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On Integral Graphs which Belong to the Class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$

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Abstract Let G be a simple graph and let $\overline{G}$ denote its complement. We say that G is integral if its spectrum consists entirely of integers. In this work we establish a characterization of integral graphs which belong to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, where mG denotes the m-fold union of the graph G.

Keywords graph; eigenvalue; Diophantine equation

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1. Introduction

Let G be a simple graph of order n and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of its $(0,1)$ adjacency matrix of G. The spectrum of G is the set of its eigenvalues and is denoted by $\sigma(G)$. We say that G is integral if its spectrum $\sigma(G)$ consists only of integers [1]. We say that an eigenvalue μ is main if and only if $\langle \mathbf{j}, \mathbf{P}\mathbf{j} \rangle = n\cos^2\alpha > 0$, where $\mathbf{j}$ is the main vector (with coordinates equal to 1) and $\mathbf{P}$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_A(\mu)$. The quantity $\beta = |\cos\alpha|$ is called the main angle of μ. In [1] was proved that the graph G and its complement $\overline{G}$ have the same number of main eigenvalues.

Let G be a graph with exactly two main eigenvalues μ_1 and μ_2 and let β_1 and β_2 be the main angles of μ_1 and μ_2, respectively. In [2] it was proved that

$$\overline{\mu}_{1,2} = \frac{n-2-\mu_1-\mu_2}{2} \pm \frac{\sqrt{(\mu_1-\mu_2+n)^2 - 4n_1(\mu_1-\mu_2)}}{2}, \tag{1.1}$$

where $\overline{\mu}_1$ and $\overline{\mu}_2$ are the main eigenvalues of its complement $\overline{G}$ and $n_1 = n\beta_1^2$. Next, let $K_{m,n,\ell}$ denote the complete three partite graph. We note that $\alpha K_{a,a,a} \cup \beta K_{b,b,b}$ is an integral graph with two main eigenvalues $\mu_1 = 2a$ and $\mu_2 = 2b$, for any $\alpha, \beta, a, b \in \mathbb{N}$ with $a > b$, where mG denotes the m-fold union of the graph G. We know that if G is an integral graph, then $\overline{G}$ is integral if and only if the main spectrum of $\overline{G}$ contains only integral values [3]. Therefore, $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is integral if and only if its largest eigenvalue $\overline{\mu}_1 \in \mathbb{N}$.

Due to relation (1.1) we have described in [2] all integral graphs which belong to the class $\overline{\alpha K_a \cup \beta K_{b,b}}$, where K_n and $K_{m,n}$ denote the complete graph and the complete bipartite graph, respectively. Besides, (i) we have described all integral graphs which belong to the class $\overline{\alpha K_a \cup \beta K_{b,b,b}}$

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and (ii) we have described all integral graphs which belong to the class $\overline{\alpha K_{a,a} \cup \beta K_{b,b,b}}$. We now proceed to establish a characterization of integral graphs which belong to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, as follows.

2. Main results

First, note that $o = 3\alpha a + 3\beta b$ is the order of $\alpha K_{a,a,a} \cup \beta K_{b,b,b}$. In the case that $a > b$ we find that $\mu_1 = 2a$ and $n_1 = 3\alpha a$. Then according to (1.1) we get implicitly

$$\overline{\mu}_{1,2} = \frac{3(\alpha a + \beta b) - 2(a + b + 1) \pm \delta}{2}, \tag{2.1}$$

where $\delta = \sqrt{((3\alpha + 2)a + (3\beta - 2)b)^2 - 24\alpha a(a - b)}$. In view of this, $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is integral if and only if $(\alpha, \beta, a, b, \delta)$ represents a positive integral solution of the Diophantine equation

$$[(3\alpha + 2)a + (3\beta - 2)b]^2 - 24\alpha a(a - b) = \delta^2. \tag{2.2}$$

Thus, the characterization of integral graphs which is related to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is reduced to the problem of finding the most general integral solution of the Eq. (2.2).

Next, $\overline{\mu}_1 \, \overline{\mu}_2 = \mu_1 \, \mu_2 - (n_2 - 1) \, \mu_1 - (n_1 - 1) \, \mu_2 - (n - 1)$ for any G with two main eigenvalues, where $n_2 = n\beta_2^2$ (see [3]). In the case that $G = \alpha K_{a,a,a} \cup \beta K_{b,b,b}$ this relation is transformed into

$$(\overline{\mu}_1 + 1)(\overline{\mu}_2 + 1) = 2ab(2 - 3\alpha - 3\beta). \tag{2.3}$$

Remark 2.1 With the condition $a > b$ note that the parameters α, β, a, b determine the graph $\alpha K_{a,a,a} \cup \beta K_{b,b,b}$ up to isomorphism, which proves that α, β, a, b also uniquely determine its complement $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$.

In the sequel (m, n) denotes the highest common divisor of integers $m, n \in \mathbb{N}$ while $m \mid n$ means that m divides n. With this notation, in order to demonstrate a method applied in this paper, we prove first the following two results:

Theorem 2.2 *If $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is an integral graph with $\overline{\mu}_1 = 2ab - 1$, then it belongs to one of the following classes of integral graphs:*

$$\overline{(2t - 1)m \, K_{a,a,a} \cup (2s - (2t - 1))n \, K_{b,b,b}}, \tag{2.4}$$

where (i) $a = 3sn - 1$, $b = 3sm - 1$ and $n > m$; (ii) $m, n, s, t \in \mathbb{N}$ and $s \geq t$; (iii) $(2s, 2t - 1) = 1$, $(2t - 1, 3) = 1$ and $(2s - (2t - 1), 3) = 1$;

$$\overline{(2t - 1)m \, K_{a,a,a} \cup (2k - 1)n \, K_{b,b,b}}, \tag{2.5}$$

where (i) $a = (t + 3k - 2)n - 1$, $b = 3(t + 3k - 2)m - 1$ and $n > 3m$; (ii) $m, n, k, t \in \mathbb{N}$ such that $s = t + 3k - 2$, $(2s, 2t - 1) = 1$ and $(2t - 1, 3) = 1$;

$$\overline{(2t - 1)m \, K_{a,a,a} \cup (2s - 3(2t - 1))n \, K_{b,b,b}}, \tag{2.6}$$

where (i) $a = 3sn - 1$, $b = sm - 1$ and $m < 3n$; (ii) $m, n, s, t \in \mathbb{N}$ and $s \geq 3t - 1$; (iii) $(2s, 2t - 1) = 1$ and $(2s, 3) = 1$;

$$\overline{4tm \, K_{a,a,a} \cup 2((2s + 1) - 2t)n \, K_{b,b,b}}, \tag{2.7}$$

where (i) $a = 3(2s+1)n-1$, $b = 3(2s+1)m-1$ *and* $n > m$; *(ii)* $m,n,s,t \in \mathbb{N}$ *and* $s \geq t$; *(iii)* $(2s+1,2t)=1$, $(t,3)=1$ *and* $((2s+1)-2t,3)=1$;

$$\overline{4tm\,K_{a,a,a} \cup 2(2k-1)n\,K_{b,b,b}}, \tag{2.8}$$

where (i) $a = (2t+6k-3)n-1$, $b = 3(2t+6k-3)m-1$ *and* $n > 3m$; *(ii)* $m,n,k,t \in \mathbb{N}$ *such that* $s = t+3k-2$, $(2s+1,2t)=1$ *and* $(t,3)=1$;

$$\overline{4tm\,K_{a,a,a} \cup 2\big((2s+1)-6t\big)n\,K_{b,b,b}}, \tag{2.9}$$

where (i) $a = 3(2s+1)n-1$, $b = (2s+1)m-1$ *and* $m < 3n$; *(ii)* $m,n,s,t \in \mathbb{N}$ *and* $s \geq 3t$; *(iii)* $(2s+1,2t)=1$ *and* $(2s+1,3)=1$;

$$\overline{2(2t-1)m\,K_{a,a,a} \cup 4(s-t+1)n\,K_{b,b,b}}, \tag{2.10}$$

where (i) $a = 3(2s+1)n-1$, $b = 3(2s+1)m-1$ *and* $n > m$; *(ii)* $m,n,s,t \in \mathbb{N}$ *and* $s \geq t$; *(iii)* $(2s+1,2t-1)=1$, $(2t-1,3)=1$ *and* $(s-t+1,3)=1$;

$$\overline{2(2t-1)m\,K_{a,a,a} \cup 4kn\,K_{b,b,b}}, \tag{2.11}$$

where (i) $a = (2t+6k-1)n-1$, $b = 3(2t+6k-1)m-1$ *and* $n > 3m$; *(ii)* $m,n,k,t \in \mathbb{N}$ *such that* $s = t+3k-1$, $(2s+1,2t-1)=1$ *and* $(2t-1,3)=1$;

$$\overline{2(2t-1)m\,K_{a,a,a} \cup 4(s-3t+2)n\,K_{b,b,b}}, \tag{2.12}$$

where (i) $a = 3(2s+1)n-1$, $b = (2s+1)m-1$ *and* $m < 3n$; *(ii)* $m,n,s,t \in \mathbb{N}$ *and* $s \geq 3t-1$; *(iii)* $(2s+1,2t-1)=1$ *and* $(2s+1,3)=1$.

Proof Let us assume that $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is an integral graph with $\overline{\mu}_1 = 2ab-1$. Using (2.3) we obtain $\overline{\mu}_2 = 1-3\alpha-3\beta$ and $\delta = 2ab+(3\alpha+3\beta-2)$. Then Diophantine Eq. (2.2) is reduced to $(b+1)(2a-(3\alpha+3\beta-2)) = 3\alpha(a-b)$. Let $b+1 = \frac{3}{2}r\alpha$ where $r = \frac{s}{t}$ such that $(s,t)=1$. Then from the last relation we obtain $a-b = \frac{r}{2}(2a-(3\alpha+3\beta-2))$. In view of this, we get

$$\alpha = \frac{2t}{3s}\,(b+1) \text{ and } \beta = \frac{2(s-t)}{3s}\,(a+1). \tag{2.13}$$

Case 1 (s is even and t is odd). Let $s \to 2s$ and $t \to 2t-1$ where $p \to q$ means that 'p is replaced with q'. In this case relation (2.13) is transformed into

$$\alpha = \frac{2t-1}{3s}\,(b+1) \text{ and } \beta = \frac{2s-(2t-1)}{3s}\,(a+1).$$

Consider the case when $(2t-1,3)=1$ and $(2s-(2t-1),3)=1$. Since $(2t-1,s)=1$ and $(2s-(2t-1),s)=1$ it must be $3s \mid (b+1)$ and $3s \mid (a+1)$. Let $b+1 = 3sm$ and $a+1 = 3sn$. So we get $\alpha = (2t-1)m$ and $\beta = (2s-(2t-1))n$, which provides the class of integral graphs represented in (2.4). Consider the case when $(2t-1,3)=1$ and $(2s-(2t-1),3)=3$. Then $2s-(2t-1) = 3(2k-1)$ where $k \in \mathbb{N}$. Since $(2s-(2t-1),2s)=1$ it follows that $(2k-1,2s)=1$. Consequently, since $(2t-1,3s)=1$ and $(2k-1,s)=1$, it must be $3s|(b+1)$ and $s|(a+1)$. Let $b+1 = 3sm$ and $a+1 = sn$. So we get $\alpha = (2t-1)m$ and $\beta = (2k-1)n$, which provides the class of integral graphs represented in (2.5). Consider the case when $(2t-1,3)=3$. Since $(2s-(2t-1),2t-1)=1$, we obtain that $(2s-(2t-1),3)=1$, which provides that $s \mid (b+1)$

and $3s|(a+1)$. Setting $2t-1 \to 3(2t-1)$, we find that $b+1 = sm$ and $a+1 = 3sn$. So we get $\alpha = (2t-1)m$ and $\beta = (2s-3(2t-1))n$, which provides the class of integral graphs represented in (2.6).

