Journal of Mathematical Research with Applications
Jul., 2024, Vol. 44, No. 4, pp. 427–436
DOI:10.3770/j.issn:2095-2651.2024.04.001
Http://jmre.dlut.edu.cn

Enumerating Pattern-avoiding Fishburn Permutations Subject to Seven Statistics

Yujie DU, Philip B. ZHANG*
College of Mathematical Science, Tianjin Normal University, Tianjin 300387*, P. R. China*

Abstract Fishburn permutations are in bijection with several important combinatorial structures including interval orders. In this paper, we use the method of generating trees to enumerate two classes of pattern-avoiding Fishburn permutations subject to 7 classical statistics simultaneously. The classes of our interest are (321,312)-avoiding and (321,4123)-avoiding Fishburn permutations. The statistics of our interest are ascents, descents, inversions, right-to-left maxima, right-to-left minima, left-to-right maxima and left-to-right minima. Our results generalize a result by Egge.

Keywords Fishburn permutation; Fishburn number; pattern avoidance; generating tree

MR(2020) Subject Classification 05A15

1. Introduction

A permutation of length n is a rearrangement of the elements in set $[n] := \{1, 2, \dots, n\}$. Denote by S_n the set of permutations of $[n]$. A permutation $\pi_1\pi_2\cdots\pi_n \in S_n$ avoids a pattern $p = p_1p_2\cdots p_k \in S_k$ if there is no subsequence $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ such that $\pi_{i_j} < \pi_{i_m}$ if and only if $p_j < p_m$. Patterns p and q are Wilf-equivalent if the number of p-avoiding permutations of length n is equal to the number of q-avoiding permutations of length n for any $n \geq 1$.

The Fishburn pattern  is a special case of bivincular patterns introduced in [1]. A permutation $\pi = \pi_1\pi_2\cdots\pi_n$ avoids  if there is no $i < j$ such that $\pi_i\pi_{i+1}\pi_j$ is an occurrence of the pattern 231 and $\pi_i = \pi_j + 1$. Fishburn permutations are -avoiding permutations. We let $F_n(\sigma_1, \dots, \sigma_k)$ denote the set of Fishburn permutations of length n avoiding the patterns $\sigma_1, \dots, \sigma_k$ simultaneously. For example, $F_n(321, 4123)$ is the set of Fishburn permutations of length n avoiding the patterns 321 and 4123.

Bousquet-Mélou et al. [1] showed that -avoiding permutations are counted by the Fishburn numbers, and also related bijectively to ascent sequences, $(2+2)$-free posets and certain chord diagrams [2–5]. In follow-up works, other combinatorial objects enumerated by Fishburn numbers were found, namely, certain upper-triangular matrices, Fishburn trees and Fishburn covers [6–8].

Gil and Weiner [9] introduced classical pattern-avoidance into the study of Fishburn permutations. They presented enumerative results on Fishburn permutations avoiding classical patterns

Received September 13, 2023; Accepted January 21, 2024
Supported by the National Natural Science Foundation of China (Grant No. 12171362).
* Corresponding author
E-mail address: duyujiemath@163.com (Yujie DU); zhang@tjnu.edu.cn (Philip B. ZHANG)

of size 3, and observed the existence of at least 13 Wilf-equivalence classes for Fishburn permutations avoiding a classical pattern of size 4. Moreover, Gil and Weiner [9] gave a Wilf-equivalence class of Fishburn permutations with eight patterns that is enumerated by the Catalan numbers.

Based on Gil and Weiner's results, Egge [10] found a bijection between pattern-avoiding Fishburn permutations and pattern-avoiding ascent sequences. He also considered avoiding two or more classical patterns on Fishburn permutations, and proved several enumerative results about 123- and 321-avoiding Fishburn permutations. Moreover, Egge [10] proposed several enumerative conjectures. Two of these conjectures (about 321-avoiding Fishburn permutations) were settled in [11].

