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An Involution over Dyck Paths Related with Stirling Statistics

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Abstract We construct an involution over Dyck paths, which implies the distribution of s-tatistics "the number of peak", "the number of returns" and "the height of the last peak". As an application, equidistributions of several Stirling statistics over 132-avoiding and 321-avoiding permutations are presented.

Keywords involutions; Dyck paths; pattern; peak; returns

MR(2020) Subject Classification 05A19; 05A05

1. Introduction

The Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ is one of the most fascinating numbers in combinatorics. It is known that the Catalan number has more than 200 different combinatorial interpretations, see the book of Stanley [1]. Two among these are dyck paths and permutations avoiding a pattern of length 3.

Consider the integer lattice in the plane $\mathbb{Z}^2 = \{(x, y)|x, y \in \mathbb{Z}\}$. A lattice path is a sequence of elements of $\mathbb{Z}^2$ written

$$P\colon (x_0, y_0), (x_1, y_1), \ldots, (x_l, y_l).$$

We say the path has length l and goes from (x_0, y_0) to (x_l, y_l) which we call endpoints. A Dyck path of semilength n is a lattice path starting at $(0, 0)$ and ending at $(2n, 0)$, consisting of n up-steps $(1, 1)$ and n down-steps $(1, -1)$ such that the path never crosses below the x-axis. Let $\mathfrak{D}_n$ be the set of all Dyck paths with semilength n. Encoding each up-step by U and each down-step by D, we may obtain a corresponding word of a Dyck path. We call this word a Dyck word. Clearly, Dyck path and Dyck word determine each other. They will be used interchangeably hereafter. As an example, see the Dyck path of semilength 8 with its corresponding Dyck word UUDUUUDUDDUDDDUD in Figure 1.

Given P $\in \mathfrak{D}_n$, assume that $(x_0, 0), (x_1, 0), \ldots, (x_s, 0)$ with $0 = x_0 < x_1 < \cdots < x_s = 2n$ are all the elements of P on the x-axis. We write ret(P) $= s$ and say that path P has s returns or s blocks. Let pk(P), val(P), dbr(P) be the number of peaks (i.e., UD's), valleys (i.e., DU's)

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and double rises (i.e., UU's) of P. Suppose that hlp(P) is the height of the last peak (i.e., the maximal length of consecutive D at the end of P) and hfp(P) is the height of the first peak (i.e., the maximal length of consecutive U in the front of P). These statistics over Dyck path have been investigated in several papers. For instance, Deutsch [2] introduced a bijection over Dyck paths which sends ret to hlp. Deutsch [3] further constructed an involution over Dyck path implying the equidistribution of (dbr, hfp) and (val, ret). In this paper, we will construct an involution over Dyck paths.

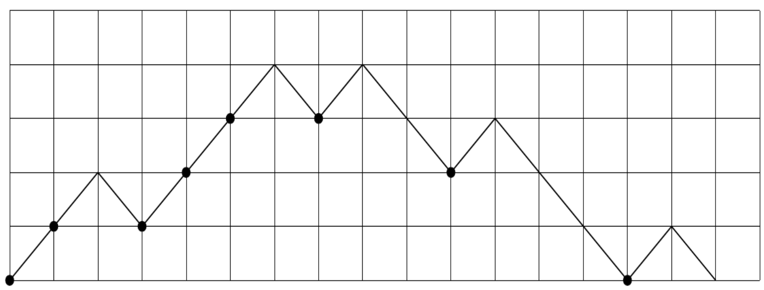

Figure 1 An Dyck path UUDUUUDUDDUDDDUD

Theorem 1.1 *There exists an involution over* $\mathfrak{D}_n$*, such that*

$$\sum_{\mathrm{P}\in\mathfrak{D}_n} x^{\mathrm{pk(P)}} y^{\mathrm{ret(P)}} t^{\mathrm{hlp(P)}} = \sum_{\mathrm{P}\in\mathfrak{D}_n} x^{n+1-\mathrm{pk(P)}} y^{\mathrm{hlp(P)}} t^{\mathrm{ret(P)}}.$$

As an application, we may derive serval equidistributions over pattern avoiding permutations. Pattern avoidance is a relatively active area in combinatorics. For more information about this topic, see the book of Bóna [4] and Kitaev [5]. Let $\mathfrak{S}_n$ be the set of all permutations of $[n] := \{1, 2, \ldots, n\}$. For two permutations $\pi = \pi_1\pi_2\cdots\pi_n$ and $p = p_1p_2\cdots p_k$ with $1 \le k \le n$, if there is a sequence $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ is order-isomorphic to p, then we say π contains pattern p. Otherwise, we say π avoids p or π is p-avoiding. For example, it is clear that the permutation $\pi = 74352681$ is 132-avoiding. Denote by $\mathfrak{S}_n(p)$ the set of all the permutations avoiding pattern p. As is well-known, $\mathfrak{S}_n(p)$, where $p = 123, 132, 213, 231, 312, 321$, are all enumerated by Catalan numbers.

