Journal of Mathematical Research with Applications
May, 2024, Vol. 44, No. 3, pp. 387–396
DOI:10.3770/j.issn:2095-2651.2024.03.009
Http://jmre.dlut.edu.cn

Characterizations of Lie Triple Derivations on the Algebra of Operators in Hilbert C^*-Modules

Guangyu AN[1], **Jun SHENG**[1], **Jun HE**[2,*]
1. *Department of Mathematics, Shaanxi University of Science and Technology, Shaanxi* 710021*, P. R. China;*
2. *Department of Mathematics, Anhui Polytechnic University, Anhui* 241000*, P. R. China*

Abstract Let $\mathcal{A}$ be a commutative unital C^*-algebra with the unit element e and $\mathcal{M}$ be a full Hilbert $\mathcal{A}$-module. Denote by $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ the algebra of all bounded $\mathcal{A}$-linear mappings on $\mathcal{M}$ and by $\mathcal{M}'$ the set of all bounded $\mathcal{A}$-linear mappings from $\mathcal{M}$ into $\mathcal{A}$. In this paper, we prove that if there exists x_0 in $\mathcal{M}$ and f_0 in $\mathcal{M}'$ such that $f_0(x_0) = e$, then every $\mathcal{A}$-linear Lie triple derivation on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is standard.

Keywords Lie triple derivation; standard; derivation; Hilbert C^*-module

MR(2020) Subject Classification 47B48; 47L35; 47C15

1. Introduction

Let $\mathcal{A}$ be an associative algebra over the complex field $\mathbb{C}$ and d be a linear mapping on $\mathcal{A}$. d is called a derivation if $d(xy) = d(x)y + xd(y)$ for each x, y in $\mathcal{A}$. And d is called an inner derivation if there exists an element m in $\mathcal{A}$ such that $d(x) = mx - xm$. Clearly, every inner derivation is a derivation.

One of the interesting problems in the theory of derivations is to identify those algebras on which every derivation is inner. The following two results are classical. In [1], Sakai proved that every derivation on a W^*-algebra is an inner derivation; and in [2], Christensen showed that every derivation on a nest algebra is an inner derivation.

A linear mapping d on $\mathcal{A}$ is called a Lie derivation if $d([x, y]) = [d(x), y] + [x, d(y)]$ for each x, y in $\mathcal{A}$, where $[x, y] = xy - yx$ is the usual Lie product on $\mathcal{A}$. A Lie derivation d on $\mathcal{A}$ is said to be standard if it can be decomposed as $d = \delta + \tau$, where δ is a derivation on $\mathcal{A}$ and τ is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ with $\tau([x, y]) = 0$ for each x, y in $\mathcal{A}$, where $\mathcal{Z}(\mathcal{A}) = \{z \in \mathcal{A} : xz = zx$ for every x in $\mathcal{A}\}$ is the center of $\mathcal{A}$.

Another interesting problem is to identify those algebras on which every Lie derivation is standard. In [3], Mathieu and Villena proved that every Lie derivation on a C^*-algebra is standard; in [4], Cheung characterized Lie derivations on triangular algebras; in [5], Lu proved

Received June 24, 2023; Accepted September 23, 2023
Supported by the Shaanxi College Students Innovation and Entrepreneurship Training Program (Grant No. S202110708069).
* Corresponding author
E-mail address: anguangyu@sust.edu.cn (Guangyu AN); 876821092@qq.com (Jun SHENG); hejun_12@163.com (Jun HE)

that every Lie derivation on a completely distributed commutative subspace lattice algebra is standard; and in [6], Benkovič proved that every Lie derivation on a matrix algebra $M_n(\mathcal{A})$ is standard, where $n \geq 2$ and $\mathcal{A}$ is a unital algebra.

A linear mapping d on $\mathcal{A}$ is called a Lie triple derivation if $d([[x,y],z]) = [[d(x),y],z] + [[x,d(y)],z] + [[x,y],d(z)]$ for each x, y and z in $\mathcal{A}$. It is clear that every Lie derivation is a Lie triple derivation. A Lie triple derivation d on $\mathcal{A}$ is said to be *standard* if it can be decomposed as $d = \delta + \tau$, where δ is a derivation on $\mathcal{A}$ and τ is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ with $\tau([[x,y],z]) = 0$ for each x, y and z in $\mathcal{A}$.

Similar to Lie derivations, the authors always consider the problem that is to identify those algebras on which every Lie triple derivation is standard. In [7], Miers proved that if $\mathcal{A}$ is a von Neumann algebra with no central abelian summands, then every Lie triple derivation on $\mathcal{A}$ is standard; in [8], Ji and Wang proved that every continuous Lie triple derivation on the TUHF algebras is standard; in [9], Zhang, Wu and Cao proved that if $\mathcal{N}$ is a nest on a complex separable Hilbert space $\mathcal{H}$, then every Lie triple derivation on the associated nest algebra Alg$\mathcal{N}$ is standard; in [10], Yu and Zhang studied the Lie triple derivations on commutative subspace lattice algebras. In [6], Benkovič showed that if $\mathcal{A}$ is a unital algebra with a nontrivial idempotent, then under suitable assumptions, every Lie triple derivation d on $\mathcal{A}$ is of the form $d = \Delta + \delta + \tau$, where Δ is a derivation on $\mathcal{A}$, δ is a Jordan derivation on $\mathcal{A}$ and τ is a linear mapping from $\mathcal{A}$ into its center $\mathcal{Z}(\mathcal{A})$ that vanished on $[[\mathcal{A},\mathcal{A}],\mathcal{A}]$.