In this paper, we use the method of generating trees (see [12] for details of the method) to enumerate (321,312)-avoiding and (321,4123)-avoiding Fishburn permutations subject to seven classical statistics. This initiates a systematic study of distributions of permutation statistics on $(321, k12\cdots(k-1))$-avoiding Fishburn permutations. Our results are a multivariate generalization of the following generating function given by Egge [10],

$$\sum_{n=0}^{\infty}|F_n(321,4123)|x^n = \frac{1-x-x^2}{1-2x-x^2+x^3+x^4}. \tag{1.1}$$

Besides, it follows from our general results that $F_n(321,312)$ is counted by the Fibonacci numbers.

In this paper we study generating functions involving the following seven statistics (asc, des, inv, rlmax, rlmin, lrmax, lrmin). For $1 \le i \le n-1$, i is an ascent (resp., descent) if $\pi_i < \pi_{i+1}$ (resp., $\pi_i > \pi_{i+1}$) and $\mathrm{asc}(\pi)$ (resp., $\mathrm{des}(\pi)$) is the number of ascents (resp., descents) in π. An ordered pair (π_i, π_j) in a permutation $\pi = \pi_1\cdots\pi_n$ is called an inversion if $\pi_i > \pi_j$ for $1 \le i < j \le n$. The number of inversions in π is denoted by $\mathrm{inv}(\pi)$. Also, π_i is a right-to-left maximum (resp., right-to-left minimum) in π if π_i is greater (resp., smaller) than any element to its right. Note that π_n is always a right-to-left maximum and a right-to-left minimum. Denote by $\mathrm{rlmax}(\pi)$ and $\mathrm{rlmin}(\pi)$ the number of right-to-left maxima and right-to-left minima in π, respectively. We can define left-to-right maximum, left-to-right minimum, $\mathrm{lrmax}(\pi)$ and $\mathrm{lrmin}(\pi)$ in a similar way.

We let

$$S_n(p,q,s,t,u,v,w) = \sum_{\pi\in F_n(321,312)} p^{\mathrm{asc}(\pi)}q^{\mathrm{des}(\pi)}s^{\mathrm{inv}(\pi)}t^{\mathrm{rlmax}(\pi)}u^{\mathrm{rlmin}(\pi)}v^{\mathrm{lrmax}(\pi)}w^{\mathrm{lrmin}(\pi)},$$

$$T_n(p,q,s,t,u,v,w) = \sum_{\pi\in F_n(321,4123)} p^{\mathrm{asc}(\pi)}q^{\mathrm{des}(\pi)}s^{\mathrm{inv}(\pi)}t^{\mathrm{rlmax}(\pi)}u^{\mathrm{rlmin}(\pi)}v^{\mathrm{lrmax}(\pi)}w^{\mathrm{lrmin}(\pi)},$$

$$S(p,q,s,t,u,v,w,x) = \sum_{n=0}^{\infty} S_n(p,q,s,t,u,v,w)x^n,$$

$$T(p,q,s,t,u,v,w,x) = \sum_{n=0}^{\infty} T_n(p,q,s,t,u,v,w)x^n.$$

In this paper we prove the following two theorems.

Theorem 1.1 *For $F_n(321, 312)$, we have*

$$S(p,q,s,t,u,v,w,x) = \frac{1 - uv(p - tw)x - qsuv(p - t^2w^2)x^2 - pqs(1-t)tu^2v^2(1-w)wx^3}{1 - puvx - pqsuvx^2}. \quad (1.2)$$

In particular, letting $p = q = s = t = u = v = w = 1$, we have

$$S(x) = \frac{1}{1 - x - x^2},$$

so that $F_n(321, 312)$ is counted by the Fibonacci numbers.