A permutation statistic is a function defined on permutations. Stirling statistic is one of the most classical permutation statistics. The Stirling number and r-Stirling number are widely investigated [6]. As an instance, some identities involving the degenerate r-Stirling numbers arising from the normal ordering of degenerate integral powers of number operator are derived [7, 8]. For $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, π_i is called a right-to-left minimum (resp., maximum) of π if $\pi_i < \pi_j$ (resp., $\pi_i > \pi_j$) for all $j > i$, while it is called a left-to-right minimum (resp., maximum) of π if $\pi_i < \pi_j$ (resp., $\pi_i > \pi_j$) for all $j < i$. Let $\mathrm{rlmin}(\pi)$ (resp., $\mathrm{rlmax}(\pi), \mathrm{lrmin}(\pi), \mathrm{lrmax}(\pi)$) denote the number of right-to-left minima (resp., right-to-left maxima, left-to-right minima, left-to-right maxima) of π.

As a consequence of Theorem 1.1 and the bijections given by Krattenthaler [9], equidistributions of Stirling statistics over 132-avoiding and 321-avoiding permutations are derived.

Theorem 1.2 *Statistic* (lrmin, rlmin, rlmax) *are equally distributed with statistic* $(n + 1 - \mathrm{lrmin}, \mathrm{rlmax}, \mathrm{rlmin})$ *over* $\mathfrak{S}_n(132)$.

Theorem 1.3 *Statistic* (lrmax, bl, F) *are equally distributed with statistic* $(n+1-\mathrm{lrmax}, \mathrm{F}, \mathrm{bl})$ *over* $\mathfrak{S}_n(321)$, *where* $\mathrm{F}(\pi)$ *is the first element of* π *and*

$$\mathrm{bl}(\pi) = |\{i : \forall j \le i, \pi_j \le i\}|.$$

The rest of this paper is organized as follows. In Section 2, we give the involution over Dyck paths. Its applications as stated in Theorems 1.2 and 1.3 will be given in Section 3.

2. An involution over dyck paths

In this section, we will introduce an involution φ defined on Dyck paths that proves Theorem 1.1.

For $\mathrm{P} \in \mathfrak{D}_n$, assume that $\mathrm{ret(P)} = s$. Write $\mathrm{P} = \mathrm{U}\alpha\mathrm{D}\beta$, where α, β are Dyck paths that may be empty. We may construct a dyck path $\varphi(\mathrm{P})$ from back to front as follows. Firstly, draw an up-step from height $s-1$ to height s and adjoin s down-steps. Then, delete the first up-step U and the last down-step D in the first block of P, we obtain a new dyck path $\mathrm{P}' = \alpha\beta$ with semilength $n-1$. Suppose P' has y returns. Draw an up-step from height $y-1$ to y and connect as many down-steps as necessary. Going like this until there is only one pair of U and D left.

We note that the map φ is well defined. Deleting the first up-step and the last down-step in the first block of a Dyck path with x blocks, we always obtain a new Dyck path with the number of blocks not smaller than $x-1$. This guarantees it is always possible to draw an up-step and several necessary down-steps at each step of the map φ.

As an example, for path P given in Figure 1 with $\mathrm{ret(P)} = 2$, $\alpha = \mathrm{UDUUUDUDDUDD}$ and $\beta = \mathrm{UD}$, we construct $\varphi(\mathrm{P})$ as follows. We first draw an up-step from height 1 to 2 and two down-steps, namely UDD. Deleting the first up-step and the last down-step in the first block of P, we obtain $\mathrm{P}' = \alpha\beta$. Clearly, P' has 3 returns. Draw an up-step from height 2 to height 3 and two down-steps to the left of UDD that is just drawn. Going like this, we may obtain $\varphi(\mathrm{P})$ as given in Figure 2.

