can deduce that the error estimate obtained by the priori regularization parameter choice rule is order optimal $O(\delta^{\frac{p}{p+1}})$ for $0 < p < 1$. When $p \geq 1$, the modified Tikhonov regularization method will cause saturation effect.

6.4. The posteriori convergence error estimation at endpoint $y = 1$

When $y = 1$, we select the regularization parameter β_2 by the following equation

$$\|K_2(1) g^{\delta}_{2,\beta_2}(x,1) - \varphi^{\delta}_2(x)\| = \tau\delta, \tag{6.28}$$

where $\tau > 1$ is a positive constant, and $\|\varphi^{\delta}_2\| \geq \tau\delta$.

Lemma 6.14 *Let $\varrho(\beta_2) = \| K_2(1) g^{\delta}_{2,\beta_2}(x,1) - \varphi^{\delta}_2(x) \|$. If $\| \varphi^{\delta}_2(x) \| \geq \tau\delta$, we have*

(a) $\varrho(\beta_2)$ is a continuous function;

(b) $\lim_{\beta_2 \to 0} \varrho(\beta_2) = 0$;

(c) $\lim_{\beta_2 \to \infty} \varrho(\beta_2) = \| \varphi^{\delta}_2 \|$;

(d) For $\beta_2 \in (0,\infty)$, $\varrho(\beta_2)$ is a strictly increasing function.

Proof The Lemma can be easily proven with expression

$$\varrho(\beta_2) = (\sum_{n=1}^{\infty} (\frac{\beta_2 k_2^2(1)}{1+\beta_2 k_2^2(1)})^2 (\varphi^{\delta}_{2n})^2)^{\frac{1}{2}}. \quad \square \tag{6.29}$$

Lemma 6.14 indicates that there exists a unique solution for (6.28).

Lemma 6.15 *For fixed $\tau > 1$, let the regularization parameter β_2 satisfy (6.28) and $g(x,y)$ satisfy (3.6). Then, we can see that the regularization parameter $\beta_2 = \beta_2(\delta, \varphi^{\delta}_2)$ satisfies*

$$\beta_2^{-1} \leq \begin{cases} (\frac{E_2}{(\tau-1)\delta})^{\frac{2}{p+1}}, & 0 < p < 1, \\ \frac{E_2}{(\tau-1)\delta}, & p \geq 1. \end{cases} \tag{6.30}$$

Proof Applying Lemma 2.3 (e) and formula (6.28), we have

$$\begin{aligned} \tau\delta &= \Big\| \sum_{n=1}^{\infty} \frac{\beta_2 k_2^2(1)}{1+\beta_2 k_2^2(1)} \varphi^{\delta}_{2n} X_n(x) \Big\| \\ &= \Big\| \sum_{n=1}^{\infty} \frac{\beta_2 k_2^2(1)}{1+\beta_2 k_2^2(1)} (\varphi^{\delta}_{2n} - \varphi_{2n}) X_n(x) + \sum_{n=1}^{\infty} \frac{\beta_2 k_2^2(1)}{1+\beta_2 k_2^2(1)} \varphi_{2n} X_n(x) \Big\| \end{aligned}$$

$$\leq \Big\| \sum_{n=1}^{\infty} \frac{\beta_2 k_2^2(1)}{1+\beta_2 k_2^2(1)} (\varphi_{2n}^{\delta} - \varphi_{2n}) X_n(x) \Big\| + \Big\| \sum_{n=1}^{\infty} \frac{\beta_2 k_2^2(1)}{1+\beta_2 k_2^2(1)} \varphi_{2n} X_n(x) \Big\|$$

$$\leq \delta + \Big\| \sum_{n=1}^{\infty} \frac{\beta_2 k_2^2(1)}{1+\beta_2 k_2^2(1)} \varphi_{2n} X_n(x) \Big\|$$

$$= \delta + \Big\| \sum_{n=1}^{\infty} \frac{\beta_2 k_2^2(1)}{1+\beta_2 k_2^2(1)} \varphi_{2n} X_n(x) \Big\|$$