Case 2 (s is odd and t is even). Let $s \to 2s+1$ and $t \to 2t$. In this case relation (2.13) is transformed into

$$\alpha = \frac{4t}{3(2s+1)}(b+1) \text{ and } \beta = \frac{2((2s+1)-2t)}{3(2s+1)}(a+1).$$

Consider the case when $(t,3) = 1$ and $((2s+1)-2t,3) = 1$. Since $(3(2s+1),4t) = 1$ and $((2s+1)-2t, 2s+1) = 1$ it must be $3(2s+1)|(b+1)$ and $3(2s+1) \mid (a+1)$. Let $b+1 = 3(2s+1)m$ and $a+1 = 3(2s+1)n$. So we get $\alpha = 4tm$ and $\beta = 2(2s-(2t-1))n$, which provides the class of integral graphs represented in (2.7). Consider the case when $(t,3) = 1$ and $((2s+1)-2t,3) = 3$. Then $(2s+1)-2t = 3(2k-1)$ where $k \in \mathbb{N}$. Since $((2s+1)-2t, 2s+1) = 1$ it follows that $(2k-1, 2s+1) = 1$. Consequently, since $(4t, 3(2s+1)) = 1$ and $(2(2k-1), 2s+1) = 1$, it must be $3(2s+1)|(b+1)$ and $(2s+1)|(a+1)$. Let $b+1 = 3(2s+1)m$ and $a+1 = (2s+1)n$. So we get $\alpha = 4tm$ and $\beta = 2(2k-1)n$, which provides the class of integral graphs represented in (2.8). Consider the case when $(t,3) = 3$. Since $((2s+1)-2t, 2t) = 1$, we obtain that $((2s+1)-2t,3) = 1$, which provides that $(2s+1)|(b+1)$ and $3(2s+1)|(a+1)$. Setting $t \to 3t$, we find that $b+1 = (2s+1)m$ and $a+1 = 3(2s+1)n$. So we get $\alpha = 4tm$ and $\beta = 2((2s+1)-6t)n$, which provides the class of integral graphs represented in (2.9).

Case 3 (s is odd and t is odd). Let $s \to 2s+1$ and $t \to 2t-1$. In this case relation (2.13) is transformed into

$$\alpha = \frac{2(2t-1)}{3(2s+1)}(b+1) \text{ and } \beta = \frac{2\big((2s+1)-(2t-1)\big)}{3(2s+1)}(a+1).$$

Consider the case when $(2t-1,3) = 1$ and $((2s+1)-(2t-1),3) = 1$. Since $(2t-1, 2s+1) = 1$ and $((2s+1)-(2t-1), 2s+1) = 1$, it must be $3(2s+1)|(b+1)$ and $3(2s+1) \mid (a+1)$. Let $b+1 = 3(2s+1)m$ and $a+1 = 3(2s+1)n$. So we get $\alpha = 2(2t-1)m$ and $\beta = 2((2s+1)-(2t-1))n$, which provides the class of integral graphs represented in (2.10). Consider the case when $(2t-1,3) = 1$ and $((2s+1)-(2t-1),3) = 3$. Then $(2s+1)-(2t-1) = 3(2k)$ where $k \in \mathbb{N}$. Since $((2s+1)-(2t-1), 2s+1) = 1$, it follows that $(2k, 2s+1) = 1$. Consequently, since $(2(2t-1), 3(2s+1)) = 1$ and $(2k, 2s+1) = 1$, it must be $3(2s+1)|(b+1)$ and $(2s+1)|(a+1)$. Let $b+1 = 3(2s+1)m$ and $a+1 = (2s+1)n$. So we get $\alpha = 2(2t-1)m$ and $\beta = 4kn$, which provides the class of integral graphs represented in (2.11). Consider the case when $(2t-1,3) = 3$. Since $((2s+1)-(2t-1), 2t-1) = 1$ we obtain that $((2s+1)-(2t-1),3) = 1$, which provides that $(2s+1)|(b+1)$ and $3(2s+1)|(a+1)$. Setting $2t-1 \to 3(2t-1)$, we find that $b+1 = (2s+1)m$ and $a+1 = 3(2s+1)n$. So we get $\alpha = 2(2t-1)m$ and $\beta = 2((2s+1)-3(2t-1))n$, which provides the class of integral graphs represented in (2.12). □

Proposition 2.3 *If* $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ *is an integral graph with* $\overline{\mu}_1 = 2ab-1$*, then it uniquely determines the parameters* k, m, n, s, t.

Proof Let us assume that k_1, m_1, n_1, s_1, t_1 and k_2, m_2, n_2, s_2, t_2 determine the same integral graph $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ with $\overline{\mu}_1 = 2ab - 1$. Since the parameters α, β, a, b determine the graph $\alpha K_{a,a,a} \cup \beta K_{b,b,b}$ up to isomorphism, using the first equality of (2.13), we have $3r\alpha = 2(b+1)$, which shows that $s_1 = s_2$ and $t_1 = t_2$ because $(s,t) = 1$. In view of this, we note that the classes represented by relations (2.4)–(2.12) are mutually disjoint. Consequently, without loss of generality, we can assume that the corresponding integral graph determined by the parameters k_1, m_1, n_1, s_1, t_1 and k_2, m_2, n_2, s_2, t_2 belongs to the class of integral graphs displayed in relation (2.5). Next, using that $s = t + 3k - 2$, we get $k_1 = k_2$. Finally, since $a = (t+3k-2)n - 1$ and $b = 3(t+3k-2)m - 1$, it follows that $n_1 = n_2$ and $m_1 = m_2$. □

Theorem 2.4 *If $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is an integral graph with $\overline{\mu}_1 = ab - 1$, then it belongs to one of the following classes of integral graphs:*

$$\overline{(2t-1)m\, K_{a,a,a} \cup (2s-(2t-1))n\, K_{b,b,b}}, \tag{2.14}$$

where (i) $a = 6sn - 2$, $b = 6sm - 2$ and $n > m$; (ii) $m, n, s, t \in \mathbb{N}$ and $s \geq t$; (iii) $(2s, 2t-1) = 1$, $(2t-1, 3) = 1$ and $(2s-(2t-1), 3) = 1$;

$$\overline{(2t-1)m\, K_{a,a,a} \cup (2k-1)n\, K_{b,b,b}}, \tag{2.15}$$

where (i) $a = 2(t+3k-2)n - 2$, $b = 6(t+3k-2)m - 2$ and $n > 3m$; (ii) $m, n, k, t \in \mathbb{N}$ such that $s = t + 3k - 2$, $(2s, 2t-1) = 1$ and $(2t-1, 3) = 1$;

$$\overline{(2t-1)m\, K_{a,a,a} \cup (2s-3(2t-1))n\, K_{b,b,b}}, \tag{2.16}$$

where (i) $a = 2(3sn-1)$, $b = 2(sm-1)$ and $m < 3n$; (ii) $m, n, s, t \in \mathbb{N}$ and $s \geq 3t-1$; (iii) $(2s, 2t-1) = 1$ and $(2s, 3) = 1$;

$$\overline{2tm\, K_{a,a,a} \cup ((2s+1)-2t)n\, K_{b,b,b}}, \tag{2.17}$$

where (i) $a = 3(2s+1)n - 2$, $b = 3(2s+1)m - 2$ and $n > m$; (ii) $m, n, s, t \in \mathbb{N}$ and $s \geq t$; (iii) $(2s+1, 2t) = 1$, $(t, 3) = 1$ and $((2s+1)-2t, 3) = 1$;

$$\overline{2tm\, K_{a,a,a} \cup (2k-1)n\, K_{b,b,b}}, \tag{2.18}$$

where (i) $a = (2t+6k-3)n - 2$, $b = 3(2t+6k-3)m - 2$ and $n > 3m$; (ii) $m, n, k, t \in \mathbb{N}$ such that $s = t + 3k - 2$, $(2s+1, 2t) = 1$ and $(t, 3) = 1$;

$$\overline{2tm\, K_{a,a,a} \cup ((2s+1)-6t)n\, K_{b,b,b}}, \tag{2.19}$$

where (i) $a = 3(2s+1)n - 2$, $b = (2s+1)m - 2$ and $m < 3n$; (ii) $m, n, s, t \in \mathbb{N}$ and $s \geq 3t$; (iii) $(2s+1, 2t) = 1$ and $(2s+1, 3) = 1$;

$$\overline{(2t-1)m\, K_{a,a,a} \cup 2(s-t+1)n\, K_{b,b,b}}, \tag{2.20}$$

where (i) $a = 3(2s+1)n - 2$, $b = 3(2s+1)m - 2$ and $n > m$; (ii) $m, n, s, t \in \mathbb{N}$ and $s \geq t$; (iii) $(2s+1, 2t-1) = 1$, $(2t-1, 3) = 1$ and $(s-t+1, 3) = 1$;

$$\overline{(2t-1)m\, K_{a,a,a} \cup 2kn\, K_{b,b,b}}, \tag{2.21}$$

where (i) $a = (2t+6k-1)n - 2$, $b = 3(2t+6k-1)m - 2$ and $n > 3m$; (ii) $m, n, k, t \in \mathbb{N}$ such *that* $s = t + 3k - 1$, $(2s+1, 2t-1) = 1$ *and* $(2t-1, 3) = 1$;

$$\overline{(2t-1)m\, K_{a,a,a} \cup 2(s-3t+2)n\, K_{b,b,b}}, \tag{2.22}$$

where (i) $a = 3(2s+1)n - 2$, $b = (2s+1)m - 2$ *and* $m < 3n$; *(ii)* $m, n, s, t \in \mathbb{N}$ *and* $s \geq 3t - 1$; *(iii)* $(2s+1, 2t-1) = 1$ *and* $(2s+1, 3) = 1$.

