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Fuzzy Quasi-Pseudo-b-Metrics on Algebraic Structures

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Abstract In this paper, we mainly study fuzzy (quasi-)pseudo-b-metrics on algebraic structures. We mainly establish the following results: (1) If a group G with a topology τ, which is induced by a left invariant fuzzy quasi-pseudo-b-metric (resp., pseudo-b-metric) on the G, is a right topological group, then G with the topology τ is a paratopological group (resp., a topological group) and (2) If a semigroup S with a topology τ is induced by an invariant fuzzy quasi-pseudo-b-metric, then S with the topology τ is a topological semigroup.

Keywords left topological group; right topological group; semitopological group; paratopological group; topological group; topological semigroup; fuzzy quasi-pseudo-b-metric; invariant fuzzy (quasi-)pseudo-b-metric

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1. Introduction

In 1906, Maurice Fréchet extracted the commonalities of real number space, complex number space, vector space, function space and other basic spaces, and gave the definition of metric [1].

Definition 1.1 ([1]) *Let X be a nonempty set. A function $d : X \times X \to [0, +\infty)$ is a metric on X if, for all $x, y, z \in X$, the following conditions hold:*

(M1) $d(x, y) = 0$ *if and only if* $x = y$*;*

(M2) $d(x, y) = d(y, x)$*;*

(M3) $d(x, z) \leq d(x, y) + d(y, z)$*.*

Then (X, d) will be called metric space, which has a wide range of applications in analytics. In addition to compact spaces, metric spaces can be regarded as another important topological spaces. Metric spaces can give many topological concepts a proper visual description.

Metrics have been generalized in many ways. Czerwik in [2] introduced b-metric. Basically the triangularity condition of metrics is relaxed as follows:

Definition 1.2 ([2]) *Let X be a nonempty set and $k \geq 1$ be a given real number. A function $d : X \times X \to [0, +\infty)$ is a b-metric on X, if for all $x, y, z \in X$, the following conditions hold:*

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(b1) $d(x, y) = 0$ *if and only if* $x = y$;

(b2) $d(x, y) = d(y, x)$;

(b3) $d(x, z) \leq k[d(x, y) + d(y, z)]$.

The tripe (X, d, k) is said to be a b-metric space. Some examples of b-metric spaces and some fixed point theorems in b-metric spaces can be found in [3–5]. We also note that the class of b-metric spaces is larger than that of metric spaces, since every b-metric is a metric when $k = 1$. An example of a b-metric space that is not a metric space was given in [6].

On the other hand, after Zadeh has introduced in his famous paper [7] the concept of a fuzzy set, one of the important problems is to obtain an adequate notion of a fuzzy metric space. Kramosil and Michálek [8] reformulated successfully the notion of probabilistic metric space, introduced by Menger in 1942, in fuzzy context.

Definition 1.3 ([9]) *A binary operation* $* : [0, 1] \times [0, 1] \to [0, 1]$ *is called triangular norm (t-norm) if it satisfies the following conditions:*

(1) $a * b = b * a, \forall a, b \in [0, 1]$;

(2) $a * 1 = a, \forall a \in [0, 1]$;

(3) $(a * b) * c = a * (b * c), \forall a, b, c \in [0, 1]$;

(4) If $a \leq c$ *and* $b \leq d$, *with* $a, b, c, d \in [0, 1]$ *then* $a * b \leq c * d$.

Three paradigmatic examples of continuous t-norms are $\wedge, \cdot$ and $*_L$ (the Lukasiewicz t-norm), which are defined by $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ and $a *_L b = \max\{a + b - 1, 0\}$, respectively. One can easily show that $* \leq \wedge$ for every continuous t-norm $*$.

Definition 1.4 ([10, Definition 2.4]) *Let* X *be an arbitrary set and* $*$ *be a continuous t-norm. A fuzzy set* M *in* $X \times X \times [0, \infty)$ *is called fuzzy metric (in the sense of Kramosil and Michalek), if, for all* $x, y, z \in X$, *the following conditions hold:*

(M1) $M(x, y, 0) = 0$;

(M2) $M(x, y, t) = 1, \forall t > 0$ *if and only if* $x = y$;

(M3) $M(x, y, t) = M(y, x, t), \forall t \geq 0$;

(M4) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ *is left continuous and* $\lim_{t\to\infty} M(x, y, t) = 1$;

(M5) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s), \forall t, s \geq 0$.

Then the triple $(X, M, *)$ is said to be a fuzzy metric space. If in Definition 1.4, (M2) is relaxed as follows:

(M2′) $M(x, y, t) = 1, \forall t > 0$ if $x = y$.

Then the M is called a fuzzy pseudo-metric on X. If M only satisfies (M1), (M2), (M4) and (M5), then the M is called a fuzzy quasi-metric on X. If M only satisfies (M1), (M2′) (M4) and (M5), then the M is a fuzzy quasi-pseudo-metric [11].