and

$$\Big\| \sum_{n=1}^{\infty} \frac{\beta_2 k_2^2(1)}{1+\beta_2 k_2^2(1)} \varphi_{2n} X_n(x) \Big\|$$

$$= \Big(\sum_{n=1}^{\infty} \big(\frac{\beta_2 k_2^2(1)}{1+\beta_2 k_2^2(1)} \big)^2 \varphi_{2n}^2 \Big)^{\frac{1}{2}}$$

$$\leq \sup_{n\geq 1} \Big| \frac{\beta_2 k_2(1)}{1+\beta_2 k_2^2(1)} e^{-np} \Big| \Big(\sum_{n=1}^{\infty} e^{2np} k_2^2(1) \varphi_{2n}^2 \Big)^{\frac{1}{2}}$$

$$\leq \begin{cases} \beta_2^{\frac{1}{2}+\frac{p}{2}} E_2, \ 0 < p < 1, \\ \beta_2 E_2, \ p \geq 1. \end{cases}$$

To sum up

$$\beta_2^{-1} \leq \begin{cases} (\frac{E_2}{(\tau-1)\delta})^{\frac{2}{p+1}}, \ 0 < p < 1, \\ \frac{E_2}{(\tau-1)\delta}, \ p \geq 1. \end{cases} \qquad \square$$

Theorem 6.16 *If expressions (3.2) and (5.12) hold and β_2 satisfies (6.28), then*

(1) If $0 < p < 1$, then the following error estimate is obtained

$$\|g_{2,\beta_2}^{\delta}(x,1) - g(x,1)\| \leq C_4 \delta^{\frac{p}{p+1}} E_2^{\frac{1}{p+1}}, \tag{6.31}$$

where $C_4 := (\frac{1}{\tau-1})^{\frac{1}{p+1}} + (\tau+1)^{\frac{p}{p+1}}$.

(2) If $p \geqslant 1$, then the following convergent estimate is obtained

$$\|g_{2,\beta_2}^{\delta}(x,1) - g(x,1)\| \leq C_5 \delta^{\frac{1}{2}} E_2^{\frac{1}{2}}, \tag{6.32}$$

where $C_5 := (\frac{1}{\tau-1})^{\frac{1}{2}} + (\tau+1)^{\frac{1}{2}}$.

Proof Using the triangle inequality, we obtain

$$\| g_{2,\beta_2}^{\delta}(x,1) - g(x,1) \| \leq \| g_{2,\beta_2}^{\delta}(x,1) - g_{2,\beta_2}(x,1) \| + \| g_{2,\beta_2}(x,1) - g(x,1) \|, \tag{6.33}$$

where $g_{2,\beta_2}(x,1)$ is the regularization solution with no error.

Case 1. $0 < p < 1$.

By (6.22), (6.30), we have

$$\| g_{2,\beta_2}^{\delta}(x,1) - g_{2,\beta_2}(x,1) \| \leq \beta_2^{-\frac{1}{2}} \delta \leq (\frac{1}{\tau-1})^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}} E_2^{\frac{1}{p+1}}. \tag{6.34}$$

Next, we estimate the second term of formula (6.33). According to the priori bound condition

(3.6), we have

$$\begin{aligned}\| g_{2,\beta_2}(x,1)-g(x,1)\|_{H^p} &= \Big\|\sum_{n=}^{\infty} e^{np}\frac{\beta_1 k_2^3(1)}{1+\beta_1 k_2^2(1)}\varphi_{2n}X_n(x)\Big\| \\ &= \Big(\sum_{n=1}^{\infty} e^{2np}(\frac{\beta_1 k_2^3(1)}{1+\beta_1 k_2^2(1)})^2\varphi_{2n}^2\Big)^{\frac{1}{2}} \\ &\le \Big(\sum_{n=1}^{\infty} e^{2np}k_2^2(1)\varphi_{2n}^2\Big)^{\frac{1}{2}} \le E_2.\end{aligned}$$