Theorem 1.2 *For $F_n(321, 4123)$, we have*

$$T(p,q,s,t,u,v,w,x) = \frac{B}{A}, \quad (1.3)$$

where

$$A = 1 - p(s+u)vx + psuv(-q - qs + pv)x^2 + p^2qs^2uv(-u + v + uv)x^3 + p^3qs^3u^2v^2x^4$$

and

$$\begin{aligned} B =& 1 - v(ps + pu - tuw)x + suv(-pq - pqs + p^2v - ptvw + qt^2w^2)x^2 - \\ & pqsuv(psu - psv - psuv + tuvw + stuvw - t^2uvw - st^2uw^2 + st^2vw^2 - tuvw^2 + t^2uvw^2)x^3 + \\ & p^2qs^2u^2v^2(ps - tuw + t^2uw + tvw - t^2vw - st^2w^2 + tuw^2 - t^2uw^2 - tvw^2 + t^2vw^2)x^4 + \\ & p^3qs^3(1-t)tu^3v^3(1-w)wx^5. \end{aligned}$$

In particular, letting $p = q = s = t = u = v = w = 1$, we obtain (1.1).

In what follows we need the following notion and notation. An active site for a permutation π in $F_n(321, 312)$ (resp., $F_n(321, 4123)$) is the space to the left or to the right of π or between two consecutive elements in π such that the permutation obtained by inserting $n+1$ into this space belongs to $F_{n+1}(321, 312)$ (resp., $F_{n+1}(321, 4123)$). As in [10], we say that π has label (x) if π is in case (x). We shall also label the active sites. If the permutation we obtain by inserting $n+1$ into an active site has label (x), then we use (x) as the superscript of this site.

For each label (y), we write

$$\begin{aligned}[\mathrm{y}]_n :=& [\mathrm{y}]_n(p,q,s,t,u,v,w) \\ =& \begin{cases} \sum p^{\mathrm{asc}(\pi)} q^{\mathrm{des}(\pi)} s^{\mathrm{inv}(\pi)} t^{\mathrm{rlmax}(\pi)} u^{\mathrm{rlmin}(\pi)} v^{\mathrm{lrmax}(\pi)} w^{\mathrm{lrmin}(\pi)}, & \text{if } n \geq 1, \\ 0, & \text{if } n \leq 0, \end{cases}\end{aligned}$$

where the sum is taken over $\pi \in F_n(321, 312)$ (resp., $\pi \in F_n(321, 4123)$) such that label$(\pi) =$ (y). Finally, let

$$[\mathrm{y}] := [\mathrm{y}](p,q,s,t,u,v,w,x) = \sum_{n=1}^{\infty} [\mathrm{y}]_n(p,q,s,t,u,v,w)x^n.$$

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by using generating trees. We first classify the permutations in $F_n(321, 312)$ according to the position of the largest element n.

Proposition 2.1 *For* $n \geq 1$ *and* $\pi = \pi_1\pi_2\cdots\pi_n \in F_n(321, 312)$, *exactly one of the following cases holds.*

(1a) $\pi = 1$.

(1b) $\pi = \cdots n$, *where* $n \geq 2$.

(2) $\pi = \cdots n(n-1)$, *where* $n \geq 2$.

Proof For $n = 1$, we have $\pi = 1$, which belongs to (1a). For $n \geq 2$, n can only appear in the rightmost two positions. Indeed, suppose that x and y are to the right of n. If $x > y$ (resp., $x < y$) then nxy is an occurrence of the forbidden pattern 321 (resp., 312), which is a contradiction.

If $\pi_n = n$, then π belongs to case (1b). If $\pi_{n-1} = n$, then $\pi_n = n - 1$. Indeed, suppose that $\pi_n \neq n - 1$ and the element immediately to the right of $\pi_n + 1 < n$ is x. If $x > \pi_n + 1$, then $(\pi_n + 1)x\pi_n$ is an occurrence of the pattern . On the other hand, if $x < \pi_n$, then $(\pi_n + 1)x\pi_n$ is an occurrence of the pattern 312. Hence, $\pi_n = n - 1$ and π belongs to case (2). □

In the following proposition, we label the active sites for each class of permutations.