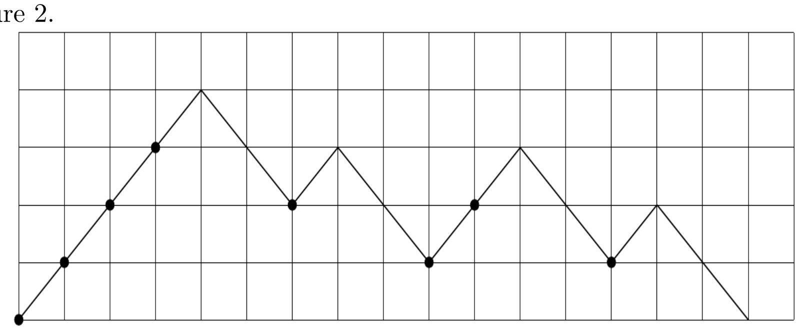

Figure 2 The corresponding Dyck path φ(UUDUUUDUDDUDDDUD)

To show that φ is an involution, we need the following two propositions.

Proposition 2.1 *Given a Dyck path* $\mathrm{P}=\mathrm{U}\alpha\mathrm{D}\beta$ *and* $\mathrm{P}'=\alpha\beta$ *as above, assume that* $\mathrm{ret}(\mathrm{P})=s$*. Then* $\varphi(\mathrm{P})$ *can be obtained from* $\varphi(\mathrm{P}')$ *by inserting* UD *in the front of its last* $s-1$ *consecutive down-steps.*

Proof Firstly, we note that the last $s-1$ consecutive down-steps of $\varphi(\mathrm{P}')$ do exist, since $\mathrm{ret}(\mathrm{P}')\geq s-1$. In the construction of $\varphi(P)$, after the first deleting of the designated U and D, the process is obviously the same as that of $\varphi(\mathrm{P}')$, as desired. □

Proposition 2.2 *Given* $\mathrm{P}\in\mathfrak{D}_n$ *with* $\mathrm{hlp}(\mathrm{P})=x$*, let* P'' *be the path obtained from* P *by deleting its last* UD*, namely the rightmost peak.*

- *If* $\mathrm{ret}(\varphi(\mathrm{P}''))=x-1$*, assume that the Dyck word of* $\varphi(\mathrm{P}'')$ *is* W*. Then,* $\varphi(\mathrm{P})=\mathrm{UD}W$*.*
- *If* $\mathrm{ret}(\varphi(\mathrm{P}''))\geq x$*, let* W_1 *be the dyck word of the path to the left of the last* $x-1$ *blocks and* W_2 *be the last* $x-1$ *blocks of* $\varphi(\mathrm{P}'')$*. Then,* $\varphi(\mathrm{P})$ *is given by* $\mathrm{U}W_1\mathrm{D}W_2$*.*

Proof Firstly, we propose Fact I: If $\mathrm{hlp}(\mathrm{P})=x$, then $\mathrm{ret}(\varphi(\mathrm{P}))=x$. It is easy to check that in the process of φ, before deleting UD with D in the last consecutive down-steps of P, there is only one block in the existing path. Otherwise, there will be more than one block. Hence, before deleting UD with D in the last consecutive down-steps of P, we draw an up-step from height 0 to height 1, which will form a new block. Thus, Fact I is verified. Furthermore, during the construction of $\varphi(\mathrm{P})$, the rightmost peak is just the remaining pair UD. We consider two cases below.