Proof Let us assume that $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is an integral graph with $\overline{\mu}_1 = ab - 1$. Using (2.3), we obtain $\overline{\mu}_2 = 3(1 - 2\alpha - 3\beta)$ and $\delta = ab + 2(3\alpha + 3\beta - 2)$. Then Diophantine Eq. (2.2) is reduced to

$$(b+2)(a - (3\alpha + 3\beta - 2)) = 3\alpha(a - b).$$

Let $b + 2 = 3r\alpha$ where $r = \frac{s}{t}$ such that $(s, t) = 1$. Then from the last relation we obtain $a - b = r(a - (3\alpha + 3\beta - 2))$. In view of this, we get

$$\alpha = \frac{t}{3s}\,(b+2) \text{ and } \beta = \frac{s-t}{3s}\,(a+2). \tag{2.23}$$

Case 1 (s is even and t is odd). Let $s \to 2s$ and $t \to 2t - 1$. In this case relation (2.23) is transformed into

$$\alpha = \frac{2t-1}{6s}\,(b+2) \text{ and } \beta = \frac{2s-(2t-1)}{6s}\,(a+2).$$

Consider the case when $(2t-1, 3) = 1$ and $(2s - (2t-1), 3) = 1$. Since $(2t-1, 2s) = 1$ and $(2s - (2t-1), 2s) = 1$, it must be $6s|(b+2)$ and $6s \mid (a+2)$. Let $b + 2 = 6sm$ and $a + 2 = 6sn$. So we get $\alpha = (2t-1)m$ and $\beta = (2s - (2t-1))n$, which provides the class of integral graphs represented in (2.14). Consider the case when $(2t-1, 3) = 1$ and $(2s - (2t-1), 3) = 3$. Then $2s - (2t-1) = 3(2k-1)$ where $k \in \mathbb{N}$. Since $(2s - (2t-1), 2s) = 1$, it follows that $(2k-1, 2s) = 1$. Consequently, since $(2t-1, 6s) = 1$ and $(2k-1, 2s) = 1$, it must be $6s|(b+2)$ and $2s \mid (a+2)$. Let $b + 2 = 6sm$ and $a + 2 = 2sn$. So we get $\alpha = (2t-1)m$ and $\beta = (2k-1)n$, which provides the class of integral graphs represented in (2.15). Consider the case when $(2t-1, 3) = 3$. Since $(2s - (2t-1), 2t-1) = 1$, we obtain that $(2s - (2t-1), 3) = 1$, which provides that $2s|(b+2)$ and $6s|(a+2)$. Setting $2t - 1 \to 3(2t-1)$, we find that $b + 2 = 2sm$ and $a + 2 = 6sn$. So we get $\alpha = (2t-1)m$ and $\beta = (2s - 3(2t-1))n$, which provides the class of integral graphs represented in (2.16).

Case 2 (s is odd and t is even). Let $s \to 2s + 1$ and $t \to 2t$. In this case relation (2.23) is transformed into

$$\alpha = \frac{2t}{3(2s+1)}\,(b+2) \text{ and } \beta = \frac{(2s+1)-2t}{3(2s+1)}\,(a+2).$$

Consider the case when $(t, 3) = 1$ and $((2s+1) - 2t, 3) = 1$. Since $(3(2s+1), 2t) = 1$ and $((2s+1) - 2t, 2s+1) = 1$, it must be $3(2s+1)|(b+2)$ and $3(2s+1)|(a+2)$. Let $b+2 = 3(2s+1)m$ and $a + 2 = 3(2s+1)n$. So we get $\alpha = 2tm$ and $\beta = (2s - (2t-1))n$, which provides the class of integral graphs represented in (2.17). Consider the case when $(t, 3) = 1$ and $((2s+1) - 2t, 3) = 3$. Then $(2s+1) - 2t = 3(2k-1)$ where $k \in \mathbb{N}$. Since $((2s+1) - 2t, 2s+1) = 1$, it follows that $(2k-1, 2s+1) = 1$. Consequently, since $(2t, 3(2s+1)) = 1$ and $(2k-1, 2s+1) = 1$, it must be $3(2s+1) \mid (b+2)$ and $(2s+1)|(a+2)$. Let $b + 2 = 3(2s+1)m$ and $a + 2 = (2s+1)n$.

So we get $\alpha = 2tm$ and $\beta = (2k-1)n$, which provides the class of integral graphs represented in (2.18). Consider the case when $(t,3) = 3$. Since $((2s+1)-2t, 2t) = 1$, we obtain that $((2s+1)-2t, 3) = 1$, which provides that $(2s+1)|(b+2)$ and $3(2s+1)|(a+2)$. Setting $t \to 3t$, we find that $b+2 = (2s+1)m$ and $a+2 = 3(2s+1)n$. So we get $\alpha = 2tm$ and $\beta = ((2s+1)-6t)n$, which provides the class of integral graphs represented in (2.19).

Case 3 (s is odd and t is odd). Let $s \to 2s+1$ and $t \to 2t-1$. In this case relation (2.23) is transformed into

$$\alpha = \frac{2t-1}{3(2s+1)}\,(b+2) \text{ and } \beta = \frac{(2s+1)-(2t-1)}{3(2s+1)}\,(a+2).$$

Consider the case when $(2t-1,3) = 1$ and $((2s+1)-(2t-1),3) = 1$. Since $(2t-1, 2s+1) = 1$ and $((2s+1)-(2t-1), 2s+1) = 1$, it must be $3(2s+1)|(b+2)$ and $3(2s+1)|(a+2)$. Let $b+2 = 3(2s+1)m$ and $a+2 = 3(2s+1)n$. So we get $\alpha = (2t-1)m$ and $\beta = ((2s+1)-(2t-1))n$, which provides the class of integral graphs represented in (2.20). Consider the case when $(2t-1,3) = 1$ and $((2s+1)-(2t-1),3) = 3$. Then $(2s+1)-(2t-1) = 3(2k)$ where $k \in \mathbb{N}$. Since $((2s+1)-(2t-1), 2s+1) = 1$, it follows that $(2k, 2s+1) = 1$. Consequently, since $(2t-1, 3(2s+1)) = 1$ and $(2k, 2s+1) = 1$, it must be $3(2s+1) \mid (b+2)$ and $(2s+1) \mid (a+2)$. Let $b+2 = 3(2s+1)m$ and $a+2 = (2s+1)n$. So we get $\alpha = (2t-1)m$ and $\beta = 2kn$, which provides the class of integral graphs represented in (2.21). Consider the case when $(2t-1,3) = 3$. Since $((2s+1)-(2t-1), 2t-1) = 1$, we obtain that $((2s+1)-(2t-1),3) = 1$, which provides that $(2s+1)|(b+2)$ and $3(2s+1)|(a+2)$. Setting $2t-1 \to 3(2t-1)$, we find that $b+2 = (2s+1)m$ and $a+2 = 3(2s+1)n$. So we get $\alpha = (2t-1)m$ and $\beta = ((2s+1)-3(2t-1))n$, which provides the class of integral graphs represented in (2.22). □

Using Remark 2.1 and using the proof of Proposition 2.3 in a quite analogous manner, we can obtain the following result.

Proposition 2.5 *If $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is an integral graph with $\overline{\mu}_1 = ab-1$ then it uniquely determines the parameters k, m, n, s, t.*

Further, using a procedure similar to the proof of Theorems 2.2 and 2.4 we proceed to establish a characterization of integral graphs for the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$. The proof is based on the following statement [4].

Theorem 2.6 *The linear Diophantine equation $ax+by = c$ has at least one solution if and only if $d|c$ where $d = (a,b)$. In that case the most general solution of this equation is given in the form*

$$x = \frac{c}{d}\,x_0 - \frac{b}{d}\,z \text{ and } y = \frac{c}{d}\,y_0 + \frac{a}{d}\,z \qquad (z \in \mathbb{Z}),$$

where (x_0, y_0) represents a particular solution[1] *of the equation $ax+by = d$.*

Theorem 2.7 *If $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is integral, then it belongs to one of the following classes of*

[1] A particular solution of the equation $ax+by = d$ may be obtained by using the EUCLID algorithm. In that case the coefficients a and b uniquely determine x_0 and y_0.

integral graphs

$$\overline{[\frac{\tau^*}{\tau}x_0 + \frac{(2t-1)m}{\tau}z]K_{a,a,a} \cup [\frac{\tau^*}{\tau}y_0 + \frac{3a}{\tau}z](2n-1)K_{b,b,b}}, \tag{2.24}$$

where (i) $a = (t+1+3(2\ell n-\ell-n))k + (2\ell-1)m$ *and* $b = (2\ell-1)m$*; (ii)* $t,k,\ell,m,n \in \mathbb{N}$ *such that* $(2n-1,m) = 1$, $(2n-1,2t-1) = 1$, $(2\ell-1,2t-1) = 1$ *and* $(2t-1,3) = 1$*; (iii)* $\tau = (3a,(2t-1)m)$ *such that* $\tau|\tau^*$ *where* $\tau^* = (2t-1)k$*; (iv)* (x_0,y_0) *is a particular solution of the linear Diophantine equation* $(3a)x - ((2t-1)m)y = \tau$ *and (v)* $z \geq z_0$ *where* z_0 *is the least integer such that* $(\frac{\tau^*}{\tau}x_0 + \frac{(2t-1)m}{\tau}z_0) \geq 1$ *and* $(\frac{\tau^*}{\tau}y_0 + \frac{3a}{\tau}z_0) \geq 1$*;*

$$\overline{[\frac{\tau^*}{\tau}x_0 + \frac{3(2t-1)m}{\tau}z]K_{a,a,a} \cup [\frac{\tau^*}{\tau}y_0 + \frac{3a}{\tau}z](2n-1)K_{b,b,b}}, \tag{2.25}$$

where (i) $a = (t+(2\ell n-\ell-n))k + (2\ell-1)m$ *and* $b = (2\ell-1)m$*; (ii)* $t,k,\ell,m,n \in \mathbb{N}$ *such that* $(2n-1,m) = 1$, $(2n-1,2t-1) = 1$, $(2\ell-1,2t-1) = 1$, $((2\ell-1),3) = 1$ *and* $((2n-1),3) = 1$*; (iii)* $\tau = (3a,3(2t-1)m)$ *such that* $\tau|\tau^*$ *where* $\tau^* = (2t-1)k$*; (iv)* (x_0,y_0) *is a particular solution of the linear Diophantine equation* $(3a)x - (3(2t-1)m)y = \tau$ *and (v)* $z \geq z_0$ *where* z_0 *is the least integer such that* $(\frac{\tau^*}{\tau}x_0 + \frac{3(2t-1)m}{\tau}z_0) \geq 1$ *and* $(\frac{\tau^*}{\tau}y_0 + \frac{3a}{\tau}z_0) \geq 1$*;*

$$\overline{[\frac{\tau^*}{\tau}x_0 + \frac{mt}{\tau}z]K_{a,a,a} \cup [\frac{\tau^*}{\tau}y_0 + \frac{3a}{\tau}z]nK_{b,b,b}}, \tag{2.26}$$

where (i) $a = (t+3\ell n)k + \ell m$ *and* $b = \ell m$*; (ii)* $t,k,\ell,m,n \in \mathbb{N}$ *such that* $(n,m) = 1$, $(n,t) = 1$, $(\ell,t) = 1$, $(t,3) = 1$ *and* $(t+3\ell n,2) = 1$*; (iii)* $\tau = (3a,mt)$ *such that* $\tau|\tau^*$ *where* $\tau^* = 2kt$*; (iv)* (x_0,y_0) *is a particular solution of the linear Diophantine equation* $(3a)x - (mt)y = \tau$ *and (v)* $z \geq z_0$ *where* z_0 *is the least integer such that* $(\frac{\tau^*}{\tau}x_0 + \frac{mt}{\tau}z_0) \geq 1$ *and* $(\frac{\tau^*}{\tau}y_0 + \frac{3a}{\tau}z_0) \geq 1$*;*

$$\overline{[\frac{\tau^*}{\tau}x_0 + \frac{3mt}{\tau}z]K_{a,a,a} \cup [\frac{\tau^*}{\tau}y_0 + \frac{3a}{\tau}z]nK_{b,b,b}}, \tag{2.27}$$

where (i) $a = (t+\ell n)k + \ell m$ *and* $b = \ell m$*; (ii)* $t,k,\ell,m,n \in \mathbb{N}$ *such that* $(n,m) = 1$, $(n,t) = 1$, $(\ell,t) = 1$, $(\ell,3) = 1$, $(n,3) = 1$ *and* $(t+\ell n,2) = 1$*; (iii)* $\tau = (3a,3mt)$ *such that* $\tau|\tau^*$ *where* $\tau^* = 2kt$*; (iv)* (x_0,y_0) *is a particular solution of the linear Diophantine equation* $(3a)x - (3mt)y = \tau$ *and (v)* $z \geq z_0$ *where* z_0 *is the least integer such that* $(\frac{\tau^*}{\tau}x_0 + \frac{3mt}{\tau}z_0) \geq 1$ *and* $(\frac{\tau^*}{\tau}y_0 + \frac{3a}{\tau}z_0) \geq 1$.