Proposition 2.2 *For* $n \geq 1$, *we have*

(1a) If π *has label (1a), then* $\pi = {}^{2}\ 1\ {}^{1\text{b}}$;

(1b) If π *has label (1b), then* $\pi = \cdots\ {}^{2}\ n\ {}^{1\text{b}}$;

(2) If π *has label (2), then* $\pi = \cdots n\ n-1\ {}^{1\text{b}}$.

Proof (1a) If $\pi = 1$, we consider the insertion of 2. Clearly, the permutations 21 and 12 belong to cases (2) and (1b), respectively.

(1b) Suppose the element immediately to the left of n is x. The element $n + 1$ cannot be inserted into the sites to the left of x since $(n + 1)xn$ would be an occurrence of 312. The remaining two sites are both active, which are labeled as (2) and (1b), respectively.

(2) In this case, n and $n - 1$ form a descent. To avoid 321, the element $n + 1$ cannot be to the left of n. Likewise, to avoid the pattern , the element $n + 1$ cannot be inserted into the site between n and $n - 1$. Thus, there is only one active site to the right of $n - 1$, which is labeled as (1b). □

It is easy to get the generating tree for $F_n(321, 312)$ by Proposition 2.2.

Corollary 2.3 *The generating tree for* $F_n(321, 312)$ *is as follows.*

Root: (1a)

Rules: (1a) → (2), (1b)

(1b) → (2), (1b)

(2) → (1b)

We next prove Theorem 1.1 using generating trees technique.

Proof of Theorem 1.1 First, we give the following expression and recurrence relations.

$$[1\mathrm{a}]_n = \begin{cases} tuvw, & \text{if } n = 1; \\ 0, & \text{if } n \geq 2. \end{cases} \tag{2.1}$$

$$[1\mathrm{b}]_n = puv[1\mathrm{a}]_{n-1} + puv[1\mathrm{b}]_{n-1} + pt^{-1}uv[2]_{n-1}. \tag{2.2}$$

$$[2]_n = qstw[1\mathrm{a}]_{n-1} + qst[1\mathrm{b}]_{n-1}. \tag{2.3}$$

The expression for label (1a) is easy to derive. By Corollary 2.3, a permutation of length n with label (1b) can be obtained from those permutations of length $n-1$ with labels (1a), (1b) and (2). For labels (1a) and (1b), the insertion of n results in that each statistic in $\{\mathrm{asc}, \mathrm{rlmin}, \mathrm{lrmax}\}$ is increased by 1 and the remaining statistics are unchanged. For label (2), the insertion of n results in that each statistic in $\{\mathrm{asc}, \mathrm{rlmin}, \mathrm{lrmax}\}$ is increased by 1, rlmax is decreased by 1, and the remaining statistics are unchanged. The recurrence relation for label (2) can be obtained in a similar way.

By definition, we get $[1\mathrm{a}] = tuvwx$. Multiplying both sides of (2.2) and (2.3) by x^n and summing over all n we get

$$[1\mathrm{b}] = ptu^2v^2wx^2 + puvx[1\mathrm{b}] + pt^{-1}uvx[2],$$

$$[2] = qst^2uvw^2x^2 + qstx[1\mathrm{b}].$$

Solving this linear system of equations yields

$$[1\mathrm{b}] = \frac{ptu^2v^2wx^2 + pqstu^2v^2w^2x^3}{1 - puvx - pqsuvx^2},$$

$$[2] = \frac{qst^2uvw^2x^2 + pqst^2u^2v^2(1-w)wx^3}{1 - puvx - pqsuvx^2}.$$

Thus, we have

$$S(p,q,s,t,u,v,w,x) = 1 + [1\mathrm{a}] + [1\mathrm{b}] + [2]$$
$$= \frac{1 - uv(p - tw)x - qsuv(p - t^2w^2)x^2 - pqs(1-t)tu^2v^2(1-w)wx^3}{1 - puvx - pqsuvx^2}.$$

This completes the proof. □

3. Proof of Theorem 1.2

The main goal of this section is to prove Theorem 1.2. First of all, we analyze the possible positions of the largest element n.