- If the step before the rightmost peak is an up-step, then we assume that $\mathrm{P}=\gamma\overline{\mathrm{U}}\mathrm{UD}\overline{\mathrm{D}}\cdots\mathrm{D}$ and $\mathrm{P}''=\gamma\overline{\mathrm{UD}}\cdots\mathrm{D}$, where γ is a necessary word of U and D. By Fact I, $\mathrm{ret}(\varphi(\mathrm{P}''))=x-1$. During the process of $\varphi(\mathrm{P})$, the pair that is last deleted in P is $\overline{\mathrm{UD}}$. Comparing the construction of $\varphi(\mathrm{P})$ and $\varphi(\mathrm{P}'')$, before the last step of deleting $\overline{\mathrm{UD}}$, the difference between $\overline{\mathrm{U}}\mathrm{UD}\overline{\mathrm{D}}$ and $\overline{\mathrm{UD}}$ brings no influence on the number of returns. After deleting $\overline{\mathrm{UD}}$, there is only one block remaining. Hence, $\varphi(\mathrm{P})$ can be obtained from $\varphi(\mathrm{P}'')$ by inserting UD in the front.
- If the step before the rightmost peak is a down-step, assume that $\mathrm{P}=\gamma\overline{\mathrm{U}}\alpha\widehat{\mathrm{U}}\beta\widehat{\mathrm{D}}\mathrm{UD}\overline{\mathrm{D}}\cdots\mathrm{D}$ and $\mathrm{P}''=\gamma\overline{\mathrm{U}}\alpha\widehat{\mathrm{U}}\beta\widehat{\mathrm{D}}\overline{\mathrm{D}}\cdots\mathrm{D}$, where γ is a word of U and D, α and β are Dyck paths that may be empty. Before deleting $\overline{\mathrm{UD}}$, there is no difference between the construction of $\varphi(\mathrm{P})$ and $\varphi(\mathrm{P}'')$. Notice that $\overline{\mathrm{D}}$ is the last but $x-1$ down-step in both P and P''. By Fact I, at the moment we have drawn $x-1$ blocks. After deleting $\overline{\mathrm{UD}}$, there is always one more block in the construction of $\varphi(\mathrm{P})$ than that of $\varphi(\mathrm{P}'')$. In the construction of $\varphi(\mathrm{P})$, after all the deletions, the rightmost peak of P is left. This implies that we draw an up-step from height 0 to 1 at the end of the construction of $\varphi(\mathrm{P})$. Hence, $\varphi(\mathrm{P})$ can be obtained from $\varphi(\mathrm{P}'')$ by inserting an up-step in the front and a down-step just before the $x-1$ blocks, as desired. □

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Firstly, we claim that φ is an involution. That is, for $\mathrm{P}\in\mathfrak{D}_n$, we have $\varphi^2(\mathrm{P})=\mathrm{P}$. If $n=1$, namely, $\mathrm{P}=\mathrm{UD}$, it is easy to check that $\varphi(\mathrm{P})=\mathrm{UD}$ and $\varphi^2(\mathrm{P})=\mathrm{UD}$. If $n\geq 2$, suppose that this claim holds for $n-1$, we wish to show that it also holds for n. Let $\mathrm{P}=\mathrm{U}\alpha\mathrm{D}\beta$ and $\mathrm{P}'=\alpha\beta$. Assume that $\mathrm{ret}(\mathrm{P})=s$, namely, $\mathrm{ret}(\beta)=s-1$.

By Proposition 2.1, we see that $\varphi(\mathrm{P})$ can be obtained from $\varphi(\mathrm{P}')$ by inserting UD in front of its last $s-1$ consecutive down-steps. Let $\mathrm{P}'' = \varphi(\mathrm{P}') = \varphi(\alpha\beta)$. By the induction hypothesis, we see that $\varphi(\mathrm{P}'') = \varphi^2(\mathrm{P}') = \alpha\beta$. Thus, following from Proposition 2.2, we deduce that $\varphi^2(\mathrm{P}) = \mathrm{U}\alpha\mathrm{D}\beta = \mathrm{P}$. The claim is verified.

By the construction of φ, it is easy to check that $\mathrm{ret}(\mathrm{P}) = \mathrm{hlp}(\varphi(\mathrm{P}))$. Fact I gives that $\mathrm{hlp}(\mathrm{P}) = \mathrm{ret}(\varphi(\mathrm{P}))$.

It remains to show that $\mathrm{pk}(\mathrm{P}) = n + 1 - \mathrm{pk}(\varphi(\mathrm{P}))$. We wish to prove it by induction on n. When $n = 1$, then $\varphi(\mathrm{UD}) = \mathrm{UD}$ and $\mathrm{pk}(\mathrm{UD}) = 2 - \mathrm{pk}(\varphi(\mathrm{UD}))$. Suppose that this formula holds for Dyck paths with semilength $n-1$, we proceed to show that it also holds for n. For $\mathrm{P} = \mathrm{U}\alpha\mathrm{D}\beta$ and $\mathrm{P}' = \alpha\beta$, we consider two cases.