Proof Let us assume that $\overline{\mu}_1 \in \mathbb{N}$ and let $\theta = \frac{\rho}{\varphi}$ so that $\overline{\mu}_1 + 1 = \theta a$ and $(\rho,\varphi) = 1$. Using (2.1) and (2.3), we obtain

$$\overline{\mu}_2 = -\frac{2(3\alpha+3\beta-2)b}{\theta} - 1 \text{ and } \delta = \theta a + \frac{2(3\alpha+3\beta-2)b}{\theta}.$$

Then by a straightforward calculation it is not difficult to see that (2.2) may be transformed in the form $\frac{\theta+2}{\theta} = \frac{3\alpha(a-b)}{\theta a-(3\alpha+3\beta-2)b}$. Let c be a constant such that $(\overline{1})$ $3\alpha(a-b) = c(\theta+2)$ and $(\overline{2})$ $\theta a - (3\alpha+3\beta-2)b = c\theta$. Combining $(\overline{1})$ and $(\overline{2})$, we find that $2c = (3\alpha-\theta)a + (3\beta-2)b$. Observe that $2c$ is an integer because $\theta a = (\overline{\mu}_1 + 1) \in \mathbb{N}$. Consequently, using $(\overline{1})$ or $(\overline{2})$, we arrive at $6\alpha(a-b) = ((3\alpha-\theta)a + (3\beta-2)b)(\theta+2)$. Hence,

$$a - b = \frac{r}{2}\left((3\alpha-\theta)a + (3\beta-2)b\right) \text{ and } (\theta+2) = 3r\alpha, \tag{2.28}$$

where $r = \frac{s}{t}$ such that $(s,t) = 1$. Making use of (2.28), by an easy calculation we obtain $(\overline{3})$ $3r\beta b = (r-1)(3r\alpha a - 2(a-b))$.

Using now the right-hand side of relation (2.28), note that $3r\alpha a = (\overline{\mu}_1+1)+2a$ which shows that $(3r\alpha a)$ is integral and $r-1=\frac{s-t}{t}>0$. Since $3\beta b = (1-\frac{1}{r})(3r\alpha a - 2(a-b))$ (see $(\overline{3})$), it turns out that $r|2(a-b)$. Let $(\overline{4})$ $2(a-b)=\gamma r$ where $(\overline{5})$ $\gamma = kt$ and $2|ks$. Then $(\overline{3})$ is reduced to the form $(\overline{6})$ $\beta = \frac{(s-t)}{b}\,\frac{(3\alpha a-kt)}{3t}$. We shall consider the following four cases:

Case 1 (s is even and $3|(s-t)$). Let $s\to 2s$ and let $t\to 2t-1$. Let $2s-(2t-1)=3(2p-1)$ where $p\in\mathbb{N}$. Since $(2s-(2t-1),2t-1)=1$, it follows that $(2p-1,2t-1)=1$. Let $(2p-1,b)=2\ell-1$ and let $m,n\in\mathbb{N}$ such that (1.1) $2p-1=(2\ell-1)(2n-1)$ and (1.2) $b=(2\ell-1)m$, where $(2n-1,m)=1$. Of course, since $3|(2s-(2t-1))$ and $(2s-(2t-1),2t-1)=1$, we find that $(2t-1,3)=1$. Next, since $(2p-1,2t-1)=1$, according to (1.1) we obtain $(2n-1,2t-1)=1$ and $(2\ell-1,2t-1)=1$. Therefore, using $(\overline{6})$ we have $\beta=\frac{(3\alpha a-(2t-1)k)(2n-1)}{(2t-1)m}$. Since $((2t-1)m,2n-1)=1$, it follows that $(2t-1)m|(3\alpha a-(2t-1)k)$. Consequently, setting (1.3) $3\alpha a-(2t-1)k=\eta(2t-1)m$, we get (1.4) $\beta=(2n-1)\eta$. We can see that (1.3) represents a linear Diophantine equation in variables α and η. Of course, if $(3a,(2t-1)m)=\tau$, then (1.3) has at least one solution if and only if $\tau|\tau^*$ where $\tau^*=(2t-1)k$. In this case, according to Theorem 2.6, we obtain that

$$\alpha=\frac{\tau^*}{\tau}\,x_0+\frac{(2t-1)m}{\tau}\,z \text{ and } \eta=\frac{\tau^*}{\tau}\,y_0+\frac{3a}{\tau}\,z,$$

where $(3a)x_0-((2t-1)m)y_0=\tau$. Finally, making use of $(\overline{4})$, $(\overline{5})$, and according to (1.1), (1.2), (1.4) and the last relation, we get easily that $a=(t+1+3(2\ell n-\ell-n))k+(2\ell-1)m$ and $\beta=[\frac{\tau^*}{\tau}\,y_0+\frac{3a}{\tau}\,z](2n-1)$, which provides the corresponding class of integral graphs represented in (2.24).

Case 2 (s is even and $3\nmid(s-t)$). Let $s\to 2s$ and let $t\to 2t-1$. Since $(2s-(2t-1),2t-1)=1$, it follows that $(2s-(2t-1),3(2t-1))=1$. Let $(2s-(2t-1),b)=2\ell-1$ and let $m,n\in\mathbb{N}$ such that (2.1) $2s-(2t-1)=(2\ell-1)(2n-1)$ and (2.2) $b=(2\ell-1)m$, where $(2n-1,m)=1$. Of course, since $3\nmid(2s-(2t-1))$, we find that $((2\ell-1)(2n-1),3)=1$. Next, since $(2s-(2t-1),2t-1)=1$, according to (2.1) we obtain $(2n-1,2t-1)=1$ and $(2\ell-1,2t-1)=1$. Therefore, using $(\overline{6})$, we have $\beta=\frac{(3\alpha a-(2t-1)k)(2n-1)}{3(2t-1)m}$. Since $((2n-1),3(2t-1)m)=1$, it follows that $3(2t-1)m|(3\alpha a-(2t-1)k)$. Consequently, setting (2.3) $3\alpha a-(2t-1)k=\eta(3(2t-1)m)$, we get (2.4) $\beta=(2n-1)\eta$. We can see that (2.3) represents a linear Diophantine equation in variables α and η. Of course, if $(3a,3(2t-1)m)=\tau$, then (2.3) has at least one solution if and only if $\tau|\tau^*$ where $\tau^*=(2t-1)k$. In this case, according to Theorem 2.6, we obtain that

$$\alpha=\frac{\tau^*}{\tau}\,x_0+\frac{3(2t-1)m}{\tau}\,z \text{ and } \eta=\frac{\tau^*}{\tau}\,y_0+\frac{3a}{\tau}\,z,$$

where $(3a)x_0-(3m(2t-1))y_0=\tau$. Finally, making use of $(\overline{4})$, $(\overline{5})$, and according to (2.1), (2.2), (2.4) and the last relation, we get easily that $a=(t+(2\ell n-\ell-n))k+(2\ell-1)m$ and $\beta=[\frac{\tau^*}{\tau}\,y_0+\frac{3a}{\tau}\,z](2n-1)$, which provides the corresponding class of integral graphs represented in (2.25).

Case 3 (s is odd and $3|(s-t)$). In this case k is an even number (see $(\overline{5})$). Let $s\to 2s+1$ and let $k\to 2k$ where $k\in\mathbb{N}$. Let $(2s+1)-t=3p$ where $p\in\mathbb{N}$. Since $((2s+1)-t,t)=1$, it follows that $(p,t)=1$. Let $(p,b)=\ell$ and let $m,n\in\mathbb{N}$ such that (3.1) $p=\ell n$ and (3.2) $b=\ell m$, where $(n,m)=1$. Of course, since $3|((2s+1)-t)$ and $((2s+1)-t,t)=1$ we find that $(t,3)=1$.

Besides, since $2s+1 = t+3p$, we find that $(t+3\ell n, 2) = 1$. Next, since $(p,t) = 1$, according to (3.1) we obtain $(n,t) = 1$ and $(\ell, t) = 1$. Therefore, using $(\overline{6})$, we have $\beta = \frac{(3\alpha a - 2kt)n}{mt}$. Since $(mt, n) = 1$, it follows that $mt|(3\alpha a - 2kt)$. Consequently, setting (3.3) $3\alpha a - 2kt = \eta(mt)$, we get (3.4) $\beta = \eta n$. We can see that (3.3) represents a linear Diophantine equation in variables α and η. Of course, if $(3a, mt) = \tau$, then (3.3) has at least one solution if and only if $\tau|\tau^*$ where $\tau^* = 2kt$. In this case, according to Theorem 2.6, we obtain that

$$\alpha = \frac{\tau^*}{\tau} x_0 + \frac{mt}{\tau} z \text{ and } \eta = \frac{\tau^*}{\tau} y_0 + \frac{3a}{\tau} z,$$

where $(3a)x_0 - (mt)y_0 = \tau$. Finally, making use of $(\overline{4})$, $(\overline{5})$, and according to (3.1), (3.2), (3.4) and the last relation, we get easily that $a = (t+3\ell n)k + \ell m$ and $\beta = [\frac{\tau^*}{\tau} y_0 + \frac{3a}{\tau} z]n$, which provides the corresponding class of integral graphs represented in (2.26).

Case 4 (s is odd and $3 \nmid (s-t)$). Let $s \to 2s+1$ and let $k \to 2k$ where $k \in \mathbb{N}$. Since $((2s+1)-t, t) = 1$, it follows that $((2s+1)-t, 3t) = 1$. Let $((2s+1)-t), b) = \ell$ and let $m, n \in \mathbb{N}$ such that (4.1) $(2s+1)-t = \ell n$ and (4.2) $b = \ell m$, where $(n,m) = 1$. Of course, since $3 \nmid ((2s+1)-t)$, we find that $(\ell n, 3) = 1$. Besides, since $2s+1 = t+\ell n$, we find that $(t+\ell n, 2) = 1$. Next, since $((2s+1)-t, t) = 1$, according to (4.1) we obtain $(n,t) = 1$ and $(\ell, t) = 1$. Therefore, using $(\overline{6})$, we have $\beta = \frac{(3\alpha a - 2kt)n}{3mt}$. Since $(n, 3mt) = 1$, it follows that $3mt|(3\alpha a - 2kt)$. Consequently, setting (4.3) $3\alpha a - 2kt = \eta(3mt)$, we get (4.4) $\beta = \eta n$. We can see that (4.3) represents a linear Diophantine equation in variables α and η. Of course, if $(3a, 3mt) = \tau$, then (4.3) has at least one solution if and only if $\tau|\tau^*$ where $\tau^* = 2kt$. In this case, according to Theorem 2.6, we obtain that

$$\alpha = \frac{\tau^*}{\tau} x_0 + \frac{3mt}{\tau} z \text{ and } \eta = \frac{\tau^*}{\tau} y_0 + \frac{3a}{\tau} z,$$

where $(3a)x_0 - (3mt)y_0 = \tau$. Finally, making use of $(\overline{4})$, $(\overline{5})$, and according to (4.1), (4.2), (4.4) and the last relation, we get easily that $a = (t+\ell n)k + \ell m$ and $\beta = [\frac{\tau^*}{\tau} y_0 + \frac{3a}{\tau} z]n$, which provides the corresponding class of integral graphs represented in (2.27). □

Proposition 2.8 *If* $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ *is an integral graph, then it uniquely determines the parameters* τ, t, k, ℓ, m, n.