Lemma 3.1 *For $n \geq 3$ and $\pi \in F_n(321, 4123)$, one of the following three cases holds.*

(1) $\pi_{n-2} = n$.

(2) $\pi_{n-1} = n$.

(3) $\pi_n = n$.

Proof We prove the lemma by contradiction. Suppose there are at least three elements to the right of n. First, the 321-avoiding condition implies that the elements to the right of n are in increasing order. Thus we can find an increasing sequence of length 3 to the right of n resulting in an occurrence of 4123 (involving n), which is not possible. Thus n must appear in the rightmost three positions in π. This completes the proof. □

Based on Lemma 3.1, we classify $F_n(321, 4123)$ into the following 9 cases according to the rightmost three elements.

Proposition 3.2 *Suppose $n \geq 1$ and $\pi \in F_n(321, 4123)$. Then exactly one of the following cases holds.*

(1a) $\pi = 1$.

(1b) $\pi = 12$.

(1c) $n \geq 3$, $\pi = \cdots x(n-1)n$ *with* $x < n - 1$.

(1d) $n \geq 3$, $\pi = \cdots xyn$ *with* $x < y < n - 1$.

(1e) $n \geq 3$, $\pi = \cdots x(x-1)n$.

(1f) $n \geq 5$, $\pi = \cdots xyn$ *with* $x > y + 1$.

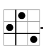
(2a) $n \geq 2$, $\pi = \cdots n(n-1)$.

(2b) $n \geq 4$, $\pi = \cdots nk$ *with* $2 \leq k \leq n - 2$.

(3) $n \geq 3$, $\pi = \cdots nx(n-1)$.

Proof If $n = 1$, then $\pi = 1$ and it belongs to case (1a). For $n = 2$, $\pi = 12$ or $\pi = 21$. The former permutation is in case (1b) and the latter is in case (2a). For $n \geq 3$, by Lemma 3.1, $n \in \{\pi_{n-2}, \pi_{n-1}, \pi_n\}$.

Let $\pi_n = n$. If $\pi_{n-1} = n - 1$, then π belongs to case (1c). If $\pi_{n-2} = \pi_{n-1} + 1$, then π belongs to case (1e). Otherwise, π is in case (1d) for $\pi_{n-2} < \pi_{n-1} < n - 1$ and in case (1f) for $\pi_{n-2} > \pi_{n-1} + 1$.

Let $\pi_{n-1} = n$. If $\pi_n = n - 1$, then π is in case (2a). Otherwise, π is in case (2b) with $n \geq 4$. Here $\pi_n \neq 1$ due to the -avoiding.

For $\pi_{n-2} = n$, we have $\pi_n = n - 1$. Indeed, if $\pi_{n-1} = n - 1$, then $n(n-1)\pi_n$ is an occurrence of 321. If $\pi_i = n - 1$ for $i < n - 3$, then there are at least 3 elements to the right of $n - 1$, which are smaller than $n - 1$. Then $n - 1$ is involved either in an occurrence of 321 or of 4123. Thus $n - 1$ is immediately to the left of n (namely, $\pi_{n-3} = n - 1$) and the Fishburn condition forces that $n - 2$ is immediately to the left of $n - 1$, and so forth until considering the element $n - \ell$. In other words, π is of the following form:

$$\pi = \cdots (n - \ell - 1) \cdots (n - \ell)(n - \ell + 1) \cdots (n - 2)(n - 1)n\pi_{n-1}\pi_n.$$

There are at least 3 elements less than $n-\ell-1$ to its right, so $n-\ell-1$ is involved in an occurrence of 321 or 4123, which is impossible. Hence, $\pi_n = n-1$ and π is in case (3).

This completes the classification of $F_n(321, 4123)$. □

Next we label the active sites for each class of permutations in $F_n(321, 4123)$.

