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Multicolor Ramsey Number of Stars Versus a Path

Xuejun ZHANG[1], Xinmin HOU[1,2,3,4,*]

1. *School of Data Science, University of Science and Technology of China, Anhui* 230026*, P. R. China;*
2. *School of Mathematical Sciences, University of Science and Technology of China, Anhui* 230026*, P. R. China;*
3. *CAS Key Laboratory of Wu Wen-Tsun Mathematics, University of Science and Technology of China, Anhui* 230026*, P. R. China;*
4. *Hefei National Laboratory, University of Science and Technology of China, Anhui* 230026*, P. R. China*

Abstract For given simple graphs $H_1, H_2, \ldots, H_c$, the multicolor Ramsey number $R(H_1, H_2, \ldots, H_c)$ is defined as the smallest positive integer n such that for an arbitrary edge-decomposition $\{G_i\}_{i=1}^c$ of the complete graph K_n, at least one G_i has a subgraph isomorphic to H_i. Let $m, n_1, n_2, \ldots, n_c$ be positive integers and $\Sigma = \sum_{i=1}^c (n_i - 1)$. Some bounds and exact values of $R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m)$ have been obtained in literature. Wang (Graphs Combin., 2020) conjectured that if $\Sigma \not\equiv 0 \pmod{m-1}$ and $\Sigma+1 \geq (m-3)^2$, then $R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m) = \Sigma+m-1$. In this note, we give a new lower bound and some exact values of $R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m)$ provided $m \leq \Sigma$, $\Sigma \equiv k \pmod{m-1}$, and $2 \leq k \leq m-2$. These results partially confirm Wang's conjecture.

Keywords Ramsey number; star; path

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1. Introduction

In this paper, all graphs are finite and simple. The vertex set, edge set, minimum degree, maximum degree, complement graph and edge chromatic number of a graph G are denoted by $V(G)$, $E(G)$, $\delta(G)$, $\Delta(G)$, $\overline{G}$ and $\chi'(G)$, respectively. For $v \in V(G)$, we use $d_G(v)$ to denote the degree of v in G. We use kG to denote k vertex disjoint copies of G. An decomposition of a graph G is a set $\{G_1, G_2, \ldots, G_c\}$ of edge-disjoint subgraphs of G such that $\bigcup_{i=1}^c G_i = G$. We use K_n^r to denote a complete n-partite graph with all partition classes having the same size r.

Let $H_1, H_2, \ldots, H_c$ be given simple graphs. The multicolor Ramsey number $R(H_1, H_2, \ldots, H_c)$ is defined as the smallest positive integer n such that for an arbitrary edge-decomposition $\{G_i\}_{i=1}^c$ of the complete graph K_n, at least one G_i has a subgraph isomorphic to H_i.

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* Corresponding author
E-mail address: tczxjwmz@mail.ustc.edu.cn (Xuejun ZHANG); xmhou@ustc.edu.cn (Xinmin HOU)

If $c \geq 3$, there are few known results about $R(H_1, H_2, \ldots, H_c)$ for very special graphs. In this note, we focus on $R(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_c}, P_m)$, where P_m is a path of order m. In the rest of this paper, let $\Sigma = \sum_{i=1}^{c}(n_i - 1)$. Some related results are listed here.

(a) ([1])

$$R(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_c}) = \begin{cases} \Sigma + 1, & \text{if } \Sigma \text{ and at least one } n_i \text{ are even,} \\ \Sigma, & \text{otherwise.} \end{cases}$$

(b) ([2]) $R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m) \leq m + \Sigma$. Moreover, if $\Sigma \equiv 0 \pmod{m-1}$ and $m + \Sigma$ is odd, then

$$R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m) = \Sigma + m.$$

(c) ([3])

(c1) If $m \geq \Sigma + 1$, then $R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m) = \max\{m, 2\Sigma + 1\}$.

(c2) If $m \leq \Sigma$, $\Sigma \equiv 0 \pmod{m-1}$ and $m + \Sigma$ is even, then

$$R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m) = \begin{cases} \Sigma + m, & \text{if all } n_i \ (1 \leq i \leq c) \text{ are odd;} \\ \Sigma + m - 1, & \text{if some } n_i \text{ is even.} \end{cases}$$

(c3) If $m \leq \Sigma$ and $\Sigma \not\equiv 0 \pmod{m-1}$, then

$$R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m) \leq \Sigma + m - 1.$$

Moreover, equality holds if $\Sigma \equiv 1 \pmod{m-1}$.

(c4) If $m \leq \Sigma$, $\Sigma \equiv k \pmod{m-1}$ and $2 \leq k \leq m-2$, then

$$R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m) \geq \Sigma + k - 2.$$

In [3], the author also proposed the following conjecture.

Conjecture 1.1 ([3]) *If $\Sigma \not\equiv 0 \pmod{m-1}$ and $\Sigma + 1 \geq (m-3)^2$, then*

$$R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m) = \Sigma + m - 1.$$

In this paper, we continue the study of the Ramsey number of stars versus a path and give a new lower bound and some exact values of $R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m)$ when $m \leq \Sigma$, $\Sigma \equiv k \pmod{m-1}$, and $2 \leq k \leq m-2$. These results partially confirm Conjecture 1.1. The main result is as follows.