In 1953, Kaplansky introduced the concepts of Hilbert C^*-modules for studying the derivations on AW^*-algebras of type I. Hilbert C^*-modules provide a natural generalization of Hilbert spaces by replacing the complex field $\mathbb{C}$ with a C^*-algebra. The theory of Hilbert C^*-modules plays an important role in the theory of operator algebras, as it can be applied in many fields, such as index theory of elliptic operators, K- and K K-theory, noncommutative geometry, and so on.

There are few results about derivations, Lie derivations and Lie triple derivations in this topic. In [11], Li, Han and Tang proved that every derivation on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is inner, where $\mathcal{M}$ is a full Hilbert C^*-module over a commutative unital C^*-algebra $\mathcal{A}$; and in [12], Moghadam, Miri and Janfada proved that every $\mathcal{A}$-linear derivation on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is inner, where $\mathcal{M}$ is a full Hilbert C^*-module over a commutative unital C^*-algebra $\mathcal{A}$ with the property that there exists x_0 in $\mathcal{M}$ and f_0 in $\mathcal{M}'$ such that $f_0(x_0) = e$.

In this paper, we study Lie triple derivations on the algebra of operators in Hilbert C^*-modules. We prove that if $\mathcal{M}$ is a full Hilbert C^*-module over a commutative unital C^*-algebra $\mathcal{A}$ containing unit e with the property that there exists x_0 in $\mathcal{M}$ and f_0 in $\mathcal{M}'$ such that $f_0(x_0) = e$, then every $\mathcal{A}$-linear Lie triple derivation on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is standard.

2. Preliminaries

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ be a left $\mathcal{A}$-module. $\mathcal{M}$ is called a *Pre-Hilbert $\mathcal{A}$-module* if there exists a mapping $\langle\cdot,\cdot\rangle : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{A}$ such that

(1) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ if and only if $x = 0$;

(2) $\langle \lambda x + y, z\rangle = \lambda\langle x, z\rangle + \langle y, z\rangle$;

(3) $\langle ax, y\rangle = a\langle x, y\rangle$;

(4) $\langle x, y\rangle = \langle y, x\rangle^*$,

where $\lambda \in \mathbb{C}, a \in \mathcal{A}, x, y, z \in \mathcal{M}$. The mapping $\langle \cdot, \cdot\rangle$ is called an $\mathcal{A}$-valued inner product. The $\mathcal{A}$-valued inner product also induces a norm on $\mathcal{M}$: $\|x\| = \|\langle x, x\rangle\|^{1/2}$. $\mathcal{M}$ is called a Hilbert $\mathcal{A}$-module (or more exactly, a Hilbert C^*-module over $\mathcal{A}$), if $\mathcal{M}$ is complete with respect to this norm.

We denote by $\langle \mathcal{M}, \mathcal{M}\rangle$ the closure of the linear span of the set $\{\langle x, y\rangle : x, y \in \mathcal{M}\}$, and $\mathcal{M}$ is called a full Hilbert $\mathcal{A}$-module if $\langle \mathcal{M}, \mathcal{M}\rangle = \mathcal{A}$.

A linear mapping T on $\mathcal{M}$ is said to be $\mathcal{A}$-linear if $T(ax) = aT(x)$ for each a in $\mathcal{A}$ and x in $\mathcal{M}$. A bounded $\mathcal{A}$-linear mapping on $\mathcal{M}$ is called an operator. Let $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ be the set of all operators on $\mathcal{M}$, and by [13] we know that $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is a Banach algebra.

Let $\mathcal{A}$ be a commutative C^*-algebra and a be in $\mathcal{A}$. Define an $\mathcal{A}$-linear mapping T_a from $\mathcal{M}$ into itself by $T_a x = ax$ for every x in $\mathcal{M}$. It is clear that T_a belongs to $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ and we should notice that if $\mathcal{A}$ is not commutative, then T_a is not $\mathcal{A}$-linear and not in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$.

Lemma 2.1 ([14, Lemma 1.4]) *Let $\mathcal{A}$ be a commutative unital C^*-algebra and $\mathcal{M}$ be a full Hilbert $\mathcal{A}$-module. Then the center of $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is $\mathcal{Z}(\mathrm{End}_{\mathcal{A}}(\mathcal{M})) = \{T_a : a \in \mathcal{A}\} = \{aI : a \in \mathcal{A}\}$, where I is the unit of $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$.*

A linear mapping f from $\mathcal{M}$ into $\mathcal{A}$ is said to be $\mathcal{A}$-linear if $f(ax) = af(x)$ for each $a \in \mathcal{A}$ and $x \in \mathcal{M}$. The set of all bounded $\mathcal{A}$-linear mappings from $\mathcal{M}$ to $\mathcal{A}$ is denoted by $\mathcal{M}'$. For each x in $\mathcal{M}$ and f in $\mathcal{M}'$, we can define a mapping $\theta_{x,f}$ on $\mathcal{M}$ by $\theta_{x,f} y = f(y)x$ for every y in $\mathcal{M}$. Obviously, $\theta_{x,f} \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$.