Proof Let us assume that $\tau_1, t_1, k_1, \ell_1, m_1, n_1$ and $\tau_2, t_2, k_2, \ell_2, m_2, n_2$ determine the same integral graph $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$. Since α, β, a, b determine the graph $\alpha K_{a,a,a} \cup \beta K_{b,b,b}$ up to isomorphism, using the second equality of (2.28), we have $3r\alpha a = (\overline{\mu}_1 + 1) + 2a$, which shows that $s_1 = s_2$ and $t_1 = t_2$ because $(s,t) = 1$. In view of this, we note that the classes represented by relations (2.24)–(2.27) are mutually disjoint. Consequently, without loss of generality, we can assume that the corresponding integral graph determined by the parameters $\tau_1, t_1, k_1, \ell_1, m_1, n_1$ and $\tau_2, t_2, k_2, \ell_2, m_2, n_2$ belongs to the class of integral graphs displayed in relation (2.27). Next, using $(\overline{4})$ and $(\overline{5})$, we get $k_1 = k_2$. Since $((2s+1)-t, b) = \ell$, we have $\ell_1 = \ell_2$. Since $b = \ell m$ and $a = (t+\ell n)k + \ell m$, we also have $m_1 = m_2$ and $n_1 = n_2$. Finally, since $(3a, 3mt) = \tau$, it follows that $\tau_1 = \tau_2$. □

Remark 2.9 If (x_0, y_0) is obtained by using the EUCLID algorithm, then a fixed integral graph $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ also uniquely determines the parameters x_0, y_0, z_0, z.

Theorem 2.10 *If $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is an integral graph with $\alpha = 1$ and $\beta = 1$, then it belongs to one of the following classes of integral graphs:*

(1^0) where (i) $a = kt(s+t)$ and $b = ks(8s-t)$; (ii) $k, s, t \in \mathbb{N}$ where $8s > t > 2s$ such that $(8s, t) = 1$ and $(s+t, 9) = 1$;

(2^0) where (i) $a = kt(3t-s)$ and $b = ks(3s-t)$; (ii) $k, s, t \in \mathbb{N}$ where $3s > t > s$ such that $(8s, 3t-s) = 1$ and $(t, 3) = 1$;

(3^0) where (i) $a = kt(9t-s)$ and $b = ks(s-t)$; (ii) $k, s, t \in \mathbb{N}$ where $s > t > \lfloor \frac{s}{3} \rfloor$ such that $(8s, 9t-s) = 1$;

(4^0) where (i) $a = kt(s+2t)$ and $b = ks(4s-t)$; (ii) $k, s, t \in \mathbb{N}$ where $4s > t > s$ such that $(4s, t) = 1$, $(4s-t, 9) = 1$ and $(s, 2) = 1$;

(5^0) where (i) $a = k(3s-2t)(4s-3t)$ and $b = kst$; (ii) $k, s, t \in \mathbb{N}$ where $t < s$ such that $(4s, 4s-3t) = 1$, $(s, 2) = 1$ and $(t, 3) = 1$;

(6^0) where (i) $a = k(s-2t)(4s-9t)$ and $b = kst$; (ii) $k, s, t \in \mathbb{N}$ where $t < \lceil \frac{s}{3} \rceil$ such that $(4s, 4s-9t) = 1$ and $(s, 2) = 1$;

(7^0) where (i) $a = kt(s+4t)$ and $b = ks(2s-t)$; (ii) $k, s, t \in \mathbb{N}$ where $2s > t > \lfloor \frac{s}{2} \rfloor$ such that $(s, 4t) = 1$, $(s+4t, 9) = 1$ and $(t, 2) = 1$;

(8^0) where (i) $a = kst$ and $b = k(2s-3t)(3s-4t)$; (ii) $k, s, t \in \mathbb{N}$ where $\lceil \frac{2s}{3} \rceil > t > \lfloor \frac{s}{2} \rfloor$ such that $(3s-4t, 4t) = 1$, $(s, 3) = 1$ and $(t, 2) = 1$;

(9^0) where (i) $a = kst$ and $b = k(2s-t)(9s-4t)$; (ii) $k, s, t \in \mathbb{N}$ where $2s > t > \lfloor \frac{3s}{2} \rfloor$ such that $(9s-4t, 4t) = 1$ and $(t, 2) = 1$;

(10^0) where (i) $a = kt(s+8t)$ and $b = ks(s-t)$; (ii) $k, s, t \in \mathbb{N}$ where $s > t > \lfloor \frac{s}{4} \rfloor$ such that $(s, 8t) = 1$ and $(s-t, 9) = 1$;

(11^0) where (i) $a = kt(s+3t)$ and $b = ks(3s+t)$; (ii) $k, s, t \in \mathbb{N}$ where $t > s$ such that $(3s+t, 8t) = 1$ and $(s, 3) = 1$;

(12^0) where (i) $a = kt(s+t)$ and $b = ks(9s+t)$; (ii) $k, s, t \in \mathbb{N}$ where $t > 3s$ such that $(9s+t, 8t) = 1$.

Proof Let us assume that $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is an integral graph with $\alpha = 1$ and $\beta = 1$. Using (2.3), we find that $(\overline{\mu}_1 + 1)(\overline{\mu}_2 + 1) = -8ab$. Let $\overline{\mu}_1 + 1 = (8a)r$ where $r = \frac{s}{t}$ such that $(s, t) = 1$. Then

$$\overline{\mu}_2 = -\frac{tb}{s} - 1 \text{ and } \delta = \frac{(8a)s}{t} + \frac{tb}{s}. \tag{2.29}$$

Case 1 ($(t, 8) = 1$). Since $(8s, t) = 1$, we obtain $\overline{\mu}_1 = \frac{(8s)a}{t} - 1$ and $\overline{\mu}_2 = -\frac{tb}{s} - 1$. Since $(s, t) = 1$ and $(8s, t) = 1$ it follows that $t|a$ and $s|b$. Setting (1.1) $a = tn$ and (1.2) $b = sm$, we obtain that $\overline{\mu}_1 = 8sn - 1$, $\overline{\mu}_2 = -tm - 1$ and $\delta = 8sn + tm$ (see (2.29)). Using (2.1), we arrive at $(8n - m)s = (n + m)t$. Since $(s, t) = 1$, we find that (1.3) $n + m = ks$ and (1.4) $8n - m = kt$, where $k \in \mathbb{N}$. Using (1.3) and (1.4), we get (1.5) $9n = k(s + t)$. Consider the case when $(s + t, 9) = 1$, which provides that $9|k$. Setting $k \to 9k$ and using (1.1), (1.2), (1.3) and

(1.5), we obtain the corresponding class of integral graphs displayed in (1^0). Consider the case when $(s+t,9)=3$. Setting $s+t=3\ell$ and using (1.5), we obtain $3n=k\ell$, which provides that $3|k$. Setting $k \to 3k$ and replacing ℓ with t, we obtain the corresponding class of integral graphs displayed in (2^0). Consider the case when $(s+t,9)=9$. Setting $s+t=9\ell$ and using (1.5), we obtain $n=k\ell$. Replacing ℓ with t, we obtain the corresponding class of integral graphs displayed in (3^0).

Case 2 ($(t,8)=2$). Setting $t \to 2t$, we have $(t,4)=1$ and $(4s,t)=1$. Since $(s,2t)=1$ and $(4s,t)=1$, it follows that $t|a$ and $s|b$. Setting (2.1) $a=tn$ and (2.2) $b=sm$, we obtain that $\overline{\mu}_1 = 4sn-1$, $\overline{\mu}_2 = -2tm-1$ and $\delta = 4sn+2tm$. Using (2.1), we arrive at $(4n-m)s=(n+2m)t$. Since $(s,t)=1$, we find that (2.3) $n+2m=ks$ and (2.4) $4n-m=kt$, where $k \in \mathbb{N}$. Using (2.3) and (2.4), we get (2.5) $9m=k(4s-t)$. Consider the case when $(4s-t,9)=1$, which provides that $9|k$. Setting $k \to 9k$ and using (2.1), (2.2), (2.3) and (2.5), we obtain the corresponding class of integral graphs displayed in (4^0). Consider the case when $(4s-t,9)=3$. Setting $4s-t=3\ell$ and using (2.5), we obtain $3m=k\ell$, which provides that $3|k$. Setting $k \to 3k$ and replacing ℓ with t, we obtain the corresponding class of integral graphs displayed in (5^0). Consider the case when $(4s-t,9)=9$. Setting $4s-t=9\ell$ and using (2.5) we obtain $m=k\ell$. Replacing ℓ with t, we obtain the corresponding class of integral graphs displayed in (6^0).

Case 3 ($(t,8)=4$). Setting $t \to 4t$, we have $(t,2)=1$ and $(2s,t)=1$. Since $(s,4t)=1$ and $(2s,t)=1$, it follows that $t|a$ and $s|b$. Setting (3.1) $a=tn$ and (3.2) $b=sm$, we obtain that $\overline{\mu}_1 = 2sn-1$, $\overline{\mu}_2 = -4tm-1$ and $\delta = 2sn+4tm$. Using (2.1), we arrive at $(2n-m)s=(n+4m)t$. Since $(s,t)=1$, we find that (3.3) $n+4m=ks$ and (3.4) $2n-m=kt$, where $k \in \mathbb{N}$. Using (3.3) and (3.4), we get (3.5) $9n=k(s+4t)$. Consider the case when $(s+4t,9)=1$, which provides that $9|k$. Setting $k \to 9k$ and using (3.1), (3.2), (3.4) and (3.5), we obtain the corresponding class of integral graphs displayed in (7^0). Consider the case when $(s+4t,9)=3$. Setting $s+4t=3\ell$ and using (3.5), we obtain $3n=k\ell$, which provides that $3|k$. Setting $k \to 3k$ and replacing ℓ with s, we obtain the corresponding class of integral graphs displayed in (8^0). Consider the case when $(s+4t,9)=9$. Setting $s+4t=9\ell$ and using (3.5), we obtain $n=k\ell$. Replacing ℓ with s, we obtain the corresponding class of integral graphs displayed in (9^0).