If in Definition 1.4, (M5) is relaxed as follows:

Definition 1.5 ([12, Definition 6]) *A fuzzy-b-metric (in the sense of Kramosil and Michalek) is a triple* $(M, *, k)$, *where* X *is an arbitrary set,* $*$ *is a continuous t-norm,* $k \geq 1$ *is a given real number and* M *is a fuzzy set in* $X \times X \times [0, \infty)$ *such that for all* $x, y, z \in X$ *we have:*

(Mb1) $M(x, y, 0) = 0$;

(Mb2) $M(x, y, t) = 1, \forall\, t > 0$ *if and only if* $x = y$;

(Mb3) $M(x, y, t) = M(y, x, t), \forall\, t \geq 0$;

(Mb4) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ *is left continuous and* $\lim_{t\to\infty} M(x, y, t) = 1$;

(Mb5) $M(x, z, k(t + s)) \geqslant M(x, y, t) * M(y, z, s), \forall\, t, s \geqslant 0$.

The quadruple $(X, M, *, k)$ is said to be a fuzzy-b-metric space.

The class of fuzzy b-metric spaces is larger than the class of fuzzy metric spaces, since a fuzzy b-metric space is a fuzzy metric space when $k = 1$.

In Definition 1.5, if (Mb2) is replaced by:

(Mb2′) $M(x, y, t) = 1, \forall\, t > 0$ if $x = y$.

Then the M is called a fuzzy pseudo-b-metric. If M only satisfies (Mb1), (Mb2), (Mb4) and (Mb5), then the M is called a fuzzy quasi-b-metric.

Definition 1.6 ([12, Definition 25]) *A fuzzy-quasi-pseudo-b-metric (in the sense of Kramosil and Michalek) on a set* X *is a triple* $(M, *, k)$, *where* $*$ *is a continuous t-norm,* $k \geq 1$ *is a given real number and* M *is a fuzzy set in* $X \times X \times [0, \infty)$ *such that for all* $x, y, z \in X$ *we have:*

(Mqpb1) $M(x, y, 0) = 0$;

(Mqpb2) $M(x, y, t) = 1, \forall t > 0$ *if* $x = y$;

(Mqpb3) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ *is left continuous and* $\lim_{t\to\infty} M(x, y, t) = 1$;

(Mqpb4) $M(x, z, k(t + s)) \geqslant M(x, y, t) * M(y, z, s), \forall t, s \geqslant 0$.

The quadruple $(X, M, *, k)$ *is said to be a fuzzy-quasi-pseudo-b-metric space.*

Definition 1.7 *A fuzzy quasi-pseudo-b-metric* $(M, *, k)$ *on a semigroup* G *is left-invariant (resp., right-invariant) if* $M(x, y, t) = M(ax, ay, t)$ *(resp.,* $M(x, y,\ t) = M(xa, ya, t)$*) whenever* $a, x, y \in G$ *and* $t > 0$. *We say that* $(M, *, k)$ *is invariant if it is both left-invariant and right-invariant.*

Left-invariant, right-invariant and invariant fuzzy quasi-pseudo-metric are defined in a similar way.

Definition 1.8 ([13]) *Let* G *be an algebraic semigroup. Pick* $x \in G$. *The function* $\lambda_x : G \to G$ *defined by* $\lambda_x(g) = xg$ *is called the left translation of* G *by* x. *Similarly,* $\rho_x : G \to G$ *defined as* $\rho_x(g) = gx$ *is known as the right translation of* G *by* x.

A topological semigroup (G, τ) is an algebraic semigroup G with a topology τ that makes the multiplication in G jointly continuous. A paratopological group G is a topological semigroup such that G is an algebraic group. We say that a paratopological group (G, τ) is a topological group if the inverse is continuous, that is, if g^{-1} stands for the inverse element of $g \in G$, then the function $g \to g^{-1}$ from G onto G is continuous. (G, τ) is said to be a left (resp., right) topological group if the translations λ_x (resp., ρ_x) are continuous in (G, τ) for all $x \in G$.

In [13], Sánchez and Sanchis stated a number of theorems about fuzzy (quasi-)pseudo-metrizable algebraic structures. They proved that:

Theorem 1.9 ([13, Theorem 3.1]) *Suppose that* (G, τ) *is a left topological group whose topolo-*

gy τ is induced by a right invariant fuzzy quasi-pseudo-metric. Then (G,τ) is a paratopological group.

Theorem 1.10 ([13, Theorem 3.2]) *Suppose that G is a left topological group whose topology is induced by a right invariant fuzzy pseudo-metric. Then G is a topological group.*

In this paper, we consider the following question: Do Theorems 1.9 and 1.10 hold for a group G with a topology τ, which is induced by a left invariant fuzzy (quasi-) pseudo-b-metric?

The paper is organized as follows. In Section 2, we study the topological properties of fuzzy quasi-pseudo-b-metric spaces. Section 3 is devoted to studying the conditions for a left (right) topological group to become a topological group. In Section 4, we mainly study fuzzy (quasi-)pseudo-b-metrics on semigroups.

2. Topological properties of fuzzy quasi-pseudo-b-metric spaces

In this section, we show some topological properties of fuzzy quasi-pseudo-b-metric spaces.