Lemma 2.2 ([13]) *Let $\mathcal{M}$ be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$. For each a in $\mathcal{A}$, x, y in $\mathcal{M}$, f, g in $\mathcal{M}'$ and A in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$, we have that*

(1) $\theta_{x,f} A = \theta_{x,f\circ A}$;

(2) $A\theta_{x,f} = \theta_{Ax,f}$;

(3) if $\mathcal{A}$ is commutative, then $\theta_{x,f}\theta_{y,g} = f(y)\theta_{x,g}$, $\theta_{ax,f} = a\theta_{x,f}$.

3. Main results

In this section, we suppose that $\mathcal{A}$ is a commutative unital C^*-algebra with the unit element e, $\mathcal{M}$ is a full Hilbert $\mathcal{A}$-module, and there exists x_0 in $\mathcal{M}$ and f_0 in $\mathcal{M}'$ such that $f_0(x_0) = e$.

For the convenience of expression, we give some symbols firstly. Denote $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ by $\mathcal{X}$ and denote by I the unit operator in $\mathcal{X}$. Let $P_1 = \theta_{x_0,f_0}$ and $P_2 = I - P_1$, it is easy to see that P_1 and P_2 are two idempotents in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$. Denote $P_i\mathcal{X}P_j$ by $\mathcal{X}_{ij}$ and P_iAP_j by A_{ij} for every A in $\mathcal{X}$, where $1 \leq i, j \leq 2$.

The following two lemmas will be used repeatedly.

Lemma 3.1 For every A in $\mathcal{X}$, we have $P_1AP_1 = f_0(Ax_0)P_1 = f_0(P_1AP_1x_0)P_1$. *Moreover, $\mathcal{X}_{11}$ is commutative.*

Proof For every A in $\mathcal{X}$, by Lemma 2.2, we have that

$$P_1AP_1 = \theta_{x_0,f_0}A\theta_{x_0,f_0} = f_0(Ax_0)\theta_{x_0,f_0} = f_0(Ax_0)P_1. \tag{3.1}$$

Replacing A by P_1AP_1 in (3.1), we get that

$$P_1AP_1 = P_1P_1AP_1P_1 = f_0(P_1AP_1x_0)P_1.$$

Notice that $f_0(Ax_0)$ belongs to $\mathcal{A}$. It follows that $\mathcal{X}_{11}$ is commutative. □

Lemma 3.2 *(1) If $BA_{21} = 0$ for every A_{21} in $\mathcal{X}_{21}$, then $BP_2 = 0$. (2) If $A_{12}B = 0$ for every A_{12} in $\mathcal{X}_{12}$, then $P_2B = 0$.*

Proof (1) Let $A_{21} = P_2\theta_{x,f_0}P_1$, where x is an arbitrary element in $\mathcal{M}$. We can obtain that

$$0 = BP_2\theta_{x,f_0}P_1x_0 = f_0(P_1x_0)BP_2x = BP_2x.$$

It follows that $BP_2 = 0$.

(2) Let $A_{12} = P_1\theta_{x_0,f}P_2$, where f is an arbitrary element in $\mathcal{M}'$. We can obtain that

$$0 = P_1\theta_{x_0,f}P_2Bx = f(P_2Bx)P_1x_0 = f(P_2Bx)x_0.$$

It follows that $f(P_2Bx) = 0$ for every $f \in \mathcal{M}'$. Define a mapping g in $\mathcal{M}'$ by $g(y) = \langle y, P_2Bx\rangle$. Hence $g(P_2Bx) = \langle P_2Bx, P_2Bx\rangle = 0$. It follows that $P_2Bx = 0$, thus $P_2B = 0$. □

The following theorem is the main result in this paper.

Theorem 3.3 *Let $\mathcal{A}$ be a commutative unital C^*-algebra with the unit element e and $\mathcal{M}$ be a full Hilbert $\mathcal{A}$-module. If there exists x_0 in $\mathcal{M}$ and f_0 in $\mathcal{M}'$ such that $f_0(x_0) = e$, then every $\mathcal{A}$-linear Lie triple derivation δ on $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ is standard.*

Before we prove Theorem 3.3, we show some lemmas.

Lemma 3.4 *Suppose that δ is the $\mathcal{A}$-linear Lie triple derivation δ that occurs in Theorem 3.3, then $\delta(I) \in \mathcal{Z}(\mathcal{X})$.*

Proof Let P be an idempotent in $\mathcal{X}$. We have that

$$0 = \delta([[I,P],P]) = [[\delta(I),P],P] = [\delta(I)P - P\delta(I),P] = \delta(I)P + P\delta(I) - 2P\delta(I)P.$$

Multiplying the above equation by P from the right side, we can obtain that $P\delta(I)P = \delta(I)P$. It means that $(I-P)\delta(I)P = 0$. Thus $P_1\delta(I)P_2 = P_2\delta(I)P_1 = 0$, it follows that $\delta(I) \in \mathcal{X}_{11} + \mathcal{X}_{22}$. By Lemma 3.1, we know that $\mathcal{X}_{11}$ is commutative, so $[\delta(I), A_{11}] = 0$ for every A_{11} in $\mathcal{X}_{11}$. In the following, we show that

$$[\delta(I), A_{22}] = [\delta(I), A_{12}] = [\delta(I), A_{21}] = 0$$

for every A_{22} in $\mathcal{X}_{22}$, A_{12} in $\mathcal{X}_{12}$ and A_{21} in $\mathcal{X}_{21}$.