Theorem 1.2 *Let $m \leq \Sigma$, $\Sigma \equiv k \pmod{m-1}$, $2 \leq k \leq m-2$, and let $s = \frac{\Sigma - k}{m-1}$. Then the following holds.*

(1) If $s + k \geq m - 2$ and $m + \Sigma$ is even, then

$$R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m) = \Sigma + m - 1.$$

(2) If $\Sigma \equiv 0 \pmod{m-2}$, $m + \Sigma$ is odd and $n_1, \ldots, n_c$ are odd, then

$$R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m) = \Sigma + m - 1.$$

(3) If $\Sigma \equiv 0 \pmod{m-2}$, $m + \Sigma$ is odd, $\Sigma < (m-2)^2$ and some n_i is even, then

$$R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m) = \Sigma + m - 2.$$

(4) $R(K_{1,n_1},\dots,K_{1,n_c},P_m)\geq \Sigma+\min\{s+k,m-2\}$.

The rest of this note is arranged as follows. We give some preliminaries in the next section. In Section 3, we prove Theorem 1.2.

2. Preliminaries

The following lemmas will be used in the proofs of this note. We first give some fundamental results in graph theory.

Lemma 2.1 ([4]) *Every graph has an even number of odd degree vertices.*

Lemma 2.2 ([4]) *For every simple graph G, $\chi'(G)\in\{\Delta(G),\Delta(G)+1\}$.*

Let $p(G)$ be the order of a longest path of a graph G.

Lemma 2.3 ([5]) *Let G be a connected graph of order $n\geq 3$ with $\delta(G)=\delta$. Then $p(G)\geq \min\{2\delta+1,n\}$.*

A graph G of order n is called overfull if $|E(G)|>\lfloor\frac{n}{2}\rfloor\Delta(G)$. Obviously, the order of an overfull graph must be odd.

Lemma 2.4 ([6]) *Let G be a complete multipartite graph. If G is not overfull, then $\chi'(G)=\Delta(G)$.*

Lemma 2.5 ([7]) *Let $n_1,\dots,n_c$ be positive integers, $\Sigma=\sum_{i=1}^{c}(n_i-1)$ and let H be a graph with $\chi'(H)\leq\Sigma$. Then H can be decomposed into edge-disjoint subgraphs $H_1,H_2,\dots,H_c$ such that $\Delta(H_i)\leq n_i-1$.*

Lemma 2.6 ([8]) *For odd n and even r, the complete multipartite graph K_n^r is the union of $\frac{n(r-1)-1}{2}$ edge-disjoint Hamilton cycles and a 1-factor. Otherwise, K_n^r is the union of $\frac{n(r-1)}{2}$ edge-disjoint Hamilton cycles.*

3. Proof of Theorem 1.2

In this section, we prove the main theorem.

Proof of Theorem 1.2 (1) By (c3), we have $R(K_{1,n_1},\dots,K_{1,n_c},P_m)\leq\Sigma+m-1$. Thus we only need to prove $R(K_{1,n_1},\dots,K_{1,n_c},P_m)>\Sigma+m-2$ in this case. Note that

$$\Sigma+m-2=s(m-1)+k+m-2=(s-m+k+2)(m-1)+(m-k)(m-2).$$

Consider a decomposition $K_{\Sigma+k-2}=G\cup\overline{G}$, where $\overline{G}=(s-m+k+2)K_{m-1}\cup(m-k)K_{m-2}$. Obviously, $\overline{G}$ contains no P_m. Since $\Sigma+m$ is even, G is not overfull. Note that G is a complete $(s+2)$-partite graph. By Lemma 2.4, we have $\chi'(G)=\Delta(G)=\Sigma$. By Lemma 2.5, G can be decomposed into edge-disjoint subgraphs $G_1,\dots,G_c$ such that $\Delta(G_i)\leq n_i-1$. Thus $R(K_{1,n_1},\dots,K_{1,n_c},P_m)>\Sigma+m-2$. Therefore, we have

$$R(K_{1,n_1},\dots,K_{1,n_c},P_m)=\Sigma+m-1.$$

(2) With the same reason as in (1), we only need to show $R(K_{1,n_1},\dots,K_{1,n_c},P_m)>\Sigma+m-2$. As $n_1,\dots,n_c$ are odd, $\Sigma=\sum_{i=1}^{c}(n_i-1)$ is even. Since $\Sigma+m$ is odd, we have m is odd. Let $\Sigma=t(m-2)$ for some positive integer t. Then $\Sigma+m-2=(t+1)(m-2)$. Consider the complete $(t+1)$-partite graph G on $\Sigma+m-2$ vertices with all parts of the same size $m-2$. By Lemma 2.6, G can be decomposed into $\frac{\Sigma}{2}$ edge-disjoint Hamilton cycles. For $1\le i\le c$, let G_i be the union of $\frac{n_i-1}{2}$ edge-disjoint Hamilton cycles. Then G_i is (n_i-1)-regular and thus it contains no K_{1,n_i}. Clearly, $\overline{G}$ contains no P_m. Therefore,