Proposition 3.3 *For $n \geq 1$ and $\pi \in F_n(321, 4123)$, we have*

(1a) If π has label (1a), *then* $\pi = {}^{\mathrm{2a}}\ 1\ {}^{\mathrm{1b}}$;

(1b) If π has label (1b), *then* $\pi = {}^{3}\ 1\ {}^{\mathrm{2a}}\ 2\ {}^{\mathrm{1c}}$;

(1c) If π has label (1c), *then* $\pi = \cdots x\ {}^{3}\ n-1\ {}^{\mathrm{2a}}\ n\ {}^{\mathrm{1c}}$;

(1d) If π has label (1d), *then* $\pi = \cdots x\ {}^{3}\ y\ {}^{\mathrm{2a}}\ n\ {}^{\mathrm{1c}}$;

(1e) If π has label (1e), *then* $\pi = \cdots x\ x-1\ {}^{\mathrm{2a}}\ n\ {}^{\mathrm{1c}}$;

(1f) If π has label (1f), *then* $\pi = \cdots x\ {}^{3}\ y\ {}^{\mathrm{2a}}\ n\ {}^{\mathrm{1c}}$;

(2a) If π has label (2a), *then* $\pi = \cdots n\ n-1\ {}^{\mathrm{1e}}$;

(2b) If π has label (2b), *then* $\pi = \cdots n\ {}^{\mathrm{2b}}\ k\ {}^{\mathrm{1f}}$;

(3) If π has label (3), *then* $\pi = \cdots n\ x\ {}^{\mathrm{2b}}\ n-1\ {}^{\mathrm{1d}}$.

Proof We consider permutations π' obtained by inserting $n+1$ into π.

(1a) If π has label (1a), then $\pi = 1$. If 2 is to the left of 1, then $\pi' = 21$, which is in case (2a). If 2 is to the right of 1, then $\pi' = 12$, which has label (1b).

(1b) In this case, $\pi = 12$. There are three active sites to insert 3. The permutation 312 has label (3), 132 has label (2a), and 123 has label (1c).

(1c) Clearly, the sites before x are not active since if we insert $n+1$ there, we would create an occurrence of 4123 involving $n+1$. It is easy to check that the remaining three sites are all active and their labels are (3), (2a), (1c) from left to right.

(1d) We can analyze the active sites of label (1d) in the same way as (1c).

(1e) There is a consecutive decreasing pair $x(x-1)$ in this case. So $n+1$ cannot be to the left of $x(x-1)$ to avoid the pattern 321 and cannot be inserted between x and $x-1$ to avoid the Fishburn pattern . If $n+1$ is to the left of n, then we have label (2a). If $n+1$ is to the right of n, then we have label (1c).

(1f) The insertion of $n+1$ to the left of x and y with $x > y+1$ would create an occurrence of 321. So the sites to the left of x are not active. The sites to the right of x are all active and it is easy to check what their labels are.

(2a) In this case, $n(n-1)$ is a consecutive decreasing pair. The element $n+1$ can only be to the right of $n-1$ to avoid 321 and , and thus we have label (1e).

(2b) Similarly to the above, n and k form a descent. To avoid 321, the element $n+1$ cannot appear to the left of n. The remaining two sites are all active and their labels are (2b) and (1f), respectively.

(3) The Fishburn condition forces that $n+1$ cannot be immediately to the right of n and the requirement to avoid 321 forces that $n+1$ cannot be to the left of n. If $n+1$ is immediately to the left of $n-1$, then we have label (2b). The elements x and $n-1$ form an ascent and inserting $n+1$ to the right of $n-1$ produces π' with label (1d).

We have accomplished the analysis of active sites for all cases in $F_n(321, 4123)$. □

Corollary 3.4 *The generating tree for* $F_n(321, 4123)$ *is as follows.*

Root: *(1a)*

Rules: *(1a)* $\to$ *(2a), (1b)*

(1b) $\to$ *(3), (2a), (1c)*

(1c) $\to$ *(3), (2a), (1c)*

(1d) $\to$ *(3), (2a), (1c)*

(1e) $\to$ *(2a), (1c)*

(1f) $\to$ *(3), (2a), (1c)*

(2a) $\to$ *(1e)*

(2b) $\to$ *(2b), (1f)*

(3) $\to$ *(2b), (1d)*

We proceed to prove Theorem 1.2 using the generating tree we obtained above.