For each A, B in $\mathcal{X}$, we have that

$$[[A,B],\delta(I)] = \delta([[A,B],I]) - [[A,\delta(B)],I] - [[\delta(A),B],I] = 0.$$

By $A_{12} = [P_1, A_{12}]$ and $A_{21} = [A_{21}, P_1]$, we have that

$$[\delta(I), A_{12}] = [\delta(I), A_{21}] = 0. \tag{3.2}$$

By (3.2), it follows that

$$0 = [\delta(I), A_{22}B_{21}] = [\delta(I), A_{22}]B_{21} + A_{22}[\delta(I), B_{21}] = [\delta(I), A_{22}]B_{21}$$

for every A_{22} in $\mathcal{X}_{22}$ and B_{21} in $\mathcal{X}_{21}$. By Lemma 3.2, we have that $[\delta(I), A_{22}]P_2 = 0$. By $\delta(I) \in \mathcal{X}_{11} + \mathcal{X}_{22}$, we can obtain that $[\delta(I), A_{22}] \in \mathcal{X}_{22}$, it follows that $[\delta(I), A_{22}] = 0$. Hence $\delta(I) \in \mathcal{Z}(\mathcal{X})$. □

Lemma 3.5 *Suppose that δ is the $\mathcal{A}$-linear Lie triple derivation δ that occurs in Theorem 3.3, then $P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 \in \mathcal{Z}(\mathcal{X})$.*

Proof By Lemma 3.1, we know that $P_1\delta(P_1)P_1 = aP_1$, where $a = f_0(P_1\delta(P_1)P_1x_0)$. For every x in $\mathcal{M}$, denote by $P_2\theta_{x,f_0}P_1 = A_{21}$, we have that

$$\begin{aligned}
-\delta(A_{21}) =& \delta([[P_2, A_{21}], P_2]) = [[\delta(P_2), A_{21}], P_2] + [[P_2, \delta(A_{21})], P_2] + [[P_2, A_{21}], \delta(P_2)] \\
=& - A_{21}\delta(P_2)P_2 - P_2\delta(P_2)A_{21} + A_{21}\delta(P_2) + 2P_2\delta(A_{21})P_2 - \\
& \delta(A_{21})P_2 - P_2\delta(A_{21}) + A_{21}\delta(P_2) - \delta(P_2)A_{21}.
\end{aligned} \tag{3.3}$$

Multiplying (3.3) by P_2 from the left side and by P_1 from the right side, we can obtain that

$$P_2\delta(P_2)A_{21} = A_{21}\delta(P_2)P_1.$$

That is

$$P_2\delta(P_2)P_2\theta_{x,f_0}P_1 = P_2\theta_{x,f_0}P_1\delta(P_2)P_1. \tag{3.4}$$

Both the two sides of (3.4) acting on x_0 in $\mathcal{M}$, we have that

$$f_0(P_1x_0)P_2\delta(P_2)P_2x = f_0(P_1\delta(P_2)P_1x_0)P_2x.$$

Since $f_0(P_1x_0) = f_0(x_0) = e$, it follows that

$$P_2\delta(P_2)P_2 = f_0(P_1\delta(P_2)P_1x_0)P_2. \tag{3.5}$$

By Lemma 3.4, we know that $\delta(I) \in \mathcal{Z}(\mathcal{X}) = \mathcal{A}I$. Since f_0 is $\mathcal{A}$-linear, we have that

$$P_2\delta(I)P_2 = \delta(I)P_2 = \delta(I)f_0(x_0)P_2 = f_0(\delta(I)x_0)P_2 = f_0(P_1\delta(I)P_1x_0)P_2.$$

Now replacing $\delta(P_2)$ by $\delta(I) - \delta(P_1)$ in (3.5), we can obtain that

$$P_2\delta(P_1)P_2 = f_0(P_1\delta(P_1)P_1x_0)P_2 = aP_2.$$

It implies that $P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 = a(P_1 + P_2) = aI$ belongs to $\mathcal{Z}(\mathcal{X})$.

Let $G = P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1$ and define a mapping Δ on $\mathcal{X}$ by

$$\Delta(A) = \delta(A) - [A, G]$$

for every A in $\mathcal{X}$. Obviously, Δ is also an $\mathcal{A}$-linear Lie triple derivation on $\mathcal{X}$. Moreover,

$$\Delta(P_1) = \delta(P_1) - [P_1, G] = P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2$$

and by Lemma 3.5, we know that $\Delta(P_1) \in \mathcal{Z}(\mathcal{X})$. □

In Lemmas 3.6–3.8, we show some properties of Δ.

Lemma 3.6 For every A_{ij} in $\mathcal{X}_{ij}$, we have $\Delta(A_{ij}) \subseteq \mathcal{X}_{ij}$, *where* $1 \le i, j \le 2$ *and* $i \ne j$.