Applying the condition stability result (3.8), we have

$$\| g_{2,\beta_2}(x,1)-g(x,1)\| \le E_2^{\frac{1}{p+1}}\|K_2(1)g_{2,\beta_2}(x,1)-K_2(1)g(x,1)\|^{\frac{p}{p+1}}, \tag{6.35}$$

where

$$\begin{aligned}&\|K_2(1)g_{2,\beta_2}(x,1)-K_2(1)g(x,1)\| \\ &=\Big\|\sum_{n=1}^{\infty}\frac{1}{1+\beta_2k_2^2(1)}\varphi_{2n}X_n(x)-\sum_{n=1}^{\infty}\varphi_{2n}X_n(x)\Big\| \\ &=\Big\|\sum_{n=1}^{\infty}\frac{-\beta_2k_2^2(1)}{1+\beta_2k_2^2(1)}\varphi_{2n}X_n(x)\Big\| \\ &=\Big\|\sum_{n=1}^{\infty}\frac{\beta_2k_2^2(1)}{1+\beta_2k_2^2(1)}(\varphi_{2n}^\delta-\varphi_{2n})X_n(x)+\sum_{n=1}^{\infty}\frac{-\beta_2k_2^2(1)}{1+\beta_2k_2^2(1)}\varphi_{2n}^\delta X_n(x)\Big\| \\ &\le\Big\|\sum_{n=1}^{\infty}\frac{\beta_2k_2^2(1)}{1+\beta_2k_2^2(1)}(\varphi_{2n}^\delta-\varphi_{2n})X_n(x)\Big\|+\Big\|\sum_{n=1}^{\infty}\frac{\beta_2k_2^2(1)}{1+\beta_2k_2^2(1)}\varphi_{2n}^\delta X_n(x)\Big\| \\ &\le\delta+\Big\|\sum_{n=1}^{\infty}\frac{\beta_2k_2^2(1)}{1+\beta_2k_2^2(1)}\varphi_{2n}^\delta X_n(x)\Big\|\le\delta+\tau\delta.\end{aligned}$$

From (6.35), we have

$$\| g_{2,\beta_2}(x,1)-g(x,1)\| \le (\tau+1)^{\frac{p}{p+1}}\delta^{\frac{p}{p+1}}E_2^{\frac{1}{p+1}}. \tag{6.36}$$

Finally, combining (6.34) with (6.36), we can obtain the error estimate (6.31).

Case 2. $p\ge 1$.

By (6.22), (6.30), we have

$$\| g_{2,\beta_2}^\delta(x,1)-g_{2,\beta_2}(x,1)\| \le \beta_2^{-\frac{1}{2}}\delta\le(\frac{1}{\tau-1})^{\frac{1}{2}}\delta^{\frac{1}{2}}E_2^{\frac{1}{2}}. \tag{6.37}$$

Now, we estimate the second term of formula (6.33)

$$\|g_{2,\beta_2}(x,1)-g(x,1)\|\le(\tau+1)^{\frac{1}{2}}\delta^{\frac{1}{2}}E_2^{\frac{1}{2}}. \tag{6.38}$$

The proof of this item is the same as that of (6.36), so it is omitted. Combining (6.37) with (6.38), we can obtain the convergence estimate (6.32). □

Next, the posteriori convergence error estimate for problem (1.2) is given. When $y=1$, we select the regularization parameter α_2 by the following equation

$$\| K_1(1)f_{2,\alpha_2}^\delta(x,1)-\varphi_1^\delta(x)\| = \tau\delta, \tag{6.39}$$

where $\tau > 1$ is a positive constant, and $\| \varphi_1^\delta(x) \| \geq \tau\delta$.