- If α is empty, then $\mathrm{pk}(\mathrm{P}) = \mathrm{pk}(\mathrm{P}') + 1$. Since $\mathrm{ret}(\mathrm{P}') = s-1$, $\mathrm{hlp}(\varphi(\mathrm{P}')) = s-1$. By Proposition 2.1 we deduce that $\mathrm{pk}(\varphi(\mathrm{P})) = \mathrm{pk}(\varphi(\mathrm{P}'))$. Based on the induction hypothesis, we have $\mathrm{pk}(\mathrm{P}') + \mathrm{pk}(\varphi(\mathrm{P}')) = n$. It follows that $\mathrm{pk}(\mathrm{P}) - 1 + \mathrm{pk}(\varphi(\mathrm{P})) = n$, as desired.
- If α is not empty, then $\mathrm{pk}(\mathrm{P}) = \mathrm{pk}(\mathrm{P}')$. As $\mathrm{ret}(\mathrm{P}') \geq s$, we have $\mathrm{hlp}(\varphi(\mathrm{P}')) \geq s$. Thus, $\mathrm{pk}(\varphi(\mathrm{P})) = \mathrm{pk}(\varphi(\mathrm{P}')) + 1$ follows immediately from Proposition 2.1. Notice that $\mathrm{pk}(\mathrm{P}') + \mathrm{pk}(\varphi(\mathrm{P}')) = n$. Then, $\mathrm{pk}(\mathrm{P}) + \mathrm{pk}(\varphi(\mathrm{P})) - 1 = n$. This completes the proof. □

It should be noted that the involution constructed by Deutsch [3] may serve to construct a different involution to prove Theorem 1.1. Deutsch [3] defined an involution $(\ )'$ inductively as follows. Let $\epsilon' = \epsilon$ where ϵ is empty. For a non-empty Dyck path P, write $\mathrm{P} = \mathrm{U}\alpha\mathrm{D}\beta$, where α and β may be empty and D is the first return step. Define $\mathrm{P}' = \mathrm{U}\beta'\mathrm{D}\alpha'$ with the property

$$(\mathrm{dbr}, \mathrm{hfp})\mathrm{P} = (\mathrm{val}, \mathrm{ret})\mathrm{P}'.$$

Since $\mathrm{val}(\mathrm{P}) = \mathrm{pk}(\mathrm{P}) - 1$ and $\mathrm{dbr}(\mathrm{P}') + \mathrm{pk}(\mathrm{P}') = n$, we have $\mathrm{pk}(\mathrm{P}) = n + 1 - \mathrm{pk}(\mathrm{P}')$. It can be concluded that

$$(\mathrm{pk}, \mathrm{ret}, \mathrm{hfp})\mathrm{P} = (n + 1 - \mathrm{pk}, \mathrm{hfp}, \mathrm{ret})\mathrm{P}'.$$

Then, define the involution $\mathrm{M}\colon \mathfrak{D}_n \to \mathfrak{D}_n$ by the mirror reflection of the Dyck paths. Notice that

$$(\mathrm{pk}, \mathrm{ret}, \mathrm{hfp})\mathrm{P} = (\mathrm{pk}, \mathrm{ret}, \mathrm{hlp})\mathrm{M}(\mathrm{P}'). \tag{2.1}$$

Combining $(\ )'$ and M, we come up with another involution to prove Theorem 1.1. □

3. Application: equidistributions of Stirling statistics

In this section, we will use Theorem 1.1 to obtain several equidistributions of Stirling statistics over pattern avoiding permutations.

3.1. The bijection between 132-avoiding permutations and Dyck paths

Krattenthaler [9] proposed a bijection Φ between 132-avoiding permutations and Dyck paths. For each permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathrm{S}_n(132)$, scan it from left to right and generate the correspongding Dyck path successively. Let h_i be the number of elements which are larger than

π_i in $\pi_{i+1}\cdots\pi_n$, then connect some up-steps necessarily and a followed down-step from height h_i+1 to h_i in the path.

As an example, for a permutation $\pi = 74352681$. Firstly, the path begins with two up-steps and a followed down-step to reach height 1 since there exists one element larger than 7 in the rest permutation 4352681. Next, 4 is scanned. The path goes on with three up-steps and a followed down-step to reach height 3 since there are three elements larger than 4 in 352681. Going like this, we finally obtain the corresponding path, shown in Figure 1.