Definition 2.1 ([12, Definition 9]) *Let $k \geq 1$ be a given real number. A function $f : \mathbb{R} \to \mathbb{R}$ is called k-nondecreasing if for $t < s$ we have $f(t) \leq f(ks)$.*

We can prove the following three conclusions by using the proof methods in [12, Example 8, Proposition 10 and Theorem 2.1], respectively.

Example 2.2 Let (X,d,k) be a (quasi-pseudo-)b-metric space. Let

$$M_d(x,y,t) = \begin{cases} \frac{t}{t+d(x,y)}, & \text{for all } x,y \in X \text{ and } t > 0; \\ 0, & \text{for all } x,y \in X \text{ and } t = 0. \end{cases}$$

Then $(X, M_d, *, k)$ is a fuzzy (quasi-pseudo-)b-metric space, for any continuous t-norm $*$, the so-called, the standard fuzzy (quasi-pseudo-)b-metric induced by d on X.

Proposition 2.3 *Let $(X,M,*,k)$ be a fuzzy quasi-pseudo-b-metric space. For all $x,y \in X$ the fuzzy quasi-pseudo-b-metric mapping $M(x,y,\cdot) : [0,\infty) \to [0,1]$ is k-nondecreasing.*

Proposition 2.4 *Let $(X,M,*,k)$ be a fuzzy quasi-pseudo-b-metric space. For $x \in X$, $r \in (0,1)$, $t > 0$, we define the ball $B(x,r,t) := \{y \in X : M(x,y,t) > 1-r\}$. Then*

$$\tau_M := \{T \subseteq X : x \in T \text{ iff } \exists\, t > 0, r \in (0,1) : B(x,r,t) \subseteq T\} \tag{2.1}$$

is a topology on X.

For a b-metric space (X,b,k) it is well known that the b-metric d can induce a topology τ_d on X, but there is an example [14] shows the ball $B(x,r) = \{y \in X : d(x,y) < r\}$, where $r > 0$, need not be an open set in τ_d. In fuzzy-b-metric spaces, this fact is also true as the following example shows.

Example 2.5 There is a fuzzy-b-metric space $(X,M,*,k)$ whose balls need not be open in the topology induced by M.

Proof Consider a fixed number $\varepsilon > 0$. For $X = \{0, 1, 2, \ldots\}$, let $d : X \times X \to [0, \infty)$ be defined by

$$d(0,1) = 1,\ d(0,m) = 1 + \varepsilon \text{ for } m \geq 2,$$

$$d(1,m) = \frac{1}{m},\ d(n,m) = \frac{1}{n} + \frac{1}{m} \text{ for } n \geq 2.$$

Then we extend d onto $X \times X$ by puting $d(n,n) = 0$ for any $n \geq 0$ and $d(m,n) = d(n,m)$ if $0 \leq m < n$. Define a fuzzy set M_d in $X \times X \times [0, \infty)$ by:

$$M_d(x,y,t) = \begin{cases} \frac{t}{t+d(x,y)}, & \text{for all } x, y \in X \text{ and } t > 0; \\ 0, & \text{for all } x, y \in X \text{ and } t = 0. \end{cases}$$

Then the ball $B(x, r, t) := \{y \in X : M(x, y, t) > 1 - r\}$ is not open.

According to the example on page 4 of reference [14], d is a b-metric, according to the Remark 2.5, M_d is a fuzzy b-metric. Let $r_1 = 1 - \frac{2t^2+2t+t\varepsilon}{2(t+1)(t+1+\varepsilon)}$. Now, note that $B(0, r_1, t) = \{0, 1\}$, while $B(1, r', t')$ contains infinitely many elements, for any $r' \in (0, 1)$ and $t' > 0$. Hence, none of $B(1, r', t')$ are contained in $B(0, r_1, t)$, which shows that $B(0, r_1, t)$ is not open. □

Example 2.5 shows that there is a fuzzy-b-metric space $(X, M, *, k)$ whose balls need not be open in the topology induced by M, but we have the following:

Proposition 2.6 *Let $(X, M, *, k)$ be a fuzzy quasi-pseudo-b-metric space. Then for $x \in X$, $r \in (0, 1)$, $t > 0$ the ball $B(x, r, t) := \{y \in X : M(x, y, t) > 1 - r\}$ is a neighbourhood of x.*

Proof Let $A = \{y \in B(x, r, t) | \exists r' \in (0, 1), t' > 0, B(y, r', t') \subseteq B(x, r, t)\}$. We will show that A is an open set in the topology τ_M.