Case 4 ($(t,8)=8$). Setting $t \to 8t$, we have $(s,t)=1$. Since $(s,8t)=1$ and $(s,t)=1$, it follows that $t|a$ and $s|b$. Setting (4.1) $a=tn$ and (4.2) $b=sm$, we obtain that $\overline{\mu}_1 = sn-1$, $\overline{\mu}_2 = -8tm-1$ and $\delta = sn+8tm$. Using (2.1), we arrive at $(n-m)s=(n+8m)t$. Since $(s,t)=1$, we find that (4.3) $n+8m=ks$ and (4.4) $n-m=kt$, where $k \in \mathbb{N}$. Using (4.3) and (4.4), we get (4.5) $9m=k(s-t)$. Consider the case when $(s-t,9)=1$, which provides that $9|k$. Setting $k \to 9k$ and using (4.1), (4.2), (4.4) and (4.5), we obtain the corresponding class of integral graphs displayed in (10^0). Consider the case when $(s-t,9)=3$. Setting $s-t=3\ell$ and using (4.5), we obtain $3m=k\ell$, which provides that $3|k$. Setting $k \to 3k$ and replacing ℓ with s, we obtain the corresponding class of integral graphs displayed in (11^0). Consider the case when $(s-t,9)=9$. Setting $s-t=9\ell$ and using (4.5), we obtain $m=k\ell$. Replacing ℓ with s, we obtain the corresponding class of integral graphs displayed in (12^0). □

Proposition 2.11 *If $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is an integral graph with $\alpha = 1$ and $\beta = 1$, then it uniquely determines the parameters k, s, t.*

Proof Assume that k_1, s_1, t_1 and k_2, s_2, t_2 determine the same integral graph $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ with $\alpha = 1$ and $\beta = 1$. Since the parameters α, β, a, b determine the graph $\alpha K_{a,a,a} \cup \beta K_{b,b,b}$ up to isomorphism, using the first equality of (2.29), we have $(\overline{\mu}_2 + 1)r = -b$, which shows that $s_1 = s_2$ and $t_1 = t_2$ because $(s, t) = 1$. In view of this, we note that the classes represented by Theorem 2.10 $(1^0)-(12^0)$ are mutually disjoint. Consequently, without loss of generality, we can assume that the corresponding integral graph determined by the parameters $k_1 s_1, t_1$ and k_2, s_2, t_2 belong to the class Theorem 2.10 (1^0). Hence, using (1.1) and (1.5), we have $9n = k_1(s_1 + t_1)$ and $9n = k_2(s_2 + t_2)$ which provides that $k_1 = k_2$. $\square$

Theorem 2.12 *There exists no integral graph from the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ with $\overline{\mu}_1 = a - 1$ for any $\alpha, \beta, a, b \in \mathbb{N}$.*

Proof Let us assume that $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is integral with $\overline{\mu}_1 = a-1$. Using that $\overline{\mu}_1 + 1 = \theta a$, we obtain $\theta = 1$. Using the right-hand side of relation (2.28), we find that $r\alpha = 1$. Since $r > 1$, it follows that $r\alpha > 1$, a contradiction. $\square$

Theorem 2.13 *If $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is an integral graph with $\overline{\mu}_1 = 2a - 1$, then it belongs to the class of integral graphs $\overline{K_{a,a,a} \cup \beta K_{b,b,b}}$ where $a = (6\beta - 1)m$ and $b = m$ for any $\beta, m \in \mathbb{N}$.*

Proof Let us assume that $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is integral with $\overline{\mu}_1 = 2a-1$. Using that $\overline{\mu}_1 + 1 = \theta a$, we obtain $\theta = 2$. Using the right-hand side of relation (2.28), we find that $3r\alpha = 4$. Since $r > 1$, it follows that $\alpha = 1$. In view of this, we have $s = 4$ and $t = 3$. Using $(\overline{4})$ and $(\overline{5})$, we find that $a = 2k + b$. Using $(\overline{6})$, we obtain $\beta = \frac{b+k}{3b}$, which provides that $3b|(b + k)$. Setting $b + k = (3b)\ell$, we obtain that $\beta = \ell$ and $k = (3\ell - 1)b$. Replacing ℓ with β and replacing b with m, we obtain the corresponding class of integral graphs represented in this statement. $\square$

Theorem 2.14 *If $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is an integral graph with $\overline{\mu}_1 = 3a - 1$, then it belongs to one of the following classes of integral graphs: (1^0) $\overline{K_{a,a,a} \cup \beta K_{b,b,b}}$ where $a = (15\beta - 4)m$ and $b = 6m$ or (2^0) $\overline{K_{a,a,a} \cup 2\beta K_{b,b,b}}$ where $a = (15\beta - 2)(2m - 1)$ and $b = 3(2m - 1)$ for any $\beta, m \in \mathbb{N}$.*

Proof Let us assume that $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is integral with $\overline{\mu}_1 = 3a-1$. Using that $\overline{\mu}_1 + 1 = \theta a$, we obtain $\theta = 3$. Using the right-hand side of relation (2.28), we find that $3r\alpha = 5$. Since $r > 1$, it follows that $\alpha = 1$. In view of this, we have $s = 5$ and $t = 3$. Setting $k \to 2k$ and using $(\overline{4})$ and $(\overline{5})$, we find that $a = 5k + b$. Using $(\overline{6})$, we obtain $\beta = \frac{2(b+3k)}{3b}$. Consider the case when b is an even number. Setting $b = 2m$, we have $\beta = \frac{2m+3k}{3m}$, which provides that $3|m$. Setting $m \to 3m$, it follows that $3m|(2m + k)$. Setting $2m + k = (3m)\ell$, we obtain that $\beta = \ell$ and $k = (3\ell - 2)m$. Replacing ℓ with β, we obtain the corresponding class of integral graphs displayed in (1^0). Consider the case when b is an odd number. Setting $b = 2m - 1$, we have $\beta = \frac{2((2m-1)+3k)}{3(2m-1)}$, which provides that $3|(2m - 1)$. Setting $(2m - 1) \to 3(2m - 1)$, it follows that $3(2m-1)|((2m-1)+k)$. Consequently, setting $(2m-1)+k = 3(2m-1)\ell$, we obtain that $\beta = 2\ell$ and $k = (3\ell-1)(2m-1)$. Replacing ℓ with β, we obtain the corresponding class of integral graphs displayed in (2^0). $\square$

Theorem 2.15 *If* $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ *is an integral graph with* $\overline{\mu}_1 = 4a-1$*, then it belongs to one of the following classes of integral graphs:* (1^0) $\overline{K_{a,a,a} \cup (\beta+1)K_{b,b,b}}$ *where* $a = (3\beta+2)m$ *and* $b = 2m$ *or* (2^0) $\overline{K_{a,a,a} \cup (2\beta+1)K_{b,b,b}}$ *where* $a = (3\beta+1)(2m-1)$ *and* $b = 2m-1$ *for any* $\beta, m \in \mathbb{N}$.

Proof Let us assume that $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is integral with $\overline{\mu}_1 = 4a-1$. Using $\overline{\mu}_1 + 1 = \theta a$, we obtain $\theta = 4$. Using the right-hand side of relation (2.28), we find that $r\alpha = 2$. Since $r > 1$, it follows that $\alpha = 1$. In view of this, we have $s = 2$ and $t = 1$. Using $(\overline{4})$ and $(\overline{5})$, we find that $a = k+b$. Using $(\overline{6})$, we obtain $\beta = 1 + \frac{2k}{3b}$. Consider the case when b is an even number. Setting $b = 2m$, we obtain $3m|k$. Setting $k = (3m)\ell$, we find that $\beta = 1+\ell$. Replacing ℓ with β, we obtain the corresponding class of integral graphs displayed in (1^0). Consider the case when b is an odd number. Setting $b = 2m-1$, we obtain $3(2m-1)|2k$. Setting $2k = 2\ell(3(2m-1))$, we find that $\beta = 1+2\ell$ and $k = 3\ell(2m-1)$. Replacing ℓ with β, we obtain the corresponding class of integral graphs displayed in (2^0). $\square$

Theorem 2.16 *There exists no integral graph from the class* $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ *with* $\overline{\mu}_1 = b-1$ *for any* $\alpha, \beta, a, b \in \mathbb{N}$.

Proof Let us assume that $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is integral with $\overline{\mu}_1 = b-1$. Using that $\overline{\mu}_1 + 1 = \theta a$, we obtain $\theta = \frac{b}{a}$. Using the right-hand side of relation (2.28), we find that $3 > \frac{b+2a}{a} = 3r\alpha > 3$, a contradiction. $\square$

Theorem 2.17 *There exists no integral graph from the class* $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ *with* $\overline{\mu}_1 = 2b-1$ *for any* $\alpha, \beta, a, b \in \mathbb{N}$.

Proof Let us assume that $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is integral with $\overline{\mu}_1 = 2b-1$. Using that $\overline{\mu}_1 + 1 = \theta a$, we obtain $\theta = \frac{2b}{a}$. Using the right-hand side of relation (2.28), we find that $\frac{2b+2a}{a} = 3r\alpha$. Since $r > 1$, it follows that $\alpha = 1$. Using (2.3), we find that $\overline{\mu}_2 + 1 = -a(3\beta+1)$, which yields $|\overline{\mu}_2| > \overline{\mu}_1$, a contradiction. $\square$

Theorem 2.18 *There exists no integral graph from the class* $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ *with* $\overline{\mu}_1 = 3b-1$ *for any* $\alpha, \beta, a, b \in \mathbb{N}$.

Proof Let us assume that $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is integral with $\overline{\mu}_1 = 3b-1$. Using that $\overline{\mu}_1 + 1 = \theta a$, we obtain $\theta = \frac{3b}{a}$. Using the right-hand side of relation (2.28), we find that $\frac{3b+2a}{a} = 3r\alpha$. Since $r > 1$, it follows that $\alpha = 1$. Using (2.3), we find that $\overline{\mu}_2 + 1 = -\frac{2a(3\beta+1)}{3}$. Since $\overline{\mu}_1 \geq |\overline{\mu}_2|$, it must be $\beta = 1$. So we obtain $\overline{\mu}_2 + 1 = -\frac{8a}{3}$ and $\delta = 3b + \frac{8a}{3}$. Using (2.1), we easily obtain $11a = 6b$, a contradiction. $\square$

Theorem 2.19 *There exists no integral graph from the class* $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ *with* $\overline{\mu}_1 = 4b-1$

for any $\alpha, \beta, a, b \in \mathbb{N}$.

Proof Let us assume that $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is integral with $\overline{\mu}_1 = 4b-1$. Using that $\overline{\mu}_1 + 1 = \theta a$, we obtain $\theta = \frac{4b}{a}$. Using the right-hand side of relation (2.28), we find that $\frac{4b+2a}{a} = 3r\alpha$. Since $r > 1$, it follows that $\alpha = 1$. Using (2.3), we find that $\overline{\mu}_2 + 1 = -\frac{a(3\beta+1)}{2}$. Since $\overline{\mu}_1 \geq |\overline{\mu}_2|$, it must be $\beta = 1$ or $\beta = 2$. Consider the case when $\beta = 1$. In this case we have $\overline{\mu}_2 + 1 = -2a$ and $\delta = 4b + 2a$. Using (2.1), we easily obtain $a = b$, a contradiction. Consider the case when $\beta = 2$. In this case we have $\overline{\mu}_2 + 1 = -\frac{7a}{2}$ and $\delta = 4b + \frac{7a}{2}$. Using (2.1), we easily obtain $9a = 0$, a contradiction. □

Theorem 2.20 *If* $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ *is an integral graph with* $\overline{\mu}_1 = 5b - 1$*, then it belongs to the class of integral graphs* $\overline{K_{a,a,a} \cup K_{b,b,b}}$ *where* $a = 20m$ *and* $b = 13m$ *for any* $m \in \mathbb{N}$.

Proof Let us assume that $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is integral with $\overline{\mu}_1 = 5b-1$. Using that $\overline{\mu}_1 + 1 = \theta a$, we obtain $\theta = \frac{5b}{a}$. Using the right-hand side of relation (2.28), we find that $\frac{5b+2a}{a} = 3r\alpha$. Since $r > 1$, it follows that $\alpha = 1$ or $\alpha = 2$.