Lemma 6.17 *Let* $\rho(\alpha_2) = \| K_1(1) f_{2,\alpha_2}^\delta(x,1) - \varphi_1^\delta(x) \|$. *If* $\| \varphi_1^\delta(x) \| \geq \tau\delta$, *we have*

(a) $\rho(\alpha_2)$ *is a continuous function;*

(b) $\lim_{\alpha_2 \to 0} \rho(\alpha_2) = 0$;

(c) $\lim_{\alpha_2 \to \infty} \rho(\alpha_2) = \| \varphi_1^\delta \|$;

(d) *For* $\alpha_2 \in (0, \infty), \rho(\alpha_2)$ *is a strictly increasing function.*

Proof The Lemma can be easily proven with expression

$$\rho(\alpha_2) = \Big(\sum_{n=1}^{\infty} (\frac{\alpha_2 k_1^2(1)}{1 + \alpha_2 k_1^2(1)})^2 (\varphi_{1n}^\delta)^2 \Big)^{\frac{1}{2}}. \quad \square \tag{6.40}$$

Lemma 6.17 indicates that there exists a unique solution for (6.39).

Lemma 6.18 *For fixed* $\tau > 1$, *let the regularization parameter* α_2 *satisfy* (6.39) *and* $f(x,y)$ *satisfy* (3.6). *Then, we can see that the regularization parameter* $\alpha_2 = \alpha_2(\delta, \varphi_1^\delta)$ *satisfies*

$$\alpha_2^{-1} \leq \begin{cases} (\frac{2^{1-p} E_2}{(\tau-1)\delta})^{\frac{2}{p+1}}, & 0 < p < 1, \\ \frac{E_2}{(\tau-1)\delta}, & p \geq 1. \end{cases} \tag{6.41}$$

Proof The proof of Lemma 6.18 is similar to Lemma 6.8, so it is omitted. □

Theorem 6.19 *If expressions* (3.1) *and* (5.9) *hold and* α_2 *satisfies the regularization parameter selection rule:*

(1) *If* $0 < p < 1$, *then the following convergent estimate is obtained*

$$\|f_{2,\alpha_2}^\delta(x,1) - f(x,1)\| \leq C_6 \delta^{\frac{p}{p+1}} E_2^{\frac{1}{p+1}}, \tag{6.42}$$

where $C_6 := (\frac{2^{1-p}}{\tau-1})^{\frac{1}{p+1}} + (\tau+1)^{\frac{p}{p+1}}$.

(2) *If* $p \geqslant 1$, *then the following convergent estimate is obtained*

$$\|f_{2,\alpha_2}^\delta(x,1) - f(x,1)\| \leq C_7 \delta^{\frac{1}{2}} E_2^{\frac{1}{2}}, \tag{6.43}$$

where $C_7 := (\frac{1}{\tau-1})^{\frac{1}{2}} + (\tau+1)^{\frac{1}{2}}$.

Proof The proof of Theorem 6.19 is similar to Theorem 6.16, so it is omitted. □

Remark 6.20 From Theorems 6.16, 6.19, 4.7 and 4.10, we can deduce that the error estimate obtained by the posteriori regularization parameter choice rule is order optimal $O(\delta^{\frac{p}{p+1}})$ for $0 < p < 1$. When $p \geq 1$, the modified Tikhonov regularization method will cause saturation effect.

7. Numerical implementation

In this section, we are going to use several numerical examples to verify the efficiency of our method. Consider the problem

$$\begin{cases} u_{xxxx}(x,y)+2u_{xxyy}(x,y)+u_{yyyy}(x,y)=0, & (x,y)\in(0,\pi)\times(0,1),\\ u(x,0)=\varphi_1(x), & x\in[0,\pi],\\ u_y(x,0)=\varphi_2(x), & x\in[0,\pi],\\ \Delta u(x,0)=0, & x\in[0,\pi],\\ \Delta u_y(x,0)=0, & x\in[0,\pi],\\ u(0,y)=u(\pi,y)=\Delta u(0,y)=\Delta u_y(\pi,y)=0, & y\in[0,1] \end{cases} \tag{7.1}$$

with the given data $\varphi_1(x)$, $\varphi_2(x)$. We define

$$x_i=i\Delta x,\ i=0,1,\ldots,N,\quad y_j=j\Delta y,\ j=0,1,\ldots,M,$$

where $\Delta x=\frac{1}{N}$ is the step size of spatial direction and Δy is the step size of temporal direction. For the simplification, we only investigate the numerical efficiency of the regularization method for (1.2), and the problem (1.3) is similar to the problem (1.2).