Proof Since $\Delta(P_1) \in \mathcal{Z}(\mathcal{X})$, for each A_{12} in $\mathcal{X}_{12}$, we have that

$$\begin{aligned}\Delta(A_{12}) =&\Delta([[A_{12}, P_1], P_1])\\ =&[[\Delta(A_{12}), P_1], P_1] + [[A_{12}, \Delta(P_1)], P_1] + [[A_{12}, P_1], \Delta(P_1)]\\ =&[[\Delta(A_{12}), P_1], P_1]\\ =&P_1\Delta(A_{12})P_2 + P_2\Delta(A_{12})P_1.\end{aligned} \tag{3.6}$$

In the following, we show that $P_2\Delta(A_{12})P_1 = 0$.

Let B_{12} be in $\mathcal{X}_{12}$. Then $[A_{12}, B_{12}] = 0$. Thus

$$\begin{aligned}0 = \Delta(0) = \Delta([[A_{12}, B_{12}], C]) =&[[\Delta(A_{12}), B_{12}], C] + [[A_{12}, \Delta(B_{12})], C]\\ =&[[\Delta(A_{12}), B_{12}] + [A_{12}, \Delta(B_{12})], C]\end{aligned}$$

for every C in $\mathcal{X}$. It means that $J = [\Delta(A_{12}), B_{12}] + [A_{12}, \Delta(B_{12})] \in \mathcal{Z}(\mathcal{X})$. Since $A_{12} = [P_1, A_{12}]$, we have that

$$\begin{aligned}[\Delta(A_{12}), B_{12}] =&J - [A_{12}, \Delta(B_{12})] = J - [[P_1, A_{12}], \Delta(B_{12})]\\ =&J - (\Delta([[P_1, A_{12}], B_{12}]) - [[\Delta(P_1), A_{12}], B_{12}] - [[P_1, \Delta(A_{12})], B_{12}])\\ =&J + [[P_1, \Delta(A_{12})], B_{12}].\end{aligned}$$

By (3.6), we have that

$$\begin{aligned}[P_1\Delta(A_{12})P_2 + P_2\Delta(A_{12})P_1, B_{12}] =&J + [[P_1, P_1\Delta(A_{12})P_2 + P_2\Delta(A_{12})P_1], B_{12}]\\ =&J + [P_1\Delta(A_{12})P_2 - P_2\Delta(A_{12})P_1, B_{12}].\end{aligned}$$

Hence

$$[P_2\Delta(A_{12})P_1, B_{12}] = \frac{1}{2}J \in \mathcal{Z}(\mathcal{X}).$$

It follows from the Kleinecke-Shirokov Theorem [15, Problem 230], we know that $[P_2\Delta(A_{12})P_1, B_{12}]$ is a quasi-nilpotent element. Since $\mathcal{Z}(\mathcal{X}) = \mathcal{A}I$ is a commutative unital C^*-algebra, it is well known that $[P_2\Delta(A_{12})P_1, B_{12}] = 0$. Thus $P_2\Delta(A_{12})B_{12} = B_{12}\Delta(A_{12})P_1 = 0$ for every B_{12} in $\mathcal{X}_{12}$. By Lemma 3.2, we know that $P_2\Delta(A_{12})P_1 = 0$. Similarly, we have that $\Delta(\mathcal{A}_{21}) \subseteq \mathcal{X}_{21}$. □

Lemma 3.7 For every A_{11} in $\mathcal{X}_{11}$, we have $\Delta(A_{11}) \subseteq \mathcal{Z}(\mathcal{X})$ *for every* A_{11} *in* $\mathcal{X}_{11}$.

Proof For every A_{11} in $\mathcal{X}_{11}$, by Lemma 3.1, we have that

$$\Delta(A_{11}) = \Delta(P_1A_{11}P_1) = \Delta(f_0(A_{11}x_0)P_1) = f_0(A_{11}x_0)\Delta(P_1).$$

Since $\Delta(P_1) \in \mathcal{Z}(\mathcal{X})$ and $f_0(A_{11}x_0) \in \mathcal{A}$, it follows that $\Delta(A_{11}) \in \mathcal{Z}(\mathcal{X})$. □

Lemma 3.8 For every A_{22} in $\mathcal{X}_{22}$, we have $\Delta(A_{22}) - f_0(\Delta(A_{22})x_0)I \in \mathcal{X}_{22}$. *Particularly,* $\Delta(P_2) = f_0(\Delta(P_2)x_0)I$.

Proof Through simple calculation, we have that

$$0 = \Delta([[P_1, A_{22}], P_1]) = [[P_1, \Delta(A_{22})], P_1] = -P_1\Delta(A_{22})P_2 - P_2\Delta(A_{22})P_1.$$

It follows that $\Delta(A_{22}) \in \mathcal{X}_{11} + \mathcal{X}_{22}$. By Lemma 3.1, we can obtain that

$$\Delta(A_{22}) = P_1\Delta(A_{22})P_1 + P_2\Delta(A_{22})P_2 = f_0(\Delta(A_{22})x_0)P_1 + P_2\Delta(A_{22})P_2.$$

It menas that

$$\Delta(A_{22}) - f_0(\Delta(A_{22})x_0)I = -f_0(\Delta(A_{22})x_0)P_2 + P_2\Delta(A_{22})P_2 \in \mathcal{X}_{22}.$$

Since $\Delta(P_2) = \Delta(I) - \Delta(P_1) \in \mathcal{Z}(\mathcal{X})$, we have that

$$\Delta(P_2) - f_0(\Delta(P_2)x_0)I \in \mathcal{Z}(\mathcal{X}) \cap \mathcal{X}_{22} = \{0\}.$$

Thus $\Delta(P_2) = f_0(\Delta(P_2)x_0)I$. □

In the following, we prove Theorem 3.3.