Alternatively, Krattenthaler [9] introduced another way to construct Φ. Let $\pi = m_1w_1m_2 w_2\cdots m_sw_s$, where $m_1, m_2, \ldots, m_s$ are the left-to-right minima and w_i is the word between m_i and m_{i+1} in π. Then, the left-to-right minima of π are transformed into $m_{i-1}-m_i$ up-steps (by convention $m_0 = n+1$) and the subword w_i is transformed into $|w_i|+1$ down-steps when we are scanning the permutation $\pi \in \mathfrak{S}_n$ from left to right.

Proposition 3.1 *For each permutation* $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n(132)$*, then we have*

$$(\mathrm{lrmin}, \mathrm{rlmax}, \mathrm{rlmin})\pi = (\mathrm{pk}, \mathrm{ret}, \mathrm{hlp})\Phi(\pi).$$

Proof Based on the second description of Φ, it is easy to see that statistic lrmin over $\mathfrak{S}_n$ is mapped to pk over $\mathfrak{D}_n$ by Φ.

If π_i is a right to left maximum of π, there will be no element to the right of π_i and larger than π_i. In view of the first description of Φ, a down-step from height 0 to height 1 will be drawn, which means a return is formed. Therefore, the statistic rlmax over $\mathfrak{S}_n$ is mapped to the statistic ret over $\mathfrak{D}_n$.

Due to $\pi \in \mathrm{S}_n(132)$, the permutation π ends in a continuous increasing sequence of elements starting with element 1. Assume that this sequence is $1\pi_x\cdots\pi_n(\pi_x < \cdots < \pi_n)$. By the first description of Φ, when we scan 1, we need to draw a down-step from height $n-x+2$ to height $n-x+1$. Next, when scanning π_x, a down-step from height $n-x+1$ to height $n-x$ is drawn. Going like this, $n-x+2$ consecutive down-steps will be drawn. Notice that $1, \pi_x, \ldots, \pi_n$ are the right-to-left minima of π. Thus, we deduce that $\mathrm{rlmin}(\pi) = \mathrm{hlp}(\Phi(\pi))$. The proof is completed. □

Combining Theorem 1.1 and Propositon 3.1, we see that Theorem 1.2 follows directly.

3.2. The bijection between 321-avoiding permutations and Dyck paths

Krattenthaler [9] also proposed a bijection Ψ between 123-avoiding permutations and Dyck paths. For each permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n(123)$, decompose the permutation π into $w_sm_sw_{s-1}m_{s-1}\cdots w_1m_1$ where $m_1, m_2, \ldots m_s$ are the right-to-left maxima of π and w_i is the subword between m_i and m_{i+1} in π. Define the bijection Ψ by reading the decomposition of π from right to left. Translate the right-to-left maximum m_i into m_i-m_{i-1} up-steps (by convention $m_0 = 0$) and the subword w_i into $|w_i|+1$ down-steps.

As an example, for a permutation $\pi = 84371652 \in \mathfrak{S}_n(123)$, its corresponding Dyck path $\Psi(\pi)$ is shown in Figure 1.

Proposition 3.2 *For each permutation* $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n(123)$, *we have*

$$(\mathrm{rlmax}, \mathrm{L})\pi = (\mathrm{pk}, \mathrm{hfp})\Psi(\pi),$$

where $\mathrm{L}(\pi)$ *is the last element of* π.

Define the bijection $\Omega\colon \mathrm{r} \circ \Psi \circ \mathrm{M}$:

$$\Omega\colon \mathfrak{S}_n(321) \xrightarrow{\mathrm{r}} \mathfrak{S}_n(123) \xrightarrow{\Psi} \mathfrak{D}_n \xrightarrow{\mathrm{M}} \mathfrak{D}_n,$$

where $\mathrm{r}(\pi)$ is the reverse of π. We may obtain the following proposition.

Proposition 3.3 *For each permutation* $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n(321)$, *we have*

$$(\mathrm{lrmax}, \mathrm{bl}, \mathrm{F})\pi = (\mathrm{pk}, \mathrm{ret}, \mathrm{hlp})\Omega(\pi).$$

Proof Based on the definition of bijection r, it is easy to figure out that $(\mathrm{rlmax}, \mathrm{L})\pi = (\mathrm{lrmax}, \mathrm{F})\mathrm{r}(\pi)$. Further, Rubey [10] claimed that $\mathrm{bl}(\pi) = \mathrm{ret}(\mathrm{r} \circ \Psi(\pi))$. In view of (2.1) and Proposition 3.2, the proof is completed. □

Combining Theorem 1.1 and Propositon 3.3, we deduce that Theorem 1.3.

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