Proof of Theorem 1.2 We have the following expressions and recurrence relations.

$$[1\mathrm{a}]_n = \begin{cases} tuvw, & \text{if } n = 1; \\ 0, & \text{if } n \geq 2. \end{cases} \tag{3.1}$$

$$[1\mathrm{b}]_n = \begin{cases} ptu^2v^2w, & \text{if } n = 2; \\ 0, & \text{if } n \neq 2. \end{cases} \tag{3.2}$$

$$[1\mathrm{c}]_n = puv[1\mathrm{b}]_{n-1} + puv[1\mathrm{c}]_{n-1} + puv[1\mathrm{d}]_{n-1} + puv[1\mathrm{e}]_{n-1} + puv[1\mathrm{f}]_{n-1}. \tag{3.3}$$

$$[1\mathrm{d}]_n = pt^{-1}uv[3]_{n-1}. \tag{3.4}$$

$$[1\mathrm{e}]_n = pt^{-1}uv[2\mathrm{a}]_{n-1}. \tag{3.5}$$

$$[1\mathrm{f}]_n = pt^{-1}uv[2\mathrm{b}]_{n-1}. \tag{3.6}$$

$$[2\mathrm{a}]_n = qstw[1\mathrm{a}]_{n-1} + qst[1\mathrm{b}]_{n-1} + qst[1\mathrm{c}]_{n-1} + qst[1\mathrm{d}]_{n-1} + qst[1\mathrm{e}]_{n-1} + qst[1\mathrm{f}]_{n-1}. \tag{3.7}$$

$$[2\mathrm{b}]_n = psv[2\mathrm{b}]_{n-1} + qsv[3]_{n-1}. \tag{3.8}$$

$$[3]_n = qs^2tv^{-1}w[1\mathrm{b}]_{n-1} + qs^2tv^{-1}[1\mathrm{c}]_{n-1} + qs^2t[1\mathrm{d}]_{n-1} + ps^2t[1\mathrm{f}]_{n-1}. \tag{3.9}$$

The formulas with labels (1a) and (1b) can be obtained directly.

For the other recurrence relations, we only explain (1c) as the remaining cases are similar. By Corollary 3.4, we can get a permutation of length n with label (1c) from those of length $n-1$ with labels (1b), (1c), (1d), (1e), (1f). We just need to determine the coefficients with these labels. For a permutation with label (1b), inserting 3 to get a permutation with label (1c) results in that three statistics (asc, rlmin, lrmax) are increased by one and the other statistics do not change. So we multiply $[1\text{b}]_{n-1}$ by puv. Similarly, it holds for permutations with labels (1c), (1d), (1e) and (1f).

The generating functions for labels (1a) and (1b) are easy to derive.

$$[1\text{a}] = \sum_{n=1}^{\infty}[1\text{a}]_n x^n = tuvwx,$$

$$[1\text{b}] = \sum_{n=1}^{\infty}[1\text{b}]_n x^n = ptu^2v^2wx^2.$$

For other labels, multiplying both sides of (3.3)–(3.9) by x^n and summing over all n, we get

$$[1\text{c}] = p^2tu^3v^3wx^3 + puvx[1\text{c}] + puvx[1\text{d}] + puvx[1\text{e}] + puvx[1\text{f}],$$

$$[1\text{d}] = pt^{-1}uvx[3],$$

$$[1\text{e}] = pt^{-1}uvx[2\text{a}],$$

$$[1\text{f}] = pt^{-1}uvx[2\text{b}],$$

$$[2\text{a}] = qst^2uvw^2x^2 + pqst^2u^2v^2wx^3 + qstx[1\text{c}] + qstx[1\text{d}] + qstx[1\text{e}] + +qstx[1\text{f}],$$

$$[2\text{b}] = psvx[2\text{b}] + qsvx[3],$$

$$[3] = pqs^2t^2u^2vw^2x^3 + qs^2tv^{-1}x[1\text{c}] + qs^2tx[1\text{d}] + ps^2tx[1\text{f}].$$