Proof of Theorem 3.3 Define two mappings τ and D on $\mathcal{X}$ by

$$\tau(A) = f_0(P_1AP_1x_0)\Delta(P_1) + f_0(\Delta(P_2AP_2)x_0)I$$

and

$$D(A) = \Delta(A) - \tau(A)$$

for every A in $\mathcal{X}$. It is clear that τ is an $\mathcal{A}$-linear mapping from $\mathcal{X}$ into $\mathcal{Z}(\mathcal{X})$ and D is an $\mathcal{A}$-linear mapping on $\mathcal{X}$. Moreover, according to the previous lemmas and the definitions of τ and D , we have that

(1) $D(A_{ij}) = \Delta(A_{ij}) \in \mathcal{X}_{ij}$ for every A_{ij} in $\mathcal{X}_{ij}$, where $1 \le i, j \le 2$ and $i \ne j$;
(2) $D(P_1) = D(P_2) = D(I) = 0$;
(3) $D(A_{11}) = 0$ for every A_{11} in $\mathcal{X}_{11}$;
(4) $D(A_{22}) \in \mathcal{X}_{22}$ for every A_{22} in $\mathcal{X}_{22}$.

To prove that Δ is standard, it is sufficient to show that D is a derivation on $\mathcal{X}$ and $\tau([[A, B], C]) = 0$ for each A, B and C in $\mathcal{X}$.

In the following we show that $D(A_{ij}B_{sk}) = D(A_{ij})B_{sk} + A_{ij}D(B_{sk})$ for every A_{ij} in $\mathcal{X}_{ij}$ and B_{sk} in $\mathcal{X}_{sk}$, where $1 \le i, j, s, k \le 2$.

Since $D(\mathcal{X}_{ij}) \in \mathcal{X}_{ij}$, we have that $D(A_{ij}B_{sk}) = D(A_{ij})B_{sk} + A_{ij}D(B_{sk})$ for $j \ne s$. Thus we only need to prove the following 8 cases:

(1) $D(A_{11}B_{11}) = D(A_{11})B_{11} + A_{11}D(B_{11})$;
(2) $D(A_{11}B_{12}) = D(A_{11})B_{12} + A_{11}D(B_{12})$;
(3) $D(A_{12}B_{22}) = D(A_{12})B_{22} + A_{12}D(B_{22})$;
(4) $D(A_{21}B_{11}) = D(A_{21})B_{11} + A_{21}D(B_{11})$;
(5) $D(A_{22}B_{21}) = D(A_{22})B_{21} + A_{22}D(B_{21})$;
(6) $D(A_{22}B_{22}) = D(A_{22})B_{22} + A_{22}D(B_{22})$;

(7) $D(A_{12}B_{21}) = D(A_{12})B_{21} + A_{12}D(B_{21})$;

(8) $D(A_{21}B_{12}) = D(A_{21})B_{12} + A_{21}D(B_{12})$.

Since $D(A_{11}) = 0$ for every A_{11} in $\mathcal{X}_{11}$, the case (1) is trivial.

For each A, B in $\mathcal{X}$, by $\Delta(A) - D(A) = \tau(A) \in \mathcal{Z}(\mathcal{X})$, we have that $[\Delta(A), B] = [D(A), B]$. It follows that

$$\begin{aligned} D(A_{11}B_{12}) =&\Delta(A_{11}B_{12}) = -\Delta([[P_1, B_{12}], A_{11}]) \\ =& - [[P_1, \Delta(B_{12})], A_{11}] - [[P_1, B_{12}], \Delta(A_{11})] \\ =& - [\Delta(B_{12}), A_{11}] - [B_{12}, \Delta(A_{11})] \\ =&[A_{11}, D(B_{12})] + [D(A_{11}), B_{12}] \\ =&A_{11}D(B_{12}) + D(A_{11})B_{12} \end{aligned}$$

for each A_{11} in $\mathcal{X}_{11}$ and B_{12} in $\mathcal{X}_{12}$. Thus the case (2) holds. The cases (3), (4) and (5) are similar to the case (2), so we omit the proofs.