$$R(K_{1,n_1},\dots,K_{1,n_c},P_m)=\Sigma+m-1.$$

(3) Let $\Sigma=t(m-2)$. For the lower bound, consider the complete $(t+1)$-partite graph G on $\Sigma+m-3$ vertices with t parts of the same size $m-2$ and one part of size $m-3$. Since $\Sigma+m$ is odd, $\Sigma+m-3$ is even. So G is not overfull. By Lemma 2.4, $\chi'(K)=\Sigma$. By Lemma 2.5, G can be decomposed into edge-disjoint subgraphs $G_1,\dots,G_c$ such that $\Delta(G_i)\le n_i-1$. Clearly, the complement of G contains no P_m. Therefore, we have

$$R(K_{1,n_1},\dots,K_{1,n_c},P_m)\ge\Sigma+m-2.$$

For the upper bound, we assume to the contrary that there is a decomposition $K_{\Sigma+m-2}=\bigcup_{i=1}^{c+1}G_i$ such that G_i contains no K_{1,n_i} $(1\le i\le c)$ and G_{c+1} contains no P_m. Then $\delta(G_{c+1})\ge m-3$ (otherwise, at least one G_i for $1\le i\le c$ has $\delta(G_i)\ge n_i$, a contradiction to the assumption that G_i contains no K_{1,n_i}). Therefore, every component of G_{c+1} has at least $m-2$ vertices. By Lemma 2.3, each component of G_{c+1} has at most $m-1$ vertices. Since $\Sigma<(m-2)^2$, we have $|V(G_{c+1})|=\Sigma+m-2<(m-2)(m-1)$. This forces that every component of G_{c+1} has exact $m-2$ vertices. Since $\delta(G_{c+1})\ge m-3$, all components of G_{c+1} are complete graphs, i.e., $G_{c+1}=(t+1)K_{m-2}$. Thus $d_{\cup_{i=1}^{c}G_i}(v)=\Sigma+m-3-d_{G_{c+1}}(v)=\Sigma=\sum_{i=1}^{c}(n_i-1)$ for all $v\in V(K_{\Sigma+m-2})$. As each G_i contains no K_{1,n_i}, we have every G_i is an (n_i-1)-regular graph for $1\le i\le c$. Note that $V(G_i)=\Sigma+m-2$ is odd. Lemma 2.2 forces that all n_i-1 are even. This contradicts the assumption that at least one n_i is even.

(4) If $s+k<m-2$, let $n=\Sigma+s+k$, then

$$\begin{aligned}n-1&=\Sigma+s+k-1=s(m-1)+k+s+k-1\\&=(s-1)(m-2)+m-3+2(s+k).\end{aligned}$$

Consider a complete $(s+2)$-partite graph K on $n-1$ vertices with $s-1$ parts of the same size $m-2$, one part of size $m-3$ and two parts of size $s+k$. By Lemma 2.2, $\chi'(K)\le\Delta(K)+1=\Sigma$. By Lemma 2.5, K can be decomposed into edge-disjoint subgraphs $G_1,\dots,G_c$ such that $\Delta(G_i)\le n_i-1$. Obviously, the complement of K contains no P_m. So $R(K_{1,n_1},\dots,K_{1,n_c},P_m)\ge\Sigma+s+k$. If $s+k>m-2$, let $n=\Sigma+m-2$, then

$$\begin{aligned}n-1&=\Sigma+m-3=s(m-1)+k+m-3\\&=(s-m+k+1)(m-1)+(m-k+1)(m-2).\end{aligned}$$

Consider a complete $(s+2)$-partite graph K on $n-1$ vertices with $s-m+k+1$ parts of the same size $m-1$ and $m-k+1$ parts of the same size $m-2$. By Lemma 2.2, $\chi'(K)\le\Delta(K)+1=\Sigma$. By

Lemma 2.5, K can be decomposed into edge-disjoint subgraphs $G_1, \ldots, G_c$ such that $\Delta(G_i) \leq n_i - 1$. Since the complement of K contains no P_m, we have $R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m) \geq \Sigma + m - 2$. If $s + k = m - 2$ and $m + \Sigma$ is even, then by (1), we have

$$R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m) = \Sigma + m - 1.$$

If $s + k = m - 2$ and $m + \Sigma$ is odd,

$$\Sigma = s(m-1) + k = s(m-2) + s + k = (s+1)(m-2) < (m-2)^2.$$

By (2) and (3), we have

$$R(K_{1,n_1}, \ldots, K_{1,n_c}, P_m) = \begin{cases} \Sigma + m - 1, & \text{if } n_i \ (1 \leq i \leq c) \text{ are odd;} \\ \Sigma + m - 2, & \text{if some } n_i \text{ is even.} \end{cases}$$

The proof of the theorem is completed. □

4. Remarks

In this note, we confirm some special cases ((1) and (2) in Theorem 1.2) of Conjecture 1.1, and result (3) also supports the conjecture, but unfortunately, Conjecture 1.1 is still open.

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