For each $a \in A$, there are $r_a \in (0, 1)$ and $t_a > 0$ such that $B(a, r_a, t_a) \subseteq B(x, r, t)$. Consider the ball $B(a, r_a, \frac{t_a}{k})$. For $y \in B(a, r_a, \frac{t_a}{k})$, we have $M(a, y, \frac{t_a}{k}) > 1 - r_a$. Since $M(a, y, \frac{t_a}{k}) > 1 - r_a$ and $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous, there is $\frac{t_0}{k} \in (0, \frac{t_a}{k})$ such that

$$M(a, y, \frac{t_0}{k}) > 1 - r_a.$$

Let $r_0 = M(a, y, \frac{t_0}{k})$. For $r_0 > 1 - r_a$, there is $s \in (0, 1)$ such that $r_0 > 1 - s > 1 - r_a$. Since the binary operation $*$ is a continuous t-norm, there is $r_1 \in (0, 1)$ such that $r_0 * r_1 \geq 1 - s$. Now consider the ball $B(y, 1 - r_1, \frac{t_a - t_0}{k})$. We claim

$$B(y, 1 - r_1, \frac{t_a - t_0}{k}) \subset B(x, r, t).$$

For $\forall z \in B(y, 1 - r_1, \frac{t_a - t_0}{k})$, we have $M(y, z, \frac{t_a - t_0}{k}) > 1 - (1 - r_1) = r_1$. Therefore,

$$\begin{aligned} M(a, z, t_a) =& M(a, z, \frac{k(t_a - t_0 + t_0)}{k}) \geq M(a, y, \frac{t_0}{k}) * M(y, z, \frac{t_a - t_0}{k}) \\ \geq& r_0 * r_1 \geq 1 - s > 1 - r_a. \end{aligned}$$

Therefore, $z \in B(x, r_a, t_a)$, this implies that $B(y, 1 - r_1, \frac{t_a - t_0}{k}) \subseteq B(a, r_a, t_a) \subseteq B(x, r, t)$. Thus we have proved that A is an open set in the topology τ_M. □

Proposition 2.7 *Let $(X, M, *, k)$ be a fuzzy quasi-pseudo-b-metric space. For $x \in X$, $\mathscr{U} =$*

$\{B(x, \frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}_+\}$, where $B(x, \frac{1}{n}, \frac{1}{n}) = \{y \in X : M(x, y, \frac{1}{n}) > 1 - \frac{1}{n}\}$. Then $\mathscr{U}$ is a local neighbourhood base at x.

Proof Let $r \in (0, 1)$, $t > 0$, for any $x \in X$, by Proposition 2.6, $B(x, r, t)$ is a neighbourhood of x. Pick $n_1 \in \mathbb{N}_+$ such that $n_1 > \frac{1}{r}$, pick $n_2 \in \mathbb{N}$ such that $n_2 > \frac{k}{t}$. Let $n = \max\{n_1, n_2\}$, $s = t - \frac{k}{n}$, Then $t = k(\frac{1}{n} + \frac{s}{k})$. We have that

$$B(x, \frac{1}{n}, \frac{1}{n}) \subset B(x, r, t). \tag{2.2}$$

Indeed, for $\forall\, y \in B(x, \frac{1}{n}, \frac{1}{n})$, we have

$$\begin{aligned} M(x, y, t) =& M(x, y, k(\frac{1}{n} + \frac{s}{k})) \geq M(x, y, \frac{1}{n}) * M(y, y, \frac{s}{k}) \\ =& M(x, y, \frac{1}{n}) > 1 - \frac{1}{n} > 1 - r. \end{aligned} \tag{2.3}$$

Hence, $y \in B(x, r, t)$, this implies that $B(x, \frac{1}{n}, \frac{1}{n}) \subset B(x, r, t)$. $\square$

Remark 2.8 Let $(X, M, *, k)$ be a fuzzy quasi-pseudo-b-metric space such that (X, M) is a right topological group. Let e be the identity of X, $\mathscr{U} = \{B(e, \frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}_+\}$, where $B(e, \frac{1}{n}, \frac{1}{n}) = \{y \in X : M(x, y, \frac{1}{n}) > 1 - \frac{1}{n}\}$. Then $\mathscr{U}$ is a local neighborhood base at e.

Proposition 2.9 *Let (X, d, k) be a quasi-pseudo-b-metric space. Then the topology τ_d, induced by d, coincides with the topology τ_{M_d}, which is induced by the induced standard fuzzy quasi-pseudo-b-metric M_d.*

Proof Let $\mathscr{B} = \{B(x, r, t) : x \in X, r \in (0, 1), t \in (0, \infty)\}$, $\mathscr{D} = \{S_\epsilon(x) : x \in X, \epsilon \in (0, \infty)\}$, where $S_\epsilon(x) = \{y : d(x, y) < \epsilon\}$. Obviously, $\mathscr{B}$ and $\mathscr{D}$ are the base of τ_{M_d} and τ_d, respectively.

For every $B \in \mathscr{B}$, suppose $B = B(x, r, t)$, let $\epsilon = \frac{t}{1-r} - t$. We have

$$S_\epsilon(x) \subset B(x, r, t). \tag{2.4}$$

Indeed, for $\forall\, y \in S_\epsilon(x)$, $M(x, y, t) = \frac{t}{t+d(x,y)} > \frac{t}{t+(\frac{t}{1-r}-t)} = 1 - r$. Hence, $y \in B(x, r, t)$, Therefore, $S_\epsilon(x) \subset B(x, r, t)$.