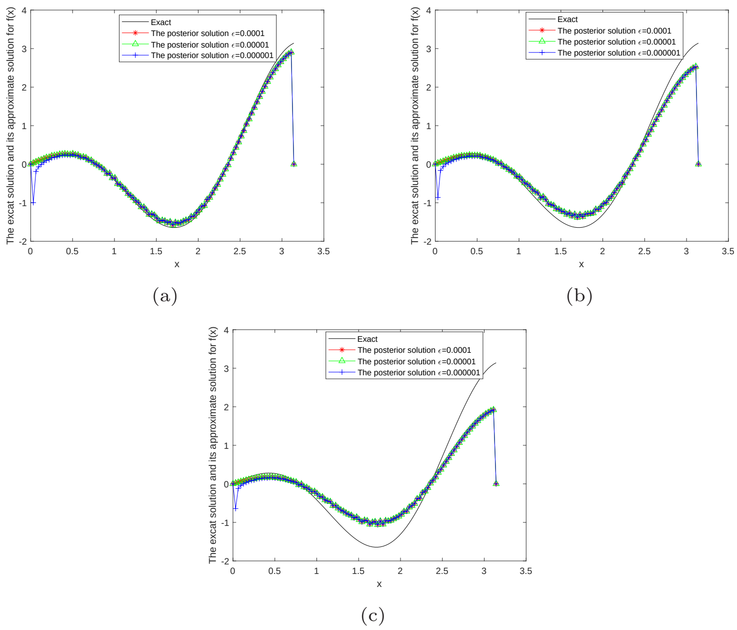


Figure 1 The exact solution and the modified Tikhonov regularization solution of Example 7.1 with (a) $y=0.1$; (b) $y=0.25$; (c) $y=0.45$ for $\varepsilon=0.0001, 0.00001, 0.000001$

According to [20], we can obtain the 13-point approximation of the biharmonic equation, which can be written as

$$\frac{1}{(\Delta x)^2(\Delta y)^2}\big(20f_i^j-8(f_{i+1}^j+f_i^{j+1}+f_{i-1}^j+f_i^{j-1})+2(f_{i+1}^j+f_{i-1}^{j+1}+f_{i-1}^{j-1}+f_{i+1}^{j-1})+ (f_{i+2}^j+f_i^{j+2}+f_{i-2}^j+f_i^{j-2})\big)=0.$$

We generate the noise-contaminated data by adding a random perturbation, i.e.,

$$f^{\delta}(x,y) = f(x,y) + \varepsilon \cdot f(x,y)(2\,\mathrm{rand}(\mathrm{size}(f)) - 1), \tag{7.2}$$

$$\varphi^{\delta} = \varphi + \varepsilon \cdot \varphi(x)(2\,\mathrm{rand}(\mathrm{size}(\varphi) - 1)), \tag{7.3}$$

here, size(f) represents the size of f in space and time, size(φ) represents the size of φ in space, the function rand$(\cdot)$ generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$, and the noise level is:

$$\delta = \|\varphi^{\delta} - \varphi\| = \sqrt{\frac{1}{N+1}\sum_{i=1}^{N+1}(\varphi_i - \varphi_i^{\delta})^2}. \tag{7.4}$$

Actually, the priori regularization parameter may consider the smooth condition of the exact solution. But it is difficult to get it in practical problem. The Tikhonov regularization method is validated based on the posteriori regularization parameter choice rule. The effectiveness and stability of this method are verified by three examples. Let us take $\tau = 1.01$. Choosing $N = 100$, $M = 1000$, we give the following three examples.