The above equations form a linear system, which can be solved by standard methods of linear algebra. We get

$$\begin{aligned}
[1\text{c}] =& \frac{1}{A}p^2tu^3v^3wx^3(1 - s(pv - qw)x - pqs^2(uv - uw + vw)x^2 - p^2qs^3uvwx^3),\\
[1\text{d}] =& \frac{1}{A}p^2qs^2tu^3v^2wx^4(1 - psvx)(w + puv(1 - w)x),\\
[1\text{e}] =& \frac{1}{A}pqstu^2v^2wx^3(w + puv(1 - w)x)(1 - psvx - pqs^2uvx^2),\\
[1\text{f}] =& \frac{1}{A}p^2q^2s^3tu^3v^3wx^5(w + puv(1 - w)x),\\
[2\text{a}] =& \frac{1}{A}qst^2uvwx^2(w + puv(1 - w)x)(1 - psvx - pqs^2uvx^2),\\
[2\text{b}] =& \frac{1}{A}pq^2s^3t^2u^2v^2wx^4(w + puv(1 - w)x),\\
[3] =& \frac{1}{A}pqs^2t^2u^2vwx^3(1 - psvx)(w + puv(1 - w)x).
\end{aligned}$$

Therefore, we have

$$T(p, q, s, t, u, v, w, x) = 1 + [1\text{a}] + [1\text{b}] + [1\text{c}] + [1\text{d}] + [1\text{e}] + [1\text{f}] + [2\text{a}] + [2\text{b}] + [3].$$

Collecting like terms we obtain the desired result. □

Acknowledgements We thank the referees for their time and comments.

References

[1] M. BOUSQUET-MÉLOU, A. CLAESSON, M. DUKES, et al. $(2+2)$*-free posets, ascent sequences and pattern avoiding permutations*. J. Combin. Theory Ser. A, 2010, **117**(7): 884–909.

[2] G. E. ANDREWS, J. A. SELLERS. *Congruences for the Fishburn numbers*. J. Number Theory, 2016, **161**: 298–310.

[3] Shishuo FU, E. Y. JIN, Zhicong LIN, et al. *A new decomposition of ascent sequences and Euler-Stirling statistics*. J. Combin. Theory Ser. A, 2020, **170**: 105141, 31 pp.

[4] P. LEVAND. *Fishburn diagrams, Fishburn numbers and their refined generating functions*. J. Combin. Theory Ser. A, 2013, **120**(1): 194–217.

[5] A. STOIMENOW. *Enumeration of chord diagrams and an upper bound for Vassiliev invariants*. J. Knot Theory Ramifications, 1998, **7**(1): 93–114.

[6] G. CERBAI, A. CLAESSON. *Fishburn trees*. Adv. in Appl. Math., 2023, **151**: Paper No. 102592, 31 pp.

[7] G. CERBAI. *Modified ascent sequences and Bell numbers*. arXiv:2305.10820.

[8] M. DUKES, R. PARVIAINEN. *Ascent sequences and upper triangular matrices containing non-negative integers*. Electron. J. Combin., 2010, **17**(1): Research Paper 53, 16 pp.

[9] J. B. GIL, M. D. WEINER. *On pattern-avoiding Fishburn permutations*. Ann. Comb., 2019, **23**(3-4): 785–800.

[10] E. S. EGGE. *Pattern-avoiding fishburn permutations and ascent sequences*. arXiv:2208.01484.

[11] Yujie DU, Philip B. ZHANG. *Enumerations of some pattern-avoiding Fishburn permutations*. arXiv:2302.13767.

[12] J. WEST. *Generating trees and forbidden subsequences*. Discrete Math., 1996, **157**(1-3): 363–374.