On the other hand, for every $S_\epsilon \in \mathscr{D}$, suppose $S_\epsilon(x) = \{y : d(x, y) < \epsilon, \epsilon > 0\}$. Let $t \in (0, \infty)$, $0 < r_0 < 1 - \frac{t}{t+\epsilon}$. Then $\frac{t}{1-r_0} - t < \epsilon$. We have

$$B(x, r_0, t) \subset S_\epsilon(x). \tag{2.5}$$

Indeed, for $\forall\, y \in B(x, r_0, t)$, $M(x, y, t) = \frac{t}{t+d(x,y)} > 1 - r_0$, then we have $\frac{t}{1-r_0} - t > d(x, y)$, thus $d(x, y) < \epsilon$. Therefore, $y \in S_\epsilon(x)$, this implies that $B(x, r_0, t) \subset S_\epsilon(x)$. $\square$

3. Fuzzy quasi-pseudo-b-metrics on right (left) topological groups

In this section, we are concerned with the continuity of the operation on right fuzzy quasi-pseudo-b-metric topological groups. It provides a sufficient condition to obtain a paratopological group.

The following result is a well known internal characterization of a (para-)topological group.

Lemma 3.1 ([13, Theorem 2.1]) *Let G be a group with identity e and $\mathscr{U}$ a family of subsets of G containing e. Consider the following conditions.*

(1) For every $U, V \in \mathscr{U}$, there exists $W \in \mathscr{U}$ such that $W \subseteq U \cap V$;

(2) For every $U \in \mathscr{U}$ and $x \in U$, there is $V \in \mathscr{U}$ such that $Vx \subseteq U$;

(3) For every $U \in \mathscr{U}$ and $x \in G$, we can find $V \in \mathscr{U}$ satisfying $xVx^{-1} \subseteq U$;

(4) For every $U \in \mathscr{U}$, there exists $V \in \mathscr{U}$ such that $V^2 \subseteq U$;

(5) For every $U \in \mathscr{U}$, we can find $V \in \mathscr{U}$ with $V^{-1} \in U$.

If $\mathscr{U}$ satisfies (1)–(4), then the family $\{Ux : x \in G, U \in \mathscr{U}\}$ is a base for a topology $\tau_{\mathscr{U}}$ on G. With this topology, G is a paratopological group, and the family $\{xU : x \in G, U \in \mathscr{U}\}$ is a base for the same topology on G. In addition, if $\mathscr{U}$ satisfies (5), then $(G, \tau_{\mathscr{U}})$ is a topological group.

Theorem 3.2 *Suppose that (G, τ_M) is a right topological group whose topology τ_M is induced by a left invariant fuzzy quasi-pseudo-b-metric $(M, *, k)$. Then (G, τ_M) is a paratopological group.*

Proof Let e be the identify of G. According to Remark 2.8, $\mathscr{U} = \{B(e, \frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}_+\}$ is a local neighborhood base at e. Let $\tau_{\mathscr{U}}$ be the topology associated to the family $\mathscr{U}$. Now, we will show $\mathscr{U}$ satisfies conditions (1)–(4) in Lemma 3.1, that is, $(G, \tau_{\mathscr{U}})$ is a paratopological group.

Item (1) follows from the fact that $\mathscr{U}$ is a local neighborhood base at e in (G, τ_M). To prove item (2), take $n \in \mathbb{N}_+$ and $x \in B(e, \frac{1}{n}, \frac{1}{n})$. Since the right translation ρ_x is continuous at e and $\rho_x(e) = ex = x \in B(e, \frac{1}{n}, \frac{1}{n})$, there exists $m \in \mathbb{N}_+$ such that $\rho_x(B(e, \frac{1}{m}, \frac{1}{m})) = B(e, \frac{1}{m}, \frac{1}{m})x \subseteq B(e, \frac{1}{n}, \frac{1}{n})$. Thus, item (2) holds.

To show item (3), we first prove that for each $n \in \mathbb{N}_+$ and $x \in G$ there holds:

$$xB(e, \frac{1}{n}, \frac{1}{n}) = B(x, \frac{1}{n}, \frac{1}{n}). \tag{3.1}$$

Indeed, take $y \in B(e, \frac{1}{n}, \frac{1}{n})$. Since $(M, *, k)$ is left-invariant, we have

$$M(x, xy, \frac{1}{n}) = M(e, y, \frac{1}{n}) > 1 - \frac{1}{n}, \tag{3.2}$$

so that $xB(e, \frac{1}{n}, \frac{1}{n}) \subseteq B(x, \frac{1}{n}, \frac{1}{n})$.

For the other inclusion, pick $z \in B(x, \frac{1}{n}, \frac{1}{n})$. Since $(M, *, k)$ is left-invariant, we conclude that

$$M(e, x^{-1}z, \frac{1}{n}) = M(x, z, \frac{1}{n}) > 1 - \frac{1}{n}. \tag{3.3}$$

This proves that $x^{-1}z \in B(e, \frac{1}{n}, \frac{1}{n})$. Thus $z \in xB(x, \frac{1}{n}, \frac{1}{n})$, this implies that $B(x, \frac{1}{n}, \frac{1}{n}) \subseteq xB(e, \frac{1}{n}, \frac{1}{n})$.