Case 1 ($\alpha = 1$). Using (2.3), we find that $\overline{\mu}_2 + 1 = -\frac{2a(3\beta+1)}{5}$. Since $\overline{\mu}_1 \geq |\overline{\mu}_2|$, it must be $\beta = 1$ or $\beta = 2$ or $\beta = 3$. Consider the case when $\beta = 1$. In this case we have $\overline{\mu}_2 + 1 = -\frac{8a}{5}$ and $\delta = 5b + \frac{8a}{5}$. Using (2.1), we easily obtain $13a = 20b$. Setting $a = 20m$ and $b = 13m$, we obtain the corresponding class of integral graphs represented in this statement. Consider the case when $\beta = 2$. In this case we have $\overline{\mu}_2 + 1 = -\frac{14a}{5}$ and $\delta = 5b + \frac{14a}{5}$. Using (2.1), we easily obtain $19a = 5b$, a contradiction. Consider the case when $\beta = 3$. In this case we have $\overline{\mu}_2 + 1 = -4a$ and $\delta = 5b + 4a$. Using (2.1), we easily obtain $5a + 2b = 0$, a contradiction.

Case 2 ($\alpha = 2$). Using (2.3), we find that $\overline{\mu}_2 + 1 = -\frac{2a(3\beta+4)}{5}$. Since $\overline{\mu}_1 \geq |\overline{\mu}_2|$, it must be $\beta = 1$ or $\beta = 2$. Consider the case when $\beta = 1$. In this case we have $\overline{\mu}_2 + 1 = -\frac{14a}{5}$ and $\delta = 5b + \frac{14a}{5}$. Using (2.1), we easily obtain $34a = 20b$, a contradiction. Consider the case when $\beta = 2$. In this case we have $\overline{\mu}_2 + 1 = -4a$ and $\delta = 5b + 4a$. Using (2.1), we easily obtain $8a = b$, a contradiction. □

Remark 2.21 If $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ is an integral graph with $\overline{\mu}_1 = 6b - 1$ or $\overline{\mu}_1 = 7b - 1$, then in both cases we also have $\alpha = 1$ or $\alpha = 2$.

Theorem 2.22 *If* $(\alpha, \beta, a, b, \delta)$ *is a positive integral solution of the Diophantine equation* (2.2)*, then it could be represented by one of the following forms:*

- $a = (t + 1 + 3(2\ell n - \ell - n))k + (2\ell - 1)m$ *and* $b = (2\ell - 1)m$;
- $\alpha = \dfrac{\tau^*}{\tau}\, x_0 + \frac{(2t-1)m}{\tau}\, z$;
- $\beta = [\dfrac{\tau^*}{\tau}\, y_0 + \frac{3a}{\tau}\, z](2n - 1)$;
- $\delta = 3k(2\ell - 1)(2n - 1) + [\frac{\tau^*}{\tau}\, y_0 + \frac{3a}{\tau}\, z]((2t - 1) + 3(2\ell - 1)(2n - 1))m$ *with the same conditions (ii)–(v) which are related to* (2.24);
- $a = (t + (2\ell n - \ell - n))k + (2\ell - 1)m$ *and* $b = (2\ell - 1)m$;
- $\alpha = \dfrac{\tau^*}{\tau}\, x_0 + \frac{3(2t-1)m}{\tau}\, z$;

- $\beta = [\frac{\tau^*}{\tau} y_0 + \frac{3a}{\tau} z](2n-1)$;
- $\delta = k(2\ell-1)(2n-1) + [\frac{\tau^*}{\tau} y_0 + \frac{3a}{\tau} z](3(2t-1) + 3(2\ell-1)(2n-1))m$ *with the same conditions (ii)–(v) which are related to* (2.25);
- $a = (t+3\ell n)k + \ell m$ and $b = \ell m$;
- $\alpha = \frac{\tau^*}{\tau} x_0 + \frac{mt}{\tau} z$;
- $\beta = [\frac{\tau^*}{\tau} y_0 + \frac{3a}{\tau} z]n$;
- $\delta = 6k\ell n + [\frac{\tau^*}{\tau} y_0 + \frac{3a}{\tau} z](t+3\ell n)m$ *with the same conditions (ii)–(v) which are related to* (2.26);
- $a = (t+\ell n)k + \ell m$ and $b = \ell m$;
- $\alpha = \frac{\tau^*}{\tau} x_0 + \frac{3mt}{\tau} z$;
- $\beta = [\frac{\tau^*}{\tau} y_0 + \frac{3a}{\tau} z]n$;
- $\delta = 2k\ell n + [\frac{\tau^*}{\tau} y_0 + \frac{3a}{\tau} z](3t+3\ell n)m$ *with the same conditions (ii)–(v) which are related to* (2.27).

Proof According to Theorem 2.7 it suffices to derive the expression for δ. First, from (2.1) we have (i) $\overline{\mu}_1 - \overline{\mu}_2 = \delta$ and (ii) $\overline{\mu}_1 + \overline{\mu}_2 = 3(\alpha a + \beta b) - 2(a+b+1)$. Using (i), (ii) and the equality $\overline{\mu}_1 = 3r\alpha a - (2a+1)$ (see (2.28)), by a straightforward calculation we obtain that $\delta = 6r\alpha a - 3(\alpha a + \beta b) - 2(a-b)$.

Case 1 (s is even and $3|(s-t)$). Using (($\overline{4}$) and ($\overline{5}$)), (1.1), (1.2), (1.3) and (1.4), we obtain that $a - b = ks$, $s = t+1+3(2\ell n - \ell - n)$, $b = (2\ell-1)m$, $3\alpha a = (2t-1)k + \eta(2t-1)m$, $3r\alpha a = 2ks + \eta(2s)m$ and $\beta = \eta(2n-1)$. So we find that $\delta = 3k(2\ell-1)(2n-1) + \eta m((2t-1) + 3(2\ell-1)(2n-1))$, which provides the statement related to (2.24).

Case 2 (s is even and $3 \nmid (s-t)$). Using (($\overline{4}$) and ($\overline{5}$)), (2.1), (2.2), (2.3) and (2.4), we obtain that $a - b = ks$, $s = t + (2\ell n - \ell - n)$, $b = (2\ell-1)m$, $3\alpha a = (2t-1)k + 3\eta(2t-1)m$, $3r\alpha a = 2ks + 3\eta(2s)m$ and $\beta = \eta(2n-1)$. So we find that $\delta = k(2\ell-1)(2n-1) + 3\eta m((2t-1) + (2\ell-1)(2n-1))$, which provides the statement related to (2.25).

Case 3 (s is odd and $3|(s-t)$). Using (($\overline{4}$) and ($\overline{5}$)), (3.1), (3.2), (3.3) and (3.4), we obtain that $a-b = k(2s+1)$, $2s+1 = t+3\ell n$, $b = \ell m$, $3\alpha a = 2kt+\eta(mt)$, $3r\alpha a = 2k(2s+1)+\eta m(2s+1)$ and $\beta = \eta n$. So we find that $\delta = 6k\ell n + \eta m(t+3\ell n)$, which provides the statement related to (2.26).

Case 4 (s is odd and $3 \nmid (s-t)$). Using (($\overline{4}$) and ($\overline{5}$)), (4.1), (4.2), (4.3) and (4.4), we obtain that $a-b = k(2s+1)$, $2s+1 = t+\ell n$, $b = \ell m$, $3\alpha a = 2kt+3\eta(mt)$, $3r\alpha a = 2k(2s+1)+3\eta m(2s+1)$ and $\beta = \eta n$. So we find that $\delta = 2k\ell n + 3\eta m(t+\ell n)$, which provides the statement related to (2.27). □

3. Appendix

In this section we present the data given in Tables 1–4, which represent the set of all integral graphs from the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, whose order does not exceed 80. In these tables an integral graph is described by the parameters α, β, a, b and ones presented in the classes of integral graphs in Theorem 2.7. In Tables 1–4 the symbol 'i' is the identification number of an integral graph.

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	$\overline{\mu}_1$	$\overline{\mu}_2$
1	0	−1	1	21	1	2	5	1	5	3	1	1	1	1	13	−6
2	1	1	0	27	1	1	7	2	7	4	1	1	2	1	15	−8
3	0	−1	1	33	1	8	3	1	1	1	1	1	1	1	29	−6
4	0	−1	1	42	1	2	10	2	10	3	2	1	2	1	27	−11
5	0	−1	2	45	2	5	5	1	5	3	1	1	1	1	37	−6
6	0	−1	1	51	1	6	11	1	11	6	1	1	1	2	37	−12
7	1	1	0	54	1	1	14	4	14	4	2	1	4	1	31	−15
8	0	−1	1	54	1	13	5	1	1	1	2	1	1	1	49	−9
9	1	1	0	57	1	3	13	2	13	7	1	1	2	2	39	−14
10	0	−1	1	63	1	2	15	3	15	3	3	1	3	1	41	−16
11	4	5	−1	66	1	1	15	7	5	3	2	1	7	1	41	−21
12	0	−1	1	66	1	8	6	2	2	1	2	1	2	1	59	−11
13	0	−1	3	69	3	8	5	1	5	3	1	1	1	1	61	−6
14	0	−1	2	69	2	17	3	1	1	1	1	1	1	1	65	−6
15	0	−1	1	75	1	3	22	1	11	6	3	1	1	1	39	−12
16	0	−1	1	75	1	5	10	3	5	3	1	2	1	1	63	−16
17	0	−1	1	75	1	18	7	1	1	1	3	1	1	1	69	−12
18	0	−1	1	78	1	5	21	1	7	4	4	1	1	1	47	−15

Table 1 Integral graphs with $o \leq 80$ belonging to the class (2.24)

The Tables 1–4 contain the integral graphs from the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ whose order does not exceed 80, which belong to the classes Theorem 2.7 (2.24), (2.25), (2.26) and (2.27), respectively. In view of this, there exist exactly $18 + 36 + 6 + 13 = 73$ non-isomorphic integral graphs which belong to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, whose order does not exceed 80.

There exist exactly 9 non-isomorphic integral graphs with $\overline{\mu}_1 = 2ab - 1$ which belong to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, whose order does not exceed 80. They are represented in Table 2 under identification numbers $i = 1, 3, 7, 9, 10, 16, 17, 28, 34$.

There exist exactly 5 non-isomorphic integral graphs with $\overline{\mu}_1 = ab - 1$ which belong to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, whose order does not exceed 80. They are represented in Table 2 under identification numbers $i = 5, 15, 31, 34$ and in Table 4 under identification number $i = 8$.

There exist exactly 13 non-isomorphic integral graphs with $\alpha = 1$ and $\beta = 1$, which belong to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$ whose order does not exceed 80. They are represented in Table 1 under identification numbers $i = 2, 7, 11$, in Table 2 under identification numbers $i = 1, 4, 5, 12, 14, 21, 22, 27$ and in Table 4 under identification numbers $i = 3, 7$.