For every C_{21} in $\mathcal{X}_{21}$, according to the case (5), we have the following two equations:

$$D(A_{22}B_{22}C_{21}) = D(A_{22}B_{22})C_{21} + A_{22}B_{22}D(C_{21}) \tag{3.7}$$

and

$$\begin{aligned} D(A_{22}B_{22}C_{21}) =&D(A_{22})B_{22}C_{21} + A_{22}D(B_{22}C_{21}) \\ =&D(A_{22})B_{22}C_{21} + A_{22}D(B_{22})C_{21} + A_{22}B_{22}D(C_{21}) \end{aligned} \tag{3.8}$$

for each A_{22}, B_{22} in $\mathcal{X}_{22}$. Comparing (3.7) and (3.8), we have that

$$D(A_{22}B_{22})C_{21} = D(A_{22})B_{22}C_{21} + A_{22}D(B_{22})C_{21}.$$

It follows that $(D(A_{22}B_{22}) - D(A_{22})B_{22} - A_{22}D(B_{22}))C_{21} = 0$ for every C_{21} in $\mathcal{X}_{21}$. By Lemma 3.2 and $D(A_{22}) \in \mathcal{X}_{22}$, we know that

$$D(A_{22}B_{22}) - D(A_{22})B_{22} - A_{22}D(B_{22}) = 0.$$

Finally, we show the cases (7) and (8). Let A_{12} be in $\mathcal{X}_{12}$ and B_{21} be in $\mathcal{X}_{21}$. Through simple calculation, we can obtain that

$$\begin{aligned} &\Delta([[A_{12}, P_2], B_{21}]) - D([[A_{12}, P_2,], B_{21}]) \\ &= [[\Delta(A_{12}), P_2], B_{21}] + [[A_{12}, P_2], \Delta(B_{21})] - D([[A_{12}, P_2,], B_{21}]) \\ &= [\Delta(A_{12}), B_{21}] + [A_{12}, \Delta(B_{21})] - D[A_{12}, B_{21}] \\ &= [D(A_{12}), B_{21}] + [A_{12}, D(B_{21})] - D(A_{12}B_{21} - B_{21}A_{12}) \\ &= D(A_{12})B_{21} - B_{21}D(A_{12}) + A_{12}D(B_{21}) - D(B_{21})A_{12} - D(A_{12}B_{21}) + D(B_{21}A_{12}) \\ &= (D(A_{12})B_{21} + A_{12}D(B_{21}) - D(A_{12}B_{21})) + (D(B_{21}A_{12}) - B_{21}D(A_{12}) - D(B_{21})A_{12}). \end{aligned}$$

Since $\Delta([A_{12}, B_{21}]) - D([A_{12}, B_{21}])$ belongs to $\mathcal{Z}(\mathcal{X})$, by Lemma 2.1, we may assume that

$$\Delta([A_{12}, B_{21}]) - D([A_{12}, B_{21}]) = \lambda I$$

for some λ in $\mathcal{A}$. That is,

$$\begin{aligned}\lambda I =&(D(A_{12})B_{21} + A_{12}D(B_{21}) - D(A_{12}B_{21}))+\\ &(D(B_{21}A_{12}) - B_{21}D(A_{12}) - D(B_{21})A_{12}).\end{aligned} \tag{3.9}$$

Since $D(\mathcal{X}_{ij}) \in \mathcal{X}_{ij}$, it follows that

$$D(A_{12})B_{21} + A_{12}D(B_{21}) - D(A_{12}B_{21}) \in \mathcal{X}_{11}$$

and

$$D(B_{21}A_{12}) - B_{21}D(A_{12}) - D(B_{21})A_{12} \in \mathcal{X}_{22}.$$

Multiplying (3.9) by P_1 and P_2, respectively, from the right side, we obtain the following two equations:

$$D(A_{12}B_{21}) = D(A_{12})B_{21} + A_{12}D(B_{21}) - \lambda P_1 \tag{3.10}$$

and

$$D(B_{21}A_{12}) = B_{21}D(A_{12}) + D(B_{21})A_{12} + \lambda P_2. \tag{3.11}$$

By the case (2) and Eq. (3.10), we can obtain that

$$\begin{aligned}D(A_{12}B_{21}A_{12}) =&D(A_{12}B_{21})A_{12} + A_{12}B_{21}D(A_{12})\\ =&D(A_{12})B_{21}A_{12} + A_{12}D(B_{21})A_{12} - \lambda A_{12} + A_{12}B_{21}D(A_{12}).\end{aligned} \tag{3.12}$$

By the case (3) and Eq. (3.11), we can obtain that

$$\begin{aligned}D(A_{12}B_{21}A_{12}) =&D(A_{12})B_{21}A_{12} + A_{12}D(B_{21}A_{12})\\ =&D(A_{12})B_{21}A_{12} + A_{12}D(B_{21})A_{12} + A_{12}B_{21}D(A_{12}) + \lambda A_{12}.\end{aligned} \tag{3.13}$$

Comparing (3.12) and (3.13), we have that $\lambda A_{12} = 0$. Noticing that D is $\mathcal{A}$-linear, we can obtain $\lambda^2 P_1 = \lambda^2 P_2 = 0$ through multiplying (3.10) and (3.11) by λ, respectively. Hence $\lambda^2 = 0$. Since $\mathcal{A}$ is a commutative C^*-algebra, it is well known that $\lambda^2 = 0$ implies $\lambda = 0$. By (3.10) and (3.11), the cases (7) and (8) hold.