Let us show item (3). Pick $n \in \mathbb{N}_+$ and $x \in G$. Note that every right translation is a homeomorphism. So, $B(e, \frac{1}{n}, \frac{1}{n})x$ is a neighborhood of x. Hence, there is an $m \in \mathbb{N}_+$ such that $B(x, \frac{1}{m}, \frac{1}{m}) \subseteq B(e, \frac{1}{n}, \frac{1}{n})x$. This and (3.1) imply

$$xB(e, \frac{1}{m}, \frac{1}{m})x^{-1} = B(x, \frac{1}{m}, \frac{1}{m})x^{-1} \subseteq B(e, \frac{1}{n}, \frac{1}{n}). \tag{3.4}$$

This proves item (3).

Finally, we show item (4). Choose $n \in \mathbb{N}_+$. Since the t-norm $*$ is continuous, there is $p \in \mathbb{N}_+$ such that for each $q \geq p$ we have $(1-\frac{1}{qk}) * (1-\frac{1}{qk}) > 1-\frac{1}{n}$. Put $m = \max\{p, 2nk+1\}$. Since $M(x,y,\cdot) : [0,\infty) \to [0,1]$ is k-nondecreasing and $\frac{1}{nk} > \frac{2}{m}$, we have:

$$M(e, yz, \frac{1}{n}) = M(e, yz, \frac{k}{nk}) \geq M(e, yz, \frac{2}{m}). \tag{3.5}$$

For each $y, z \in B(e, \frac{1}{mk}, \frac{1}{mk})$, the following inequalities hold.

$$\begin{aligned} M(e, yz, \frac{2}{m}) &= M(e, yz, \frac{k(1+1)}{mk}) \geq M(e, y, \frac{1}{mk}) * M(y, yz, \frac{1}{mk}) \\ &= M(e, y, \frac{1}{mk}) * M(e, z, \frac{1}{mk}) > (1-\frac{1}{mk}) * (1-\frac{1}{mk}) \\ &> 1-\frac{1}{n}. \end{aligned} \tag{3.6}$$

(3.5) and (3.6) imply that $B(e, \frac{1}{mk}, \frac{1}{mk})B(e, \frac{1}{mk}, \frac{1}{mk}) \subseteq B(e, \frac{1}{n}, \frac{1}{n})$. For $\mathscr{U}$ is a local neighborhood base at e, there is an $m_1 \in \mathbb{N}_+$, such that $B(e, \frac{1}{m_1}, \frac{1}{m_1}) \subseteq B(e, \frac{1}{mk}, \frac{1}{mk})$. Then we have $B(e, \frac{1}{m_1}, \frac{1}{m_1})B(e, \frac{1}{m_1}, \frac{1}{m_1}) \subseteq B(e, \frac{1}{n}, \frac{1}{n})$, this proves item (4).

By Lemma 3.1, $(G, \tau_{\mathscr{U}})$ is a paratopological group and $\{xB(e, \frac{1}{n}, \frac{1}{n}) : x \in G, n \in \mathbb{N}\}$ is a base for $\tau_{\mathscr{U}}$. Notice that Eq. (3.1) implies that $\{xB(e, \frac{1}{n}, \frac{1}{n}) : x \in G, n \in \mathbb{N}\}$ also is a base for τ_M so that $\tau_M = \tau_{\mathscr{U}}$. This shows that (G, τ_M) is a paratopological group. □

If we replace fuzzy quasi-pseudo-b-metrics by fuzzy pseudo-b-metrics in Theorem 3.2, we obtain a symmetric structure.

Theorem 3.3 *Suppose that (G, τ_M) is a right topological group whose topology τ_M is induced by a left invariant fuzzy pseudo-b-metric $(M, *, k)$. Then (G, τ_M) is a topological group.*

Proof It is obvious that every left invariant fuzzy pseudo-b-metric is a left invariant fuzzy quasi-pseudo-b-metric. By Theorem 3.2, (G, τ_M) is a paratopological group. We only need to prove that the family $\mathscr{U} = \{B(e, \frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}_+\}$ satisfies item (5) of Lemma 3.1. Choose $n \in \mathbb{N}_+$. If we take $x \in B(e, \frac{1}{n}, \frac{1}{n})$, then

$$M(e, x^{-1}, \frac{1}{n}) = M(x, e, \frac{1}{n}) = M(e, x, \frac{1}{n}) > 1-\frac{1}{n}. \tag{3.7}$$

We conclude that $x^{-1} \in B(e, \frac{1}{n}, \frac{1}{n})$. So $B(e, \frac{1}{n}, \frac{1}{n}) = B^{-1}(e, \frac{1}{n}, \frac{1}{n})$ for every $n \in \mathbb{N}_+$. Let's take any neighborhood U of y^{-1}, there is an $m \in \mathbb{N}_+$, such that $y^{-1}B(e, \frac{1}{m}, \frac{1}{m}) \subseteq U$. Then $B(e, \frac{1}{m}, \frac{1}{m})y$ is a neighborhood of y.