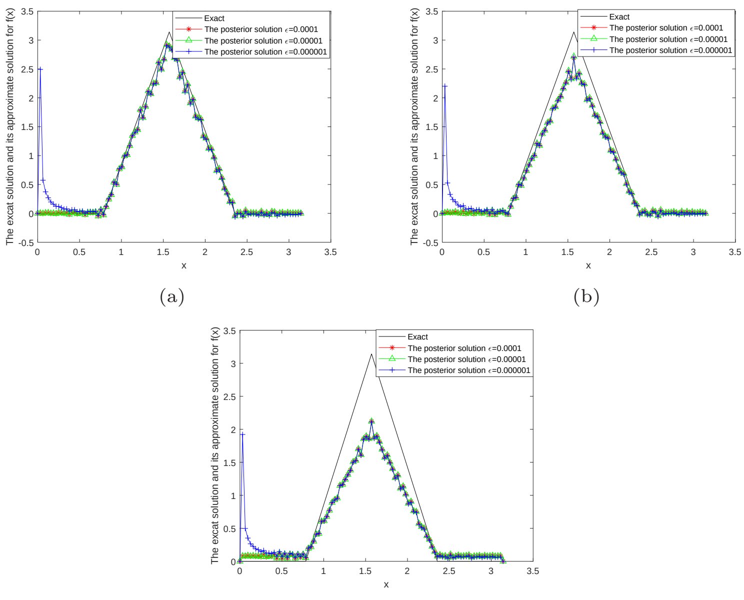


Figure 2 The exact solution and the modified Tikhonov regularization solution of Example 7.2 with (a) $y = 0.1$; (b) $y = 0.25$; (c) $y = 0.45$ for $\varepsilon = 0.0001, 0.00001, 0.000001$

Example 7.1 Consider the function

$$f(x) = x\cos(2x), \quad x \in [0, \pi].$$

Example 7.2 Consider the piecewise smooth function

$$f(x)=\begin{cases}0, & x\in[0,\frac{\pi}{4}),\\ 4(x-\frac{1}{4}), & x\in[\frac{\pi}{4},\frac{\pi}{2}),\\ -4(x-\frac{3}{4}), & x\in[\frac{\pi}{2},\frac{3}{4}\pi),\\ 0, & x\in[\frac{3}{4}\pi,\pi].\end{cases}$$

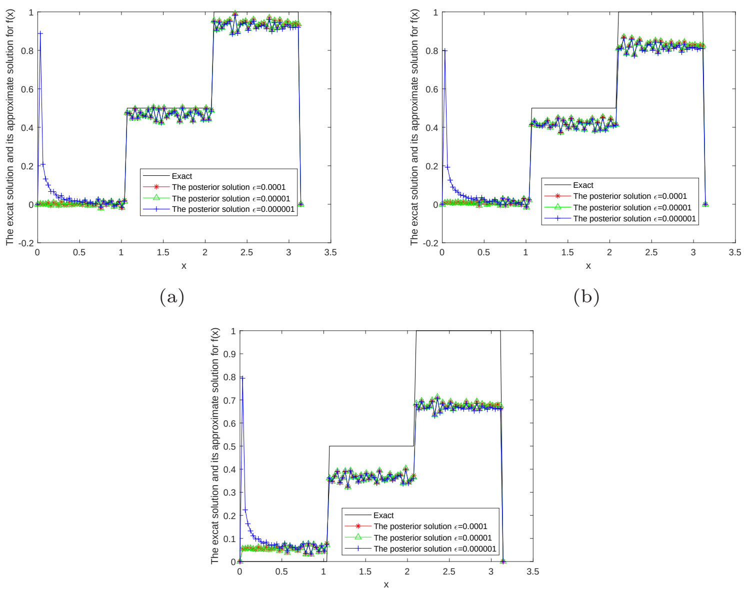
Example 7.3 Consider the function

$$f(x)=\begin{cases}0, & x\in[0,\frac{\pi}{3}),\\ \frac{1}{2}, & x\in[\frac{\pi}{3},\frac{2}{3}\pi),\\ 1, & x\in[\frac{2}{3}\pi,\pi].\end{cases}$$

Example 7.4 Consider the non-smooth function

$$f(x)=\begin{cases}0, & x\in[0,\frac{\pi}{4}],\\ 1, & x\in(\frac{\pi}{4},\frac{\pi}{2}],\\ 0, & x\in(\frac{\pi}{2},\frac{3}{4}\pi],\\ -1, & x\in(\frac{3}{4}\pi,\pi].\end{cases}$$

Figures 1–3 show the error of the exact solution and the approximate solution of the modified Tikhonov regularization method.