By the cases (1)–(8), it implies that D is a derivation immediately. Now we show that $\tau([[A, B], C]) = 0$ for each A, B and C in $\mathcal{X}$. Indeed,

$$\begin{aligned}\tau([[A, B], C]) =&\Delta([[A, B], C]) - D([[A, B], C])\\ =&[[\Delta(A), B], C] + [[A, \Delta(B)], C] + [[A, B], \Delta(C)] - D([[A, B], C])\\ =&[[D(A), B], C] + [[A, D(B)], C] + [[A, B], D(C)] - D([[A, B], C])\\ =&0.\end{aligned}$$

It follows that $\Delta(A) = D(A) + \tau(A)$ is a standard Lie triple derivation on $\mathcal{X}$. Define an $\mathcal{A}$-linear mapping on $\mathcal{X}$ by $d(A) = D(A) + [A, G]$ for every A in $\mathcal{X}$. Thus we have that

$$\delta(A) = \Delta(A) + [A, G] = D(A) + \tau(A) + [A, G] = d(A) + \tau(A),$$

where d is a derivation on $\mathcal{X}$ and τ is an $\mathcal{A}$-linear mapping from $\mathcal{X}$ into $\mathcal{Z}(\mathcal{X})$ such that $\tau([[A, B], C]) = 0$ for each A, B and C in $\mathcal{X}$. □

Remark 3.9 In [6], Benkovic supposed that $\mathcal{X}$ is a unital algebra with a nontrivial idempotent P_1 and $P_2 = I - P_1$, and denoted $P_i \mathcal{X} P_j$ by $\mathcal{X}_{ij}$ and $P_i A P_j$ by A_{ij} for every A in $\mathcal{X}$, where $1 \le i, j \le 2$. He showed that if

$$A_{22}\mathcal{X}_{21} = 0 \text{ or } \mathcal{X}_{12}A_{22} = 0 \text{ implies } A_{22} = 0,$$

and

$$A_{11}\mathcal{X}_{12} = 0 \text{ or } \mathcal{X}_{21}A_{11} = 0 \text{ implies } A_{11} = 0,$$

then every Lie triple derivation d on $\mathcal{X}$ is of the form $d = \Delta + \delta + \tau$, where Δ is a derivation on $\mathcal{X}$, δ is a Jordan derivation on $\mathcal{X}$ and τ is a linear mapping from $\mathcal{X}$ into its center $\mathcal{Z}(\mathcal{X})$ that vanishes on $[[\mathcal{X}, \mathcal{X}], \mathcal{X}]$.

In this paper, by Lemma 3.2, we know that

$$A_{22}\mathcal{X}_{21} = 0 \text{ or } \mathcal{X}_{12}A_{22} = 0 \text{ implies } A_{22} = 0.$$

But it is also a question that whether $A_{11}\mathcal{X}_{12} = 0$ or $\mathcal{X}_{21}A_{11} = 0$ implies that $A_{11} = 0$.

Acknowledgements We thank the referees for their time and comments.

References

[1] S. SAKAI. *Derivations of W^*-algebras*. Ann. Math., 1966, **83**: 273–279.

[2] E. CHRISTENSEN. *Derivations of nest algebras*. Math. Ann., 1977, **229**(2): 155–161.

[3] M. MATHIEU, A. VILLENA. *The structure of Lie derivations on C^*-algebras*. J. Funct. Anal., 2003, **202**(2): 504–525.

[4] W. CHEUNG. *Lie derivations of triangular algebra*. Linear Multilinear Algebra, 2003, **51**(3): 299–310.

[5] Fangyan LU. *Lie derivation of certain CSL algebras*. Israel J. Math., 2006, **155**: 149–156.

[6] D. BENKOVIČ. *Lie triple derivations of unital algebras with idempotents*. Linear Multilinear Algebra, 2015, **63**(1): 141–165.

[7] C. MIERS. *Lie triple derivations of von Neumann algebras*. Proc. Amer. Math. Soc., 1978, **71**(1): 57–61.

[8] Peisheng JI, Lin WANG. *Lie triple derivations of TUHF algebras*. Linear Algebra Appl., 2005, **403**: 399–408.

[9] Jianhua ZHANG, Baowei WU, Huaixin CAO. *Lie triple derivations of nest algebras*. Linear Algebra Appl., 2006, **416**(2-3): 559–567.

[10] Weiyan YU, Jianhua ZHANG. *Lie triple derivations of CSL algebras*. Internat. J. Theoret. Phys., 2013, **52**(6): 2118–2127.

[11] Pengtong LI, Deguang HAN, Waishing TANG. *Derivations on the algebra of operators in Hilbert C^*-modules*. Acta Math. Sin. (Engl. Ser.), 2012, **28**(8): 1615–1622.

[12] M. MOGHADAM, M. MIRI, A. JANFADA. *A note on derivations on the algebra of operators in Hilbert C^*-modules*. Mediterr. J. Math., 2016, **13**(3): 1167–1175.

[13] V. MANUILOV, E. TROITSKY. *Hilbert C^*-Modules*. American Mathematical Society, Providence, RI, 2005.

[14] Jun HE, Jiankui LI, Danjun ZHAO. *Derivations, local and 2-local derivations on some algebras of operators on Hilbert C^*-modules*. Mediterr. J. Math., 2017, **14**(6): Paper No. 230, 11 pp.

[15] P. HALMOS. *Hilbert Space Problem Book*. Springer-Verlag, New York-Berlin, 1982.