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	$\overline{\mu}_1$	$\overline{\mu}_2$
1	2	3	−1	18	1	1	5	1	3	2	2	1	1	1	9	−5
2	0	−1	1	21	1	3	4	1	3	1	3	1	1	1	15	−6
3	1	2	0	27	1	2	5	2	3	1	3	1	2	1	19	−8
4	0	−1	1	33	1	1	10	1	15	3	3	1	1	1	15	−6
5	2	3	−1	36	1	1	10	2	6	2	4	1	2	1	19	−9
6	0	−1	1	36	1	5	7	1	3	1	6	1	1	1	27	−9
7	2	7	−3	39	1	2	11	1	3	2	5	1	1	1	21	−8
8	−1	−1	1	39	2	1	6	1	9	5	1	1	1	1	27	−4
9	2	1	0	39	2	1	5	3	3	2	1	1	3	1	29	−8
10	0	−1	1	42	1	3	8	2	6	1	6	1	2	1	31	−11
11	0	−1	2	45	2	7	4	1	3	1	3	1	1	1	39	−6
12	1	1	0	51	1	1	12	5	9	5	1	3	1	1	31	−16
13	0	−1	1	51	1	7	10	1	3	1	9	1	1	1	39	−12
14	2	3	−1	54	1	1	15	3	9	2	6	1	3	1	29	−13
15	1	2	0	54	1	2	10	4	6	1	6	1	4	1	39	−15
16	1	5	−1	57	1	4	11	2	3	1	9	1	2	1	43	−14
17	2	11	−5	60	1	3	17	1	3	2	8	1	1	1	33	−11
18	2	3	−1	60	1	5	15	1	9	5	2	1	1	3	39	−13
19	0	−1	1	63	1	3	12	3	9	1	9	1	3	1	47	−16
20	1	1	0	63	1	7	7	2	3	2	1	1	2	4	55	−12
21	1	2	−1	66	1	1	21	1	9	5	4	1	1	1	27	−7
22	0	−1	1	66	1	1	20	2	30	3	6	1	2	1	31	−11
23	0	−1	1	66	1	9	13	1	3	1	12	1	1	1	51	−15
24	0	−1	2	69	2	3	10	1	15	3	3	1	1	1	51	−6
25	13	5	−2	69	3	1	7	2	3	5	1	1	2	1	55	−6
26	0	−1	3	69	3	11	4	1	3	1	3	1	1	1	63	−6
27	2	3	−1	72	1	1	20	4	12	2	8	1	4	1	39	−17
28	0	−1	1	72	1	5	14	2	6	1	12	1	2	1	55	−17
29	1	1	0	75	1	13	12	1	9	5	1	1	1	7	63	−16
30	2	3	0	75	2	15	5	1	3	2	1	1	1	3	69	−8
31	2	7	−3	78	1	2	22	2	6	2	10	1	2	1	43	−15
32	−1	−1	1	78	2	1	12	2	18	5	2	1	2	1	55	−7
33	2	1	0	78	2	1	10	6	6	2	2	1	6	1	59	−15
34	5	6	−1	78	1	5	11	3	3	2	2	1	3	3	65	−17
35	2	3	0	78	4	6	5	1	3	2	2	1	1	1	69	−5
36	0	−1	2	78	2	12	7	1	3	1	6	1	1	1	69	−9

Table 2 Integral graphs with $o \leq 80$ belonging to the class (2.25)

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	$\overline{\mu}_1$	$\overline{\mu}_2$
1	0	−1	1	36	1	4	8	1	4	4	1	1	1	1	25	−9
2	0	−1	1	39	1	7	6	1	2	2	1	1	1	1	32	−9
3	0	−1	1	66	1	8	14	1	7	7	1	1	1	2	49	−15
4	0	−1	1	72	1	4	16	2	8	4	2	1	2	1	51	−17
5	0	−1	1	78	1	7	12	2	4	2	2	1	2	1	65	−17
6	0	−1	2	78	2	10	8	1	4	4	1	1	1	1	67	−9

Table 3 Integral graphs with $o \leq 80$ belonging to the class (2.26)

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	$\overline{\mu}_1$	$\overline{\mu}_2$
1	0	−1	1	39	1	3	10	1	6	2	3	1	1	1	24	−9
2	2	5	−1	48	1	8	8	1	3	3	1	1	1	4	39	−11
3	5	6	−1	51	1	1	11	6	3	3	1	2	3	1	32	−17
4	0	−1	1	54	1	2	16	1	12	4	3	1	1	1	27	−9
5	1	1	0	54	2	2	8	1	6	6	1	1	1	1	39	−5
6	2	3	−1	57	1	2	15	2	9	9	1	2	1	2	34	−13
7	3	4	−1	57	1	1	14	5	6	2	3	1	5	1	34	−17
8	7	10	−3	57	1	2	13	3	3	3	2	1	3	2	38	−15
9	1	2	0	66	2	4	7	2	3	3	1	2	1	1	55	−9
10	2	7	−1	69	1	3	11	4	3	3	1	4	1	1	54	−17
11	0	−1	1	78	1	3	20	2	12	2	6	1	2	1	49	−17
12	2	5	−1	78	1	2	16	5	6	6	1	5	1	1	55	−21
13	0	−1	1	78	1	16	10	1	3	1	3	1	1	2	69	−15

Table 4 Integral graphs with $o \leq 80$ belonging to the class (2.27)

There exist exactly 7 non-isomorphic integral graphs with $\overline{\mu}_1 = 2a-1$ which belong to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, whose order does not exceed 80. They are represented in Table 2 under identification numbers $i = 1, 5, 7, 14, 17, 27, 31$.

There exist exactly 2 non-isomorphic integral graphs with $\overline{\mu}_1 = 3a-1$ which belong to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, whose order does not exceed 80. They are represented in Table 4 under identification numbers $i = 3, 8$.

There exist exactly 10 non-isomorphic integral graphs with $\overline{\mu}_1 = 4a-1$ which belong to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, whose order does not exceed 80. They are represented in Table 2 under identification numbers $i = 2, 3, 6, 10, 13, 15, 16, 19, 23, 28$.

There exists no integral graph with $\overline{\mu}_1 = 5b-1$ which belongs to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, whose order does not exceed 80. This completes my explanation on the Tables 1–4.

018^{01}	021^{02}	027^{02}	033^{02}	036^{03}	039^{05}	042^{02}	045^{02}	048^{01}	051^{04}	054^{06}	057^{05}
060^{02}	063^{03}	066^{07}	069^{06}	072^{03}	075^{05}	078^{12}	081^{08}	084^{06}	087^{05}	090^{03}	093^{05}
096^{06}	099^{06}	102^{09}	105^{04}	108^{11}	111^{06}	114^{09}	117^{13}	120^{05}	123^{07}	126^{10}	129^{09}
132^{10}	135^{06}	138^{13}	141^{07}	144^{10}	147^{10}	150^{06}	153^{06}	156^{18}	159^{14}	162^{15}	165^{10}
168^{09}	171^{17}	174^{14}	177^{14}	180^{13}	183^{14}	186^{14}	189^{12}	192^{10}	195^{13}	198^{16}	201^{13}
204^{16}	207^{18}	210^{08}	213^{13}	216^{21}	219^{17}	222^{18}	225^{13}	228^{18}	231^{11}	234^{26}	237^{13}
240^{09}	243^{23}	246^{21}	249^{15}	252^{19}	255^{12}	258^{25}	261^{11}	264^{25}	267^{15}	270^{17}	273^{13}
276^{23}	279^{16}	282^{10}	285^{22}	288^{21}	291^{28}	294^{20}	297^{16}	300^{15}	303^{10}	306^{22}	309^{23}
312^{32}	315^{12}	318^{23}	321^{14}	324^{31}	327^{19}	330^{20}	333^{28}	336^{24}	339^{13}	342^{31}	345^{15}
348^{28}	351^{29}	354^{34}	357^{22}	360^{22}	363^{10}	366^{23}	369^{18}	372^{25}	375^{14}	378^{30}	381^{13}
384^{24}	387^{18}	390^{32}	393^{16}	396^{29}	399^{26}	402^{29}	405^{19}	408^{23}	411^{22}	414^{37}	417^{23}
420^{16}	423^{18}	426^{32}	429^{27}	432^{35}	435^{12}	438^{35}	441^{24}	444^{30}	447^{20}	450^{23}	453^{31}
456^{40}	459^{26}	462^{22}	465^{16}	468^{41}	471^{24}	474^{31}	477^{35}	480^{27}	483^{22}	486^{44}	489^{26}
492^{37}	495^{30}	498^{29}	501^{26}	504^{28}	507^{32}	510^{22}	513^{32}	516^{37}	519^{15}	522^{35}	525^{16}
528^{42}	531^{36}	534^{34}	537^{17}	540^{46}	543^{21}	546^{33}	549^{32}	552^{37}	555^{21}	558^{44}	561^{27}
564^{30}	567^{29}	570^{31}	573^{17}	576^{39}	579^{38}	582^{43}	585^{41}	588^{37}	591^{19}	594^{43}	597^{23}
600^{24}											

Table 5 Distribution of integral graphs with $o = 1, 2, \dots, 600$

Finally, there exist exactly 3730 non-isomorphic integral graphs which belong to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, whose order does not exceed 600. In particular, there exist exactly 876, 1404, 525 and 925 non-isomorphic[2] integral graphs which belong to the classes Theorem 2.7 (2.24), (2.25), (2.26) and (2.27), respectively, whose order does not exceed 600. In view of this, there exist[3] exactly $876 + 1404 + 525 + 925 = 3730$ non-isomorphic integral graphs which belong to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, whose order does not exceed 600. Table 5 contains a distribution of those graphs in respect to their orders. In Table 5 the symbol o^n denotes the number of integral graphs of the corresponding order $o = 1, 2, \dots, 600$. In this table o^n is omitted if the

[2]In this work the data given in Tables 1–5 are obtained in two different ways: (i) they are generated by using relations ((2.24), (2.25), (2.26) and (2.27)) and (ii) by varying the parameters α, β, a, b in all possible ways in Eq. (2.2).

[3]Besides, there exists exactly 172483 non-isomorphic integral graphs which belong to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, whose order does not exceed 6000. In particular, (i) there exists exactly 1397, 973, 1802, 106, 93, 108, 109, 33 and 235 integral graphs which belong to the classes Theorem 2.2 (2.4), (2.5), (2.6), (2.7), (2.8), (2.9), (2.10), (2.11) and (2.12), respectively, whose order does not exceed 6000. In view of this, there exists exactly $1397+973+1802+106+93+108+109+33+235 = 4856$ non-isomorphic integral graphs with $\overline{\mu}_1 = 2ab-1$ which belong to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, whose order does not exceed 6000; (*ii*) there exists exactly 592, 386, 761, 288, 271, 307, 300, 109 and 618 integral graphs which belong to the classes Theorem 2.4 (2.14), (2.15), (2.16), (2.17), (2.18), (2.19), (2.20), (2.21) and (2.22), respectively, whose order does not exceed 6000. In view of this, there exists exactly $592+386+761+288+271+307+300+109+618 = 3632$ non-isomorphic integral graphs with $\overline{\mu}_1 = ab-1$ which belong to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, whose order does not exceed 6000 and (iii) there exists exactly 376, 336, 482, 200, 48, 159, 429, 87, 196, 382, 264 and 247 integral graphs which belong to the classes Theorem 2.10 (1^0), (2^0), (3^0), (4^0), (5^0), (6^0), (7^0), (8^0), (9^0), (10^0), (11^0) and (12^0), respectively, whose order does not exceed 6000. In view of this, there exists exactly $376 + 336 + 482 + 200 + 48 + 159 + 429 + 87 + 196 + 382 + 264 + 247 = 3206$ non-isomorphic integral graphs with $\alpha = 1$ and $\beta = 1$ which belong to the class $\overline{\alpha K_{a,a,a} \cup \beta K_{b,b,b}}$, whose order does not exceed 6000.

corresponding number $n = 0$.

Conclusion. Using results and a similar procedure presented in this work, it is possible to investigate any class of integral graphs which have exactly two main eigenvalues.

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