$$(B(e, \frac{1}{m}, \frac{1}{m})y)^{-1} = y^{-1}B^{-1}(e, \frac{1}{m}, \frac{1}{m}) = y^{-1}B(e, \frac{1}{m}, \frac{1}{m}) \subseteq U.$$

It follows that (G, τ_M) is a topological group. □

Corollary 3.4 ([13, Theorems 3.1 and 3.2]) *Suppose that (G, τ) is a right topological group whose topology τ is induced by a left invariant fuzzy (quasi-)pseudo-metric. Then (G, τ) is a (para-)topological group.*

Since one can easily show that the standard fuzzy quasi-pseudo-b-metric induced by a right (left) invariant quasi-pseudo-b-metric is right (left) invariant, by Proposition 2.9 and Theorems 3.2 and 3.3 we can obtain:

Corollary 3.5 *Suppose that (G, τ) is a left topological group whose topology τ is induced by a right invariant (quasi-)pseudo-b-metric. Then (G, τ) is a (para-)topological group.*

Arguing as in Theorems 3.2 and 3.3 we can obtain:

Theorem 3.6 *Suppose that G is a left topological group whose topology is induced by a right invariant fuzzy quasi-pseudo-b-metric $(M, *, k)$. Then G is a paratopological group. If in addition, $(M, *, k)$ is a right invariant fuzzy pseudo-b-metric then G is a topological group.*

Corollary 3.7 ([13, Theorems 3.3]) *Suppose that (G, τ) is a left topological group whose topology τ is induced by a right invariant fuzzy (quasi-)pseudo-metric. Then (G, τ) is a (para-)topological group.*

Recall that a semitopological group G is a group G with a topology that makes the multiplication separately continuous. Note that a semitopological group is both a left and right topological group. Thus from Theorems 3.2 and 3.3 it follows that:

Corollary 3.8 *Suppose that G is a semitopological group whose topology is induced by a left invariant fuzzy (quasi-)pseudo-b-metric. Then G is a (para-)topological group.*

It is easy to verify that the same conclusion holds if the left invariant fuzzy (quasi-)pseudo-b-metric is replaced by the right invariant fuzzy (quasi-)pseudo-b-metric in Corollary 3.8.

4. Fuzzy quasi-pseudo-b-metrics on semigroups

In this section, we mainly study fuzzy (quasi-)pseudo-b-metrics on semigroups.

Theorem 4.1 *Suppose that $(M, *, k)$ is an invariant fuzzy quasi-pseudo-b-metric on a semigroup S. Then (S, τ_M) is a topological semigroup, where τ_M is the topology induced by $(M, *, k)$.*

Proof Take $y, z \in S$, $n \in \mathbb{N}_+$. Since the *t-norm* $*$ is continuous, there is $p \in \mathbb{N}_+$ such that for each $q \geq p$ we have:

$$(1 - \frac{1}{qk}) * (1 - \frac{1}{qk}) > 1 - \frac{1}{n}. \tag{4.1}$$

Put $m = \max\{p, 2nk + 1\}$. Since $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is k-nondecreasing and $\frac{1}{nk} > \frac{2}{m}$. For $\forall\, a \in B(y, \frac{1}{mk}, \frac{1}{mk})$ and $\forall\, b \in B(z, \frac{1}{mk}, \frac{1}{mk})$, we have

$$\begin{aligned} M(yz, ab, \frac{1}{n}) &= M(yz, ab, \frac{k}{nk}) > M(yz, ab, \frac{2}{m}) = M(yz, ab, k(\frac{1}{mk} + \frac{1}{mk})) \\ &\geq M(yz, yb, \frac{1}{mk}) * M(yb, ab, \frac{1}{mk}) = M(z, b, \frac{1}{mk}) * M(y, a, \frac{1}{mk}) \\ &> (1 - \frac{1}{mk}) * (1 - \frac{1}{mk}) > 1 - \frac{1}{n}. \end{aligned} \tag{4.2}$$

Therefore, $B(y, \frac{1}{mk}, \frac{1}{mk})B(z, \frac{1}{mk}, \frac{1}{mk}) \subseteq B(yz, \frac{1}{n}, \frac{1}{n})$. We have proved that multiplication is continuous in (S, τ_M), i.e., (S, τ_M) is a topological semigroup. □

Corollary 4.2 *Suppose that d is an invariant quasi-pseudo-b-metric on a semigroup S. Then (S,τ_d) is a topological semigroup, where τ_d is the topology induced by d.*

Corollary 4.3 ([13, Theorem 3.10]) *Suppose that $(M,*)$ is an invariant fuzzy quasi-pseudo-metric on a semigroup S. Then (S,τ_M) is a topological semigroup, where τ_M is the topology induced by $(M,*)$.*

Let us recall that a monoid is a semigroup with a neutral element.