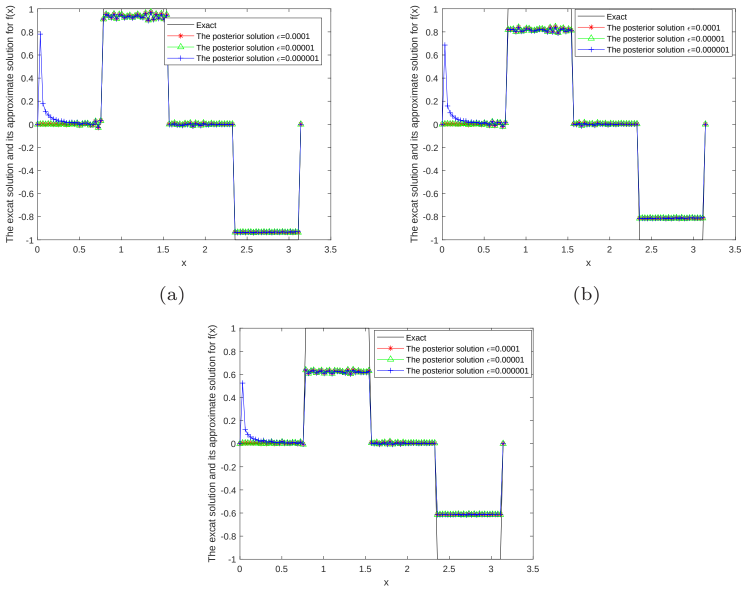


(a) (b) (c)

Figure 3 The exact solution and the modified Tikhonov regularization solution of Example 7.3 with (a) $y=0.1$; (b) $y=0.25$; (c) $y=0.45$ for $\varepsilon=0.0001,0.00001,0.000001$

Figure 1 shows the exact solution $f(x)$ and the modified Tikhonov regularization solution $f_\alpha^\delta(x)$ of Example 7.1 for the relative error levels $\varepsilon=0.0001,0.00001,0.000001$ with various values $y=0.1,0.25,0.45$. Figure 2 shows the exact solution $f(x)$ and the modified Tikhonov regularization solution $f_\alpha^\delta(x)$ of Example 7.2 for the relative error levels $\varepsilon=0.0001,0.00001,0.000001$

with various values $y = 0.1, 0.25, 0.45$. Figure 3 shows the exact solution $f(x)$ and the modified Tikhonov regularization solution $f_\alpha^\delta(x)$ of Example 7.3 for the relative error levels $\varepsilon = 0.0001, 0.00001, 0.000001$ with various values $y = 0.1, 0.25, 0.45$. From above three figures we can see, the same numerical example, the smaller the value of ε and y, the better the fitting effect of the exact solution $f(x)$ and the corresponding regular solution $f_\alpha^\delta(x)$ will be. For different numerical examples, the fitting results of the function with better smoothness are better than that of the function with worse smoothness. Above four examples show that the modified Tikhonov regularization method is very effective.



(a) (b) (c)

Figure 4 The exact solution and the modified Tikhonov regularization solution of Example 7.4 with (a) $y = 0.1$; (b) $y = 0.25$; (c) $y = 0.45$ for $\varepsilon = 0.0001, 0.00001, 0.000001$

8. Conclusion

This paper investigates the Cauchy problem of biharmonic equations and the condition stability is given under the a priori bound assumption for the exact solution. A modified Tikhonov regularization method is used to solve this ill-posed problem. For the choice of regularization parameter, we give the priori and the posteriori rules. Under the priori regularization parameter selection rules and the posteriori regularization parameter selection rules, the corresponding error estimates are obtained respectively. Finally, we verify the feasibility of our method by doing the corresponding numerical experiments.

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