Theorem 4.4 *Suppose $(M,*,k)$ is a left-invariant fuzzy quasi-pseudo-b-metric on a monoid G such that for each $x\in G$, the left translation λ_x is open and the right translation ρ_x is continuous at the identity e of G. Then (G,τ_M) is a topological semigroup, where τ_M is the topology induced by $(M,*,k)$.*

Proof We claim that for each $n\in\mathbb{N}_+$ and $x\in G$ we have:

$$xB(e,\frac{1}{n},\frac{1}{n})\subseteq B(x,\frac{1}{n},\frac{1}{n}). \tag{4.3}$$

Indeed, take $y\in B(e,\frac{1}{n},\frac{1}{n})$. Since $(M,*,k)$ is left invariant, we have:

$$M(x,xy,\frac{1}{n})=M(e,y,\frac{1}{n})>1-\frac{1}{n}. \tag{4.4}$$

This proves (4.3). As a consequence of (4.3), we have that left translations are continuous at e. Now, we will show that for every $n\in\mathbb{N}_+$, there is $m\in\mathbb{N}_+$ satisfying

$$B(e,\frac{1}{mk},\frac{1}{mk})B(e,\frac{1}{mk},\frac{1}{mk})\subseteq B(e,\frac{1}{n},\frac{1}{n}). \tag{4.5}$$

Since the t-norm is continuous, there is $j\in\mathbb{N}_+$ such that for each $q\geq j$ we have that $(1-\frac{1}{qk})*(1-\frac{1}{qk})>1-\frac{1}{n}$. Put $m=\max\{j,2nk+1\}$. $M(x,y,\cdot):[0,\infty)\to[0,1]$ is k-nondecreasing and $\frac{1}{nk}>\frac{2}{m}$. Then for each $y,z\in B(e,\frac{1}{mk},\frac{1}{mk})$, the following inequalities hold:

$$\begin{aligned}M(e,yz,\frac{1}{n})&>M(e,yz,\frac{2}{m})=M(e,yz,\frac{2k}{mk})\geq M(e,y,\frac{1}{mk})*M(y,yz,\frac{1}{mk})\\&=M(e,y,\frac{1}{mk})*M(e,z,\frac{1}{mk})>(1-\frac{1}{mk})*(1-\frac{1}{mk})>1-\frac{1}{n}.\end{aligned} \tag{4.6}$$

Therefore,

$$B(e,\frac{1}{mk},\frac{1}{mk})B(e,\frac{1}{mk},\frac{1}{mk})\subseteq B(yz,\frac{1}{n},\frac{1}{n}).$$

Now, we will prove that the multiplication is continuous in (G,τ_M). Take $x,y\in G$ and $n\in\mathbb{N}_+$. By (4.3), we have $xyB(e,\frac{1}{n},\frac{1}{n})\subseteq B(xy,\frac{1}{n},\frac{1}{n})$. For some $m\in\mathbb{N}_+$, it follows from (4.5) that

$$B(e,\frac{1}{mk},\frac{1}{mk})B(e,\frac{1}{mk},\frac{1}{mk})\subseteq B(e,\frac{1}{n},\frac{1}{n}).$$

Hence

$$xyB(e,\frac{1}{mk},\frac{1}{mk})B(e,\frac{1}{mk},\frac{1}{mk})\subseteq xyB(e,\frac{1}{n},\frac{1}{n})\subseteq B(xy,\frac{1}{n},\frac{1}{n}). \tag{4.7}$$

By hypothesis, left translations are open. So, $yB(e,\frac{1}{mk},\frac{1}{mk})$ is a neighborhood of y. Also by hypothesis, ρ_y is continuous at e. Hence, there is $i\in\mathbb{N}$ satisfying

$$\rho_y(B(e,\frac{1}{i},\frac{1}{i}))=B(e,\frac{1}{i},\frac{1}{i})y\subseteq yB(e,\frac{1}{mk},\frac{1}{mk}). \tag{4.8}$$

Therefore, (4.7) and (4.8) imply

$$xB(e,\frac{1}{i},\frac{1}{i})yB(e,\frac{1}{mk},\frac{1}{mk}) \subseteq xyB(e,\frac{1}{mk},\frac{1}{mk})B(e,\frac{1}{mk},\frac{1}{mk}) \subseteq xyB(e,\frac{1}{n},\frac{1}{n}) \subseteq B(xy,\frac{1}{n},\frac{1}{n}). \tag{4.9}$$

Since left translations are open, $xB(e,\frac{1}{i},\frac{1}{i})$ and $yB(e,\frac{1}{mk},\frac{1}{mk})$ are neighborhoods of x and y, respectively. Therefore, multiplication in (G,τ_M) is continuous. This completes the proof. □

Corollary 4.5 *Suppose d is a left-invariant quasi-pseudo-b-metric on a monoid G such that for each $x \in G$, λ_x is open and ρ_x is continuous at the identity e of G. Then (G,τ_d) is a topological semigroup, where τ_d is the topology induced by d.*

Corollary 4.6 ([13, Theorem 3.11]) *Suppose $(M,*)$ is a left-invariant fuzzy quasi-pseudo-metric on a monoid G such that for each $x \in G$, λ_x is open and ρ_x is continuous at the identity e of G. Then (G,τ_M) is a topological semigroup, where τ_M is the topology induced by $(M,*